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A class of Second Order BVPs On Infinite Intervals

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Abstract

In this work, we are concerned with a boundary value problem associated with a generalized Fisher-like equation. This equation involves an eigenvalue and a parameter which may be viewed as a wave speed. According to the behavior of the nonlinear source term, existence results of bounded solutions, positive solutions, classical as well as weak solutions are provided. We mainly use fixed point arguments.

1 INTRODUCTION

The aim of this paper is to prove existence theorems for the boundary value problem

$$\begin{cases} -u'' + cu' + \lambda u &= h(x,u), \qquad -\infty < x < +\infty. \\ \lim\limits_{|x|\to+\infty} u(x) &= 0. \end{cases} \tag{1.1}$$

The parameter $c > 0$ is a real positive constant while $h\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying $\lim\limits_{|x|\to+\infty} h(x,0) = 0$; the parameter $\lambda > 0$ may be seen as an eigenvalue of the problem. In the linear case $h(x,u) = f(x)u$, the problem arises in the study of a reaction-diffusion system involved in disease propagation throughout a given population [2]; the sublinear case $h(x,u) \leq f(x)u$ was also studied in [2] where existence and non existence results are given. For other recent developments in solvability to boundary value problems on unbounded domains see [1, 7] and references therein.

In this work, we investigate the nonlinear case; the study of problem (1.1) depends on the growth type of the source term h with respect to the second argument. In Section 2, we prove existence of bounded classical solutions in case the nonlinear right-hand term h obeys a generalized polynomial growth condition. Section 3 is devoted to proving existence of positive solutions under integral restrictions on the nonlinear function h. A general

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existence principle is given in Section 4. In Section 5, we show existence of positive solutions on the half-line under polynomial-like growth condition on the function h. Finally, existence of weak solutions is discussed in Section 6.

Our arguments will be based on fixed point theory. So, let us recall for the sake of completeness, respectively Schauder's and Schauder-Tichonoff's fixed point theorems [11]:

Theorem 1. Let E be a Banach space and $K \subset E$ a bounded, closed and convex subset of E. Let $F : K \longrightarrow K$ be a completely continuous operator. Then F has a fixed point in K.

Theorem 2. Let K be a closed, convex subset of a locally convex, Hausdorff space E. Assume that $T\colon K \longrightarrow K$ is continuous, and $T(K)$ is relatively compact in E. Then T has at least one fixed point in K.

In sequel, $C^k(I, \mathbb{R})$ ($k \in \mathbb{N}$) will refer to the space of k^{th} continuously differentiable functions defined on an interval I of the real line. $C_0(\mathbb{R}, \mathbb{R})$ stands for the space of continuous functions defined on the real line and vanishing at infinity; throughout this article, we will shorten the notation of this space to E_0. Endowed with the sup-norm $\|u\| = \sup_{x\in\mathbb{R}} |u(x)|$, it is a Banach space. Recall that $L^p(\mathbb{R})$ is the Banach space of p^{th} power integrable functions on $\mathbb{R}$. Hereafter, $\mathbb{R}_*^+$ refers to the set of positive real numbers and the notation $:=$ means throughout to be defined equal to.

2 A GENERALIZED POLYNOMIAL GROWTH CONDITION

The main existence result of this section is

Theorem 2.1 *The Green function being defined by (2.3), assume the following assumptions hold true:*

$$\begin{cases} \exists\, \Psi : [0, +\infty[\longrightarrow [0, +\infty[\\ \textit{continuous and nondecreasing;} \\ \exists\, q \in E_0 \textit{ positive, continuous such that} \\ |h(x,u)| \leq q(x)\Psi(|u|),\ \forall\,(x,u) \in \mathbb{R}^2; \\ \exists\, M_0 \in \mathbb{R}_*^+,\ \frac{\alpha\Psi(M_0)}{M_0} \leq 1 \\ \textit{with } \alpha := \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} G(x,y)q(y)\,dy < \infty. \end{cases} \tag{2.1}$$

Then Problem (1.1) admits a solution $u \in E_0$.

Example 2.1 *Consider a separated-variable nonlinear function* $h(x,y) = f(x)g(y)$ *with* $|f(x)| \leq \frac{1}{x^2+1} := q(x)$ *and* $|g(y)| \leq \sqrt{|y|+1} := \Psi(|y|)$*. The real numbers* r_1, r_2 *being defined in (2.4), we have* $0 \leq G(x,y) \leq \frac{1}{r_1-r_2}$*; then the constant* α *introduced in Assumptions 2.1 satisfies the estimate* $0 < \alpha \leq \frac{1}{r_1-r_2}\int_{-\infty}^{+\infty}\frac{1}{x^2+1}\,dx = \frac{\pi}{r_1-r_2}$*. We infer the existence of some positive number* M_0 *large enough such that* $\frac{\pi}{r_1-r_2}\sqrt{M_0+1} \leq M_0$*. Therefore,* $\alpha\Psi(M_0) \leq \frac{\pi}{r_1-r_2}\Psi(M_0) \leq M_0$ *and so Assumptions 2.1 are satisfied. For instance, the following problem has at least one nontrivial solution:*

$$\begin{cases} -u'' + cu' + \lambda u &= \frac{1}{(x^2+1)(u^2+1)}, \qquad -\infty < x < +\infty. \\ \lim\limits_{|x|\to+\infty} u(x) &= 0. \end{cases}$$

Remark 2.1 **(a)** *Assumptions (2.1) encompass the case where the nonlinear function* h *satisfies the polynomial growth condition*

$$\begin{array}{l} \exists\, f \in E_0,\ \exists\, \rho > 0,\ |h(x,y)| \leq |f(x)||y|^\rho,\ \forall\,(x,y) \in \mathbb{R}^2 \\ \textit{with either } (\rho \neq 1) \textit{ or } (\rho = 1 \textit{ and } \sup_{x\in\mathbb{R}}|f(x)| \leq \lambda). \end{array} \tag{2.2}$$

(b) *If, in Assumptions (2.1), the function* h *rather satisfies* $|h(x,y)| \leq |f(x)|\,|y|^\rho + \beta$, $\forall\,(x,y) \in \mathbb{R}^2$*, with some* $(f,\beta) \in E_0^2$*, then we can only take* $\rho < 1$ *in Assumption (2.2). This particular case was studied in [9]; Theorem 2.1 then improves a similar result obtained in [9].*

(c) *Since we work on the whole real line, Theorem 2.1, as well as the other existence theorems in this paper, provide solutions which are not in general known to be nontrivial. To ensure existence of nontrivial solutions, one must add assumptions on the nonlinear function* h *such as* $h(x,0) \not\equiv 0$ *further to* $\lim\limits_{|x|\to\pm\infty} h(x,0) = 0$*. Example 2.1 shows existence of at least one nontrivial solution.*

(d) *If we consider instead the autonomous case* $h(x,u) = g(u)$ *with* $g(0) = 0$*, then the trivial solution* $u \equiv 0$ *is the unique solution. Let us prove this in two steps:*

- *For any solution* u *to Problem (1.1), note that* $\lim\limits_{|x|\to+\infty} u'(x) = 0$*. We check this when* $x \to +\infty$*. Indeed, let*

$$\underline{\ell} := \liminf_{x\to+\infty} u'(x) \leq \bar{\ell} := \limsup_{x\to+\infty} u'(x).$$

Then by a classical fluctuation lemma [6], there exist two sequences $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *converging to positive infinity such that* $\underline{\ell} = \lim\limits_{n\to\infty} u'(x_n)$ *and* $\bar{\ell} = \lim\limits_{n\to\infty} u'(y_n)$ *whereas* $\lim\limits_{n\to\infty} u''(x_n) = \lim\limits_{n\to\infty} u''(y_n) = 0$*. Inserting into the equation in Problem (1.1), we find that* $c\underline{l} = c\bar{l} = 0$*; hence* $\underline{\ell} = \bar{\ell} = 0$ *for* $c > 0$*.*

- *Define the energy*

$$E(x) = \frac{1}{2}|u'(x)|^2 - \frac{\lambda}{2}|u(x)|^2 + G(u(x))$$

with $G(u) := \int_0^u g(s)\,ds$*. Then, multiplying the equation in Problem (1.1) by* u' *and making an integration by part, we find that* $E(x) = c\int_{-\infty}^{x}|u'|^2\,dx$ *and so* $E'(x) = c|u'(x)|^2 \geq 0$*. As* $\lim\limits_{x\pm\infty} E(x) = 0$*, we deduce that* E *is identically zero. From* $u'(x)^2 = \frac{1}{c}E'(x) = 0$*,* u *is constant and hence* $u \equiv 0$*.*

(d) *As for the separated-variable case* $h(x,u) = f(x)g(u)$*, we must impose* $g(0) \neq 0$ *both with* $f(\pm\infty) = 0$ *otherwise we could also obtain* $u \equiv 0$ *as a solution.*

Under Hypothesis (2.1), we will make use of Schauder's fixed point theorem to prove existence of a solution in a closed ball $B(0,R)$ with some radius $R > 0$.

Proof of Theorem 2.1 It is clear that Problem (1.1) is equivalent to the integral equation:

$$u(x) = \int_{-\infty}^{+\infty} G(x,y)h(y,u(y))\,dy$$

with Green function

$$G(x,y) = \frac{1}{r_1 - r_2}\begin{cases} e^{r_1(x-y)} & \text{if} \quad x \leq y \\ e^{r_2(x-y)} & \text{if} \quad x \geq y \end{cases} \tag{2.3}$$

and characteristic roots

$$r_1 = \frac{c + \sqrt{c^2 + 4\lambda}}{2} \quad \text{and} \quad r_2 = \frac{c - \sqrt{c^2 + 4\lambda}}{2}\cdot \tag{2.4}$$

Define the mapping $T\colon E_0 \to E_0$ by

$$Tu(x) = \int_{-\infty}^{+\infty} G(x,y)h(y,u(y))\,dy. \tag{2.5}$$

In view of Schauder's fixed point theorem, we look for fixed points for the operator T in the Banach space E_0. The proof is split into four steps.

- **Claim 1:** The mapping T is well defined; indeed, for any $u \in E_0$, we get, by Assumptions (2.1), the following estimates:

$$\begin{aligned} |Tu(x)| &\leq \int_{-\infty}^{+\infty} G(x,y)|h(y,u(y))|\,dy \\ &\leq \int_{-\infty}^{+\infty} G(x,y)q(y)\Psi(|u(y)|)\,dy \\ &\leq \Psi(\|u\|)\int_{-\infty}^{+\infty} G(x,y)q(y)\,dy, \ \forall\, x \in \mathbb{R} \\ &\leq \alpha\Psi(\|u\|). \end{aligned}$$

The convergence of the integral defining $Tu(x)$ is then established. In addition for any $y \in \mathbb{R}$, $G(\pm\infty, y) = 0$, and then, taking the limit in $Tu(x)$, we get, by l'Hospital Theorem, $Tu(\pm\infty) = 0$. Therefore, the mapping $T : E_0 \to E_0$ is well defined.

Claim 2: The operator T is continuous.
Let be a sequence $(u_n)_n \in E_0$ converge uniformly to u_0 on all compact subinterval of $\mathbb{R}$. For some fixed $a > 0$, we will prove the uniform convergence of $(Tu_n)_n$ to some limit Tu_0 on the interval $[-a, a]$. Let $\varepsilon > 0$ and choose some $b > a$ large enough. By the uniform convergence of the sequence $(u_n)_n$ on $[-b, b]$, there exists an integer $N = N(\varepsilon, b)$ satisfying

$$n \geq N \Longrightarrow I_1 := \sup_{x\in\mathbb{R}} \int_{-b}^{+b} G(x,y)|h(y,u_n(y)) - h(y,u_0(y))|\, dy < \frac{\varepsilon}{2}.$$

For $x \in [-a, a]$, we have that $|Tu_n(x) - Tu_0(x)| \leq I_1 + I_2 + I_3$ with:

$$I_2 := \sup_{x\in\mathbb{R}} \int_{\mathbb{R}-[-b,+b]} G(x,y)|h(y,u_0(y))|\, dy \leq \tfrac{\varepsilon}{4}$$
(by Cauchy Convergence Criterion and $\lim\limits_{|y|\to+\infty} h(y,u_0(y)) = 0$.)

$$I_3 := \sup_{x\in\mathbb{R}} \int_{\mathbb{R}-[-b,+b]} G(x,y)|h(y,u_n(y))|\, dy \leq \tfrac{\varepsilon}{4}.$$
(by Lebesgue Dominated Convergence Theorem.)

This proves the uniform convergence of the sequence $(Tu_n)_n$ to the limit Tu_0 on the interval $[-a, a]$.

Claim 3: For any $M > 0$, the set $\{Tu, \|u\| \leq M\}$ is relatively compact in E_0. By Ascoli-Arzela Theorem, it is sufficient to prove that all the functions of this set are equicontinuous on every subinterval $[-a, a]$ and that there exists a function $\gamma \in E_0$ such that for any $x \in \mathbb{R}$, $|Tu(x)| \leq \gamma(x)$. Let $x_1, x_2 \in [-a, a]$; we have successively the estimates:

$$\begin{array}{rcl} |Tu(x_2) - Tu(x_1)| & \leq & \int_{-\infty}^{+\infty} |G(x_2,y) - G(x_1,y)||h(y,u(y))|\, dy \\ & \leq & \int_{-\infty}^{+\infty} |G(x_2,y) - G(x_1,y)|q(y)\Psi(|u(y)|)\, dy \\ & \leq & \Psi(M) \int_{-\infty}^{+\infty} |G(x_2,y) - G(x_1,y)|q(y)\, dy. \end{array}$$

By continuity of the Green function G, the latter term tends to 0, when x_2 tends x_1; whence comes the equicontinuity of the functions $\{T(u); \|u\| \leq M\}$. Now, we check analogously the second statement:

$$\begin{array}{rcl} |Tu(x)| & \leq & \int_{-\infty}^{+\infty} G(x,y)|h(y,u(y))|\, dy \\ & \leq & \int_{-\infty}^{+\infty} G(x,y)q(y)\Psi(|u(y)|)\, dy \\ & \leq & \Psi(M) \int_{-\infty}^{+\infty} G(x,y)q(y)\, dy := \gamma(x), \ \forall\, x \in \mathbb{R}. \end{array}$$

By l'Hopital Theorem, we have that $\gamma \in E_0$.

Claim 4: There exists some $R > 0$ such that T maps the closed ball $B(0, R)$ into itself. From assumption (2.1), we know that there is some positive number M_0 such that $\frac{\alpha\Psi(M_0)}{M_0} \leq 1$. If $\|u\| \leq M_0$, then

$$\begin{aligned} \|T(u)\| &\leq \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} G(x,y)q(y)\Psi(|u(y)|)\, dy \\ &\leq \alpha\Psi(M_0) \\ &\leq M_0, \end{aligned}$$

so that it is enough to take $R = M_0$. The proof of Theorem 2.1 then follows from Schauder's fixed point theorem.

3 EXISTENCE OF POSITIVE SOLUTIONS

Making use of Schauder-Tichonov's theorem, we prove here existence of a positive solution under an integral condition on the nonlinear term:

Theorem 3.1 *Problem (1.1) has a positive solution provided the following mean growth assumption on the nonlinear function h is fulfilled*

$$\left\{\begin{array}{l} \textit{The function } h \textit{ is positive and satisfies } h(x,u) \leq H(x,|u|) \\ \textit{where } H : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+ \textit{is continuous, nondecreasing} \\ \textit{ with respect to the second argument and verifies} \\ \exists\, c_* > 0,\ \int_{-\infty}^{+\infty} H(x,c_*)\, dx \leq c_*(r_1 - r_2). \end{array}\right. \tag{3.1}$$

Remark 3.1 *It is easy to check that in the separated-variable case, Assumption (3.1) leads to Assumption (2.1).*

Example 3.1 *The problem*

$$\left\{\begin{array}{rcl} -u'' + cu' + \lambda u &=& \frac{u^n}{x^2+u^2} + \frac{1}{x^2+1},\ (\ n \in \mathbb{N},\ n > 2) \quad -\infty < x < +\infty; \\ \lim\limits_{|x|\to+\infty} u(x) &=& 0 \end{array}\right.$$

has at least one positive nontrivial solution. Indeed, the function $H(x,y) = \frac{y^n}{x^2+y^2} + \frac{1}{x^2+1}$ satisfies $\lim\limits_{|x|\to 0} H(x,0) = 0$, $H(x,0) \not\equiv 0$ and is nondecreasing in the second argument y for any integer $n > 2$. Moreover, $\int_{-\infty}^{+\infty} H(x,y)\, dx = \pi(1 + y^{n-1})$ so that Assumption (3.1) may be satisfied. For instance, if we take $n = 3$, then there exists $c_* \in]c_1, c_2[$ with $c_1 = \frac{k-\sqrt{k^2-1}}{2}$, $c_2 = \frac{k+\sqrt{k^2-1}}{2}$ assuming $k := \frac{r_1-r_2}{\pi} > 2$.

To prove Theorem 3.1, we proceed as in Theorem 2.1, and reformulate Problem (1.1) as a fixed point problem for the mapping T defined in 2.5. Here, we appeal to Schauder-Tichonov's fixed point theorem. Let K be the closed convex subset of E_0 defined by:

$$K = \{u \in E_0,\ 0 \leq u(x) \leq c_*,\ \forall\, x \in \mathbb{R}\}.$$

Using Assumption (3.1) and the fact that the mapping H is nondecreasing in the second argument, we find that T maps K into itself. Indeed, taking into account the bound $0 < G(x,y) \leq \frac{1}{r_1 - r_2}$, we derive the straightforward estimates:

$$\begin{array}{rcl} 0 \leq Tu(x) & \leq & \int_{-\infty}^{+\infty} G(x,y)H(y,|u(y)|)\,dy \\ & \leq & \frac{1}{r_1-r_2}\int_{-\infty}^{+\infty} H(y,c_*)\,dy \\ & \leq & c_*. \end{array}$$

Since $0 \leq Tu(x) \leq \int_{-\infty}^{+\infty} G(x,y)H(y,c_*)\,dy$ and $G(\pm\infty, y) = 0,\ \forall\, y \in \mathbb{R}$, we have that $Tu(\pm\infty) = 0$ and so $T(E_0) \subset E_0$. In addition, the mapping T is continuous as can easily be seen. It remains to check that $T(K)$ is relatively compact. By Ascoli-Arzela Theorem, it is sufficient to prove that all the functions of this set are equicontinuous on every subinterval $[-a,a]$ and that there exists a function $\gamma \in E_0$ such that for any $x \in \mathbb{R}$, $|Tu(x)| \leq \gamma(x)$. Let $x_1, x_2 \in [-a,a]$;

$$\begin{array}{rcl} |Tu(x_2) - Tu(x_1)| & \leq & \int_{-\infty}^{+\infty} |G(x_2,y) - G(x_1,y)||h(y,u(y))|\,dy \\ & \leq & \int_{-\infty}^{+\infty} |G(x_2,y) - G(x_1,y)|H(y,c_*)\,dy. \end{array}$$

By continuity of the function G, we deduce from Lebesgue dominated convergence theorem that the last right-hand term tends to 0 when x_2 tends to x_1. Whence comes the compactness of $T(K)$ by Ascoli-Arzela Lemma and then the claim of Theorem 3.1 follows.
Now, we check analogously the second statement:

$$|Tu(x)| \leq \int_{-\infty}^{+\infty} G(x,y)H(y,c_*)\,dy \equiv \gamma(x)$$

with $\gamma \in E_0$ for $G(\pm\infty, y) = 0,\ \forall\, y \in \mathbb{R}$.

4 A FURTHER TYPE OF GROWTH

In this section, we prove existence of bounded, solutions to Problem (1.1) under new growth conditions on the nonlinearity h; by the way we show that polynomial-like growth condition may be relaxed. The proof of our existence result relies on the following fixed point theorem by Furi and Pera [3]. This theorem was also used in [1] to deal with a BVP on an infinite interval.

Theorem 3. Let E be a Fréchet space, Q a closed convex subset of E, $0 \in Q$ and let $T : Q \to E$ be a continuous compact mapping. Assume further that, for any sequence $(u_j, \mu_j)_{j\geq 1}$ from $\partial Q \times [0,1]$ that converges to (u, μ) with $u = \mu Tu$, $0 \leq \mu < 1$, one has $\mu_j Tu_j \in Q$ for all j large enough.

Then, T has a fixed point in Q.

Our aim is now to prove

Theorem 4.1 *First, assume the following sign assumption is fulfilled:*

(H1) $\exists\, M_0 > 0$ *such that* $|y| > M_0 \;\Rightarrow\; yh(x,y) \le 0,\ \forall\, x \in \mathbb{R}$.
Then Problem (1.1) has a bounded solution provided either one of the following growth assumptions is satisfied:
(H2)$_1$ *There exists a function* $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ *continuous and nondecreasing with respect to the second argument such that:*
$|h(x,y)| \le H(x,|y|),\ \forall\, (x,y) \in \mathbb{R}^2$ *with* $\int_{-\infty}^{+\infty} H(x, M_0+1)\, dx < \infty$;
or (H2)$_2$ $|h(x,y)| \le q(x)\psi(|y|),\ \forall\, (x,y) \in \mathbb{R}^2$
with $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *continuous and nondecreasing,* $q \in E_0$ *positive, continuous and* $\alpha := \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} G(x,y)q(y)\, dy < \infty$;
or (H2)$_3$ $\lim\limits_{x\to+\infty} \sup_{|y|\le M_0+1} |h(x,y)| = 0$.

Example 4.1 *Consider the function* h *defined by*

$$h(x,y) = \begin{cases} \frac{1-y}{x^2+1}, & y \ge +1 \\ \frac{1}{x^2+1}, & -1 \le y \le +1 \\ \frac{-1-y}{x^2+1}, & y \le -1 \end{cases}$$

Then, $|h(x,y)| \le \frac{1+|y|}{x^2+1} = H(x,|y|)$ *with* $H(x,s) = \frac{1+s}{x^2+1}$. *Since* $\int_{-\infty}^{+\infty} \frac{2}{x^2+1}\, dx = 2\pi < \infty$, *Assumptions (H1) and* $(H2)_1$ *are satisfied with* $M_0 = 1$. *Since* $h(x,0) \not\equiv 0$, *Problem (1.1) has a nontrivial solution for such a nonlinear right-hand term. We may notice that the function* h *can be written as* $h(x,y) = \frac{\theta(y)-y}{x^2+1}$ *where* $\theta(y) = \mathcal{H}(y-1) - \mathcal{H}(-y-1)$, *the function* $\mathcal{H}$ *being the Heaviside function.*

Remark 4.1 (a) *The sign condition (H1) implies that any solution* u *of Problem (1.1) satisfies* $|u(x)| \le M_0,\ \forall\, x \in \mathbb{R}$. *Indeed, on the contrary, assume* $\max_{x\in\mathbb{R}} |u(x)| = |u(x_0)| > M_0$ *for some* $x_0 \in \mathbb{R}$. *Then* $u'(x_0) = 0$ *and* $-u''(x_0)u(x_0) + cu'(x_0)u(x_0) + \lambda u(x_0)^2 = u(x_0)f(x_0, u(x_0)) \le 0$; *this yields* $u''(x_0)u(x_0) \ge 0$, *leading to a contradiction.*

(b) *Assumption* $(H2)_1$ *is weaker than the one in Theorem 3.1.*

(c) *Assumption* $(H2)_2$ *is weaker than the one in Theorem 2.1.*

(d) *Then, when Assumption (H1) is satisfied, Theorem 4.1 improves both of Theorems 2.1 and 3.1.*

Proof of Theorem 4.1. For the sake of clarity, we only do the proof in case Assumptions (H1) and (H2) are simultaneously satisfied. The other cases can be treated similarly. In the Fréchet space $E := C(\mathbb{R},\mathbb{R})$, set $r_0 := M_0+1$, consider the closed, convex set $Q = \{u \in E :\ \sup_{x\in\mathbb{R}} |u(x)| \le r_0\}$, and define the mapping $T\colon Q \to E$ as in (2.5). In three steps, we carry over the proof.

• **Claim 1.** T is continuous. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in Q such that $u_n \to u$ in Q; we show that $Tu_n \to Tu$ in Q, as n goes to infinity. We have that for any $x \in \mathbb{R}$, $|h(x, u_n(x))| \le H(x, r_0)$, $|h(x, u(x))| \le H(x, r_0)$ and that $\lim_{n\to\infty} h(x, u_n(x)) = h(x, u(x))$. By the Dominated Convergence Lebesgue's Theorem, we deduce that $\lim_{n\to\infty} Tu_n(x) = Tu(x)$ on each subinterval $[-x_m, x_m]$.

In addition, for $x_1, x_2 \in [-x_m, x_m]$, the following estimates hold true

$$\begin{array}{rcl} |Tu_n(x_2) - Tu_n(x_1)| & \le & \int_{-\infty}^{+\infty} |G(x_2, y) - G(x_1, y)| H(y, r_0)\, dy; \\ |Tu(x_2) - Tu(x_1)| & \le & \int_{-\infty}^{+\infty} |G(x_2, y) - G(x_1, y)| H(y, r_0)\, dy. \end{array}$$

Therefore, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|x_2 - x_1| < \delta \Rightarrow |Tu(x_2) - Tu(x_1)| < \varepsilon \text{ and } |Tu_n(x_2) - Tu_n(x_1)| < \varepsilon, \forall\, n \in \mathbb{N}.$$

Furthermore, the convergence is uniform since $Tu_n(x) \to Tu(x)$ on $[-x_m, x_m]$ as $n \to +\infty$, and the claim follows.

• **Claim 2.** Using Ascoli-Arzela Theorem, we are going to prove that $T(Q)$ is relatively compact in E, that is $T(Q)$ is uniformly bounded and equicontinuous on each subinterval $[-x_m, x_m]$. For any $x \in [-x_m, x_m]$ and any $u \in Q$, we have:

$$\begin{array}{rcl} |Tu(x)| & \le & \int_{-\infty}^{+\infty} G(x, y) H(y, |u(y)|)\, dy \\ & \le & \int_{-\infty}^{+\infty} G(x, y) H(y, r_0)\, dy := \psi_{r_0}(x), \end{array}$$

with, by l'Hospital rule, $\lim_{x\to\pm\infty} \psi_{r_0}(x) \to 0$; that is $\psi_{r_0} \in E_0$. Moreover, $T(Q)$ is equicontinuous on each subinterval $[-x_m, x_m]$. Indeed, let $x_1, x_2 \in [-x_m, x_m]$ with $x_1 < x_2$; we have:

$$|Tu(x_2) - Tu(x_1)| \le \int_{-\infty}^{+\infty} |G(x_2, y) - G(x_1, y)| H(y, r_0)\, dy$$

which also tends to 0 as x_2 tends to x_1, for any $u \in Q$.

• **Claim 3.** Now, we check the last assumption in Furi-Pera's Theorem. Consider some sequence $(u_j, \mu_j)_{j\ge 0} \in \partial Q \times [0, 1]$ such that, when $j \to \infty$, $\mu_j \to \mu$ and $u_j \to u$ with $u = \mu Tu$ and $\mu \in [0, 1]$. We must show that $\mu_j Tu_j \in Q$ as $j \to \infty$. Let $v \in E$ be such that $|v(x)| \le r_0$ for $x \in \mathbb{R}$, then we have that $|Tv(x)| \le \int_{-\infty}^{+\infty} G(x, y) H(y, r_0)\, dy := \psi_{r_0}(x)$ and $\lim_{|x|\to+\infty} \psi_{r_0}(x) = 0$. Since $u_j \in Q$, there exists some $x^* \in \mathbb{R}$ such that

$$\forall\, x \in \mathbb{R} \setminus [-x^*, x^*],\ |\mu_j Tu_j(x)| \le r_0.$$

Let us consider the case $x \in [-x^*, x^*]$. Since $\mu_j \to \mu$ and $T(Q)$ is bounded in E, the sequence $\mu_j T u_j$ converges uniformly to $\mu T u$ on $[-x^*, x^*]$; thus there exists some $j_0 \in \mathbb{N}^*$ such that $\forall\, j \geq j_0,\ |\mu_j T u_j(x)| \leq |\mu T u(x)| + 1,\ \forall\, x \in [-x^*, x^*]$. We have also by Remak 4.1(a) that $|\mu T u(x)| \leq M_0$. Therefore, for j large enough, it holds that

$$|\mu_j T u_j(x)| \leq M_0 + 1 = r_0,\ \forall\, x \in [-x^*, x^*].$$

We then conclude the estimate $|\mu_j T u_j(x)| \leq M_0 + 1 = r_0,\ \forall\, x \in \mathbb{R}$.

The claim of Theorem 4.1 then follows from Theorem 3.

5 THE PROBLEM ON THE POSITIVE HALF LINE

5.1 Setting of the problem

In this section, we consider the problem posed on the positive half-line:

$$\left\{ \begin{array}{rcl} -u'' + cu' + \lambda u & = & h(x, u(x)), \quad x \in I \\ u(0) = u(+\infty) & = & 0. \end{array} \right. \tag{5.1}$$

Hereafter, I denotes $]0, +\infty[$, the set of positive real numbers. Setting $k\colon= \sqrt{\lambda + c^2/4}$, we rewrite the problem for the function $v(x) = e^{\frac{-c}{2}x} u(x)$:

$$\left\{ \begin{array}{rcl} -v'' + k^2 v & = & e^{\frac{-c}{2}x} h(x, e^{\frac{c}{2}x} v(x)), \quad x \in I \\ v(0) = v(+\infty) & = & 0. \end{array} \right. \tag{5.2}$$

Equivalently, the unknown v satisfies the integral equation:

$$v(x) = \int_0^{+\infty} K(x,s) e^{\frac{-c}{2}s} h(s, e^{\frac{c}{2}s} v(s))\, ds$$

with new Green function

$$K(x,s) = \frac{1}{2k} \left\{ \begin{array}{ll} e^{-ks}(e^{kx} - e^{-kx}) & x \leq s \\ e^{-kx}(e^{ks} - e^{-ks}) & x \geq s. \end{array} \right. \tag{5.3}$$

Notice that K is different from the one in (2.3) and that the unknown u is now solution of the integral equation:

$$u(x) = \int_0^{+\infty} e^{\frac{c}{2}(x-s)} K(x,s) h(s, u(s))\, ds.$$

The following lemma provides estimates of the Green function K and will play an important role in the sequel; we omit the proof:

Lemma 5.1 *We have*

$$\begin{array}{ll} (a) & K(x,s) \leq \frac{1}{2k},\ K(x,s)e^{-\mu x} \leq K(s,s)e^{-ks},\ \ \forall\, x, s \in I,\ \forall\, \mu \geq k. \\ (b) & \forall\, s \in I, \forall\, (0 < \gamma < \delta),\ \forall\, x \in [\gamma, \delta]\,,\ K(x,s) \geq mK(s,s)e^{-ks}. \end{array} \tag{5.4}$$

Here $m\colon= \min\left\{e^{-k\delta}, e^{k\gamma} - e^{-k\gamma}\right\}$.

Under suitable assumptions on the nonlinear function h, we shall prove the existence of a positive solution to Problem (5.1). The proof relies on Krasnosels'kii fixed point theorem in cones ([5], [8]) and Zima's compactness criterion [12]; but first of all, let us recall some

5.2 Preliminaries

Definition 5.1 *A nonempty subset $\mathcal{C}$ of a Banach space X is called a cone if $\mathcal{C}$ is convex, closed, and satisfies*
(i) $\alpha x \in \mathcal{C}$ for all $x \in \mathcal{C}$ and any real positive number α,
(ii) $x, -x \in \mathcal{C}$ imply $x = 0$.

Definition 5.2 *A set of functions $u \in \Omega \subset X$ are said to be almost equicontinuous on I if they are equicontinuous on each interval $[0, T]$, $0 < T < +\infty$.*

Next we state Krasnosels'kii Fixed Point Theorem in cones.

Theorem 4. ([8]) Let X be a Banach space and $\mathcal{C} \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open sets in X such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let $F : \mathcal{C} \cap (\overline{\Omega}_2 - \Omega_1) \longrightarrow \mathcal{C}$ be a completely continuous operator such that either
(i) $\|Fx\| \leq \|x\|$ for $x \in \mathcal{C} \cap \partial\Omega_1$ and $\|Fx\| \geq \|x\|$ for $x \in \mathcal{C} \cap \partial\Omega_2$,
or (ii) $\|Fx\| \geq \|x\|$ for $x \in \mathcal{C} \cap \partial\Omega_1$ and $\|Fx\| \leq \|x\|$ for $x \in \mathcal{C} \cap \partial\Omega_2$
is satisfied.
Then F has at least one fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 - \Omega_1)$.

Now, let p: $I \longrightarrow]0, +\infty[$ be a continuous function. Denote by X the Banach space consisting of all weighted functions u continuous on I and satisfying

$$\sup_{x \in I}\{|u(x)|p(x)\} < \infty,$$

equipped with the norm $\|u\| = \sup_{x \in I}\{|u(x)|p(x)\}$. We have

Lemma 5.2 *([13]) If the functions $u \in \Omega$ are almost equicontinuous on I and uniformly bounded in the sense of the norm*

$$\|u\|_q = \sup_{x \in I}\{|u(x)|q(x)\}$$

where the function q is positive, continuous on I and satisfies

$$\lim_{x \to +\infty} \tfrac{p(x)}{q(x)} = 0,$$

then Ω is relatively compact in X.

Having disposed of these auxiliary results, we are ready to prove

Theorem 5.1 *Suppose that:*

$$\begin{cases} h : I \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \ \text{ is a continuous function,} \\ \exists\, p > 0 : \ p \neq 1, \ h(x,u) \leq a(x) + b(x)u^p, \quad \forall\,(x,u) \in I \times \mathbb{R}^+, \\ \ \text{where } a, b : I \longrightarrow \mathbb{R}^+ \ \text{ are continuous positive functions} \\ \ \text{vanishing at positive infinity and} \end{cases} \tag{5.5}$$

$$\begin{cases} \text{there exists } \theta > k + \frac{c}{2} \ \text{ such that} \\ M_1 := \int_0^{+\infty} e^{-(k+\frac{c}{2})s} a(s)\, ds < \infty, \\ M_2 := \int_0^{+\infty} e^{(p\theta - k - \frac{c}{2})s} b(s)\, ds < \infty. \end{cases} \tag{5.6}$$

$$2k\left(\frac{2k}{pM_2}\right)^{\frac{1}{p-1}} - M_2\left(\frac{2k}{pM_2}\right)^{\frac{p}{p-1}} - M_1 \geq 0, \quad \text{when } p > 1. \tag{5.7}$$

$$2k\left(\frac{2k}{pM_2}\right)^{\frac{1}{p-1}} - M_2\left(\frac{2k}{pM_2}\right)^{\frac{p}{p-1}} - M_1 \leq 0, \quad \text{when } 0 < p < 1. \tag{5.8}$$

$$\begin{cases} \ \text{There exist } \alpha > 0, \ \gamma, \delta > 0 \ \text{and } x_0 \in I \ \text{such that:} \\ \min_{x \in [\gamma,\delta], u \in \left[m\alpha, \alpha e^{\theta\delta}\right]} h(x,u) \geq \alpha e^{\theta x_0} \left[\int_\gamma^\delta e^{\frac{c}{2}(x_0 - s)} K(x_0, s)\, ds\right]^{-1}, \end{cases} \tag{5.9}$$

the Green function K being defined in (5.3) and $m := \min\left\{e^{-k\delta}, e^{k\gamma} - e^{-k\gamma}\right\}$.
Then Problem (5.1) has at least one positive solution $u \in C(I;\mathbb{R}^+)$.

Remark 5.1 *The case $p = 1$ is treated in [2].*

Proof of Theorem 5.1:
We follow the method used in [2, 14]. Let $\theta \in \mathbb{R}$ be as in Hypothesis (5.6) and consider the weighted space $X = \{u \in C(I;\mathbb{R}) \colon \sup_{x\in I}\{e^{-\theta x}|u(x)|\} < \infty\}$ endowed with the weighted sup-norm:

$$\|u\|_\theta = \sup_{x\in I}\{e^{-\theta x}|u(x)|\},$$

as well as the positive cone in X:

$$\mathcal{C} = \{u \in X; u \geq 0 \ \text{ on } \ I \ \text{ and } \ \min_{x\in[\gamma,\delta]} u(x) \geq m\,\|u\|_\theta\}.$$

Next define on $\mathcal{C}$ the operator F by:

$$Fu(x) = \int_0^{+\infty} e^{\frac{c}{2}(x-s)} K(x,s) h(s,u(s))\, ds.$$

(a) First step. In the following, we study the properties of this operator:

- **Claim 1**:

For any $u \in X$, $\sup_{x\in I} e^{-\theta x}\,|Fu(x)| < \infty$, that is $F(X) \subset X$.

Indeed, choosing $\mu = \theta - \frac{c}{2}$ in (5.4) (note that $\mu \geq k$ by (5.6)), and using (5.4)(a), (5.5), we have the estimates

$$
\begin{aligned}
&0 \leq e^{-\theta x} Fu(x) = \\
&= e^{-\theta x} \int_0^{+\infty} e^{\frac{c}{2}(x-s)} K(x,s) h(s,u(s))\, ds \\
&= e^{-(\theta-\frac{c}{2})x} \int_0^{+\infty} e^{\frac{-c}{2}s} K(x,s) h(s,u(s))\, ds \\
&\leq \int_0^{+\infty} e^{-(k+\frac{c}{2})s} K(s,s) \left[a(s) + b(s)|u(s)|^p\right]\, ds \\
&\leq \frac{1}{2k} \int_0^{+\infty} e^{-(k+\frac{c}{2})s} a(s)\, ds \;+\; \frac{1}{2k} \|u\|_\theta \int_0^{+\infty} e^{(p\theta-k-\frac{c}{2})s} b(s)\, ds \\
&\leq \frac{1}{2k}(M_1 + M_2 \|u\|_\theta^p) < \infty.
\end{aligned}
$$

- **Claim 2**:

For any $u \in \mathcal{C}$, let us check that $\min_{x\in[\gamma,\delta]} Fu(x) \geq m \|Fu\|_\theta$ that is $F(\mathcal{C}) \subset \mathcal{C}$. Indeed, $Fu(x) \geq 0 \quad \forall\, x \in I$ and $\forall\, x \in [\gamma,\delta]$, $\forall\, s,\tau \in I$ we have, choosing $\mu = \theta - \frac{c}{2}$ in (5.4)(a):

$$
\begin{aligned}
\min_{x\in[\gamma,\delta]} Fu(x) \;&\geq\; m \int_0^{+\infty} e^{\frac{c}{2}(\gamma-s)} K(s,s) e^{-ks} h(s,u(s))\, ds \\
&\geq\; m e^{\gamma\frac{c}{2}} e^{-\theta\tau} \int_0^{+\infty} e^{\frac{c}{2}(\tau-s)} K(\tau,s) h(s,u(s))\, ds \\
&\geq\; m e^{\frac{c}{2}\gamma} \|Fu\|_\theta \geq m \|Fu\|_\theta\, .
\end{aligned}
$$

Next, we prove that F is completely continuous:

- **Claim 3**:

Let $\Omega_1 = \{u \in X, \|u\|_\theta < r\}, \Omega_2 = \{u \in X, \|u\|_\theta < R\}$, the constants $0 < r < R$ being real positive numbers to be selected later on. Consider some $u \in \mathcal{C} \cap \overline{\Omega}_2$; then Fu is uniformly bounded. Indeed, as in claim 1, we have that $\|Fu\|_\theta \leq \frac{1}{2k}(M_1 + M_2 R^p)$, for any $u \in \mathcal{C} \cap \overline{\Omega}_2$.

- **Claim 4**:

The functions $\{Fu\}$ for $u \in \mathcal{C}K \cap \overline{\Omega}_2$ are almost equicontinuous on I; indeed

$$
\begin{aligned}
|Fu(x_2) - Fu(x_1)| \;&\leq\; \int_0^{+\infty} \left| e^{\frac{c}{2}(x_2-s)} K(x_2,s) - e^{\frac{c}{2}(x_1-s)} K(x_1,s) \right| h(s,u(s))\, ds \\
&=\; \int_0^{x_1} \left| e^{\frac{c}{2}(x_2-s)} K(x_2,s) - e^{\frac{c}{2}(x_1-s)} K(x_1,s) \right| h(s,u(s))\, ds \\
&+\; \int_{x_1}^{x_2} \left| e^{\frac{c}{2}(x_2-s)} K(x_2,s) - e^{\frac{c}{2}(x_1-s)} K(x_1,s) \right| h(s,u(s))\, ds \\
&+\; \int_{x_2}^{+\infty} \left| e^{\frac{c}{2}(x_2-s)} K(x_2,s) - e^{\frac{c}{2}(x_1-s)} K(x_1,s) \right| h(s,u(s))\, ds.
\end{aligned}
$$

In the following, we derive lengthy estimates of each of the summands in the right-hand side. We have successively:

$$
\int_0^{x_1} \left| e^{\frac{c}{2}(x_2-s)} K(x_2,s) - e^{\frac{c}{2}(x_1-s)} K(x_1,s) \right| h(s,u(s))\, ds \leq
$$

$$
\begin{aligned}
\leq\ & \frac{1}{2k}\left|e^{(\frac{c}{2}-k)x_2}-e^{(\frac{c}{2}-k)x_1}\right|\int_0^{x_1} e^{-\frac{c}{2}s}(e^{ks}-e^{-ks})a(s)\,ds \\
+\ & \frac{1}{2k}\left|e^{(\frac{c}{2}-k)x_2}-e^{(\frac{c}{2}-k)x_1}\right|\int_0^{x_1} e^{-\frac{c}{2}s}(e^{ks}-e^{-ks})b(s)|u(s)|^p\,ds \\
\leq\ & \frac{1}{2k}\left|e^{(\frac{c}{2}-k)x_2}-e^{(\frac{c}{2}-k)x_1}\right|\int_0^{x_1} e^{-\frac{c}{2}s}(e^{ks}-e^{-ks})a(s)\,ds \\
+\ & \frac{1}{2k}\left|e^{(\frac{c}{2}-k)x_2}-e^{(\frac{c}{2}-k)x_1}\right|\|u\|_\theta^p\int_0^{x_1} e^{(p\theta-c)s}(e^{ks}-e^{-ks})b(s)\,ds \\
\leq\ & \frac{1}{2k}\left|e^{(\frac{c}{2}-k)x_2}-e^{(\frac{c}{2}-k)x_1}\right|\int_0^{x_1} e^{-\frac{c}{2}s}(e^{ks}-e^{-ks})a(s)\,ds \\
+\ & \frac{1}{2k}R^p\left|e^{(\frac{c}{2}-k)x_2}-e^{(\frac{c}{2}-k)x_1}\right|\int_0^{x_1} e^{(p\theta-\frac{c}{2})s}(e^{ks}-e^{-ks})b(s)\,ds.
\end{aligned}
$$

The right-hand term in the last inequality tends to 0 when $x_2 \longrightarrow x_1$, for any $u \in \mathcal{C}\cap\bar{\Omega}_2$. In addition, we have the bounds:

$$
\begin{aligned}
& \int_{x_1}^{x_2}\left|e^{\frac{c}{2}(x_2-s)}K(x_2,s)-e^{\frac{c}{2}(x_1-s)}K(x_1,s)\right|h(s,u(s))\,ds \leq \\
\leq\ & \frac{1}{2k}e^{(\frac{c}{2}-k)x_2}\int_{x_1}^{x_2} e^{-\frac{c}{2}s}(e^{ks}-e^{-ks})a(s)\,ds \\
+\ & \frac{1}{2k}e^{(\frac{c}{2}-k)x_2}\|u\|_\theta^p\int_{x_1}^{x_2} e^{(p\theta-\frac{c}{2})s}(e^{ks}-e^{-ks})b(s)\,ds \\
+\ & \frac{1}{2k}e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})\int_{x_1}^{x_2} e^{-(\frac{c}{2}+k)s}a(s)\,ds \\
+\ & \frac{1}{2k}e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})\|u\|_\theta^p\int_{x_1}^{x_2} e^{(p\theta-\frac{c}{2}-k)s}b(s)\,ds \\
\leq\ & \frac{1}{2k}e^{(\frac{c}{2}-k)x_2}\int_{x_1}^{x_2} e^{-\frac{c}{2}s}(e^{ks}-e^{-ks})a(s)\,ds \\
+\ & \frac{1}{2k}e^{(\frac{c}{2}-k)x_2}R^p\int_{x_1}^{x_2} e^{(p\theta-\frac{c}{2})s}(e^{ks}-e^{-ks})b(s)\,ds \\
+\ & \frac{1}{2k}e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})\int_{x_2}^{x_1} e^{-(\frac{c}{2}+k)s}a(s)\,ds \\
+\ & \frac{1}{2k}e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})R^p\int_{x_1}^{x_2} e^{(p\theta-\frac{c}{2}-k)s}b(s)\,ds.
\end{aligned}
$$

Again, all of the terms in the right side tend to 0 when $x_2 \longrightarrow x_1$, for all $u \in \mathcal{C}\cap\bar{\Omega}_2$.

At last, we have:

$$
\int_{x_2}^{+\infty}\left|e^{\frac{c}{2}(x_2-s)}K(x_2,s)-e^{\frac{c}{2}(x_1-s)}K(x_1,s)\right|h(s,u(s))\,ds \leq
$$

$$\leq \frac{1}{2k}\left|e^{\frac{c}{2}x_2}(e^{kx_2}-e^{-kx_2})-e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})\right|\int_{x_2}^{+\infty} e^{-(\frac{c}{2}+k)s}a(s)\,ds$$
$$+ \frac{1}{2k}\left|e^{\frac{c}{2}x_2}(e^{kx_2}-e^{-kx_2})-e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})\right|\|u\|_\theta^p\int_{x_2}^{+\infty} e^{(p\theta-\frac{c}{2}-k)s}b(s)\,ds$$
$$\leq \frac{1}{2k}\left|e^{\frac{c}{2}x_2}(e^{kx_2}-e^{-kx_2})-e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})\right|\int_{x_2}^{+\infty} e^{-(\frac{c}{2}+k)s}a(s)\,ds$$
$$+ \frac{1}{2k}\left|e^{\frac{c}{2}x_2}(e^{kx_2}-e^{-kx_2})-e^{\frac{c}{2}x_1}(e^{kx_1}-e^{-kx_1})\right|R^p\int_{x_2}^{+\infty} e^{(p\theta-\frac{c}{2}-k)s}b(s)\,ds.$$

And all of the terms in the right side tend to 0 when $x_2 \longrightarrow x_1$, for all $u \in \mathcal{C}\cap\bar{\Omega}_2$.

According to Lemma 5.2, we conclude that the operator F is completely continuous on $\mathcal{C}\cap\Omega_2$.

(b) Second step. Now, we check the first alternative in Theorem 4.

• If $u \in \mathcal{C}\cap\partial\Omega_1$, then $e^{-\theta x}Fu(x) \leq \frac{1}{2k}(M_1+M_2\|u\|_\theta^p) \leq \frac{1}{2k}(M_1+M_2r^p) \leq r$ which is fulfilled by Assumptions (5.7), (5.8). We have then proved that $\|Fu\|_\theta \leq \|u\|_\theta$.

• Moreover, if $u \in \mathcal{C}\cap\partial\Omega_2$, then take $\|u\|_\theta = R = \alpha$ where α is as defined in Assumption (5.9) and find that $\min_{x\in[\gamma,\delta]} u(x) \geq m\alpha$. Hence, for any $x \in [\gamma,\delta]$, $m\alpha \leq u(x) \leq \alpha e^{\theta\delta}$. Furthermore, it holds that:

$$\begin{array}{rcl} Fu(x_0) & = & \int_0^{+\infty} e^{\frac{c}{2}(x_0-s)}K(x_0,s)h(s,u(s))\,ds \\ & \geq & \int_\gamma^\delta e^{\frac{c}{2}(x_0-s)}K(x_0,s)h(s,u(s))\,ds \\ & \geq & \left[\min_{x\in[\gamma,\delta],u\in[m\alpha,\alpha e^{\theta\delta}]} h(x,u)\right]\alpha e^{\theta x_0}\int_\gamma^\delta e^{\frac{c}{2}(x_0-s)}K(x_0,s)\,ds \\ & \geq & \alpha e^{\theta x_0}. \end{array}$$

Consequently, $e^{-\theta x_0}Fu(x_0) \geq \alpha$ that is $\|Fu\|_\theta \geq \|u\|_\theta$ for any $u \in K\cap\partial\Omega_2$.

Thanks to Theorem 4, the operator F has a fixed point in $\mathcal{C}\cap(\overline{\Omega}_2\setminus\Omega_1)$ and so Problem (5.1) admits a positive solution u in the cone $\mathcal{C}$.

6 WEAK SOLUTIONS

In contrast with the previous results, we look here for solutions in the Lebesgue Space $L^p(\mathbb{R})$. We first need a compactness criterion in $L^p(\mathbb{R})$ due to Fréchet-Kolmogorov:

Theorem 5. ([10], p. 275) A set $S \subset L^p(\mathbb{R})$ $(1 \leq p < +\infty)$ is relatively compact if and only if S is bounded and for every $\varepsilon > 0$, we have:

(i) $\exists\, \delta > 0$ such that $\int_{-\infty}^{+\infty} |u(x+h) - u(x)|^p\, dx < \varepsilon,\ \forall\, u \in S,\ \forall\, 0 < h < \delta.$
(ii) There exists a number $N > 0$ such that $\int_{\mathbb{R}\setminus[-N,N]} |u(x)|^p\, dx < \varepsilon,\ \forall\, u \in S.$

Let us recall a

Definition 6.1 *We say that $f\colon I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function if*
(i) the map $x \longrightarrow f(x,y)$ is measurable for all $y \in \mathbb{R}$.
(ii) the map $y \longrightarrow f(x,y)$ is continuous for almost every $x \in I$.
(iii) there exists $h \in L^1(I)$ such that $|f(x,y)| \leq h(x)$, for a.e. $x \in I$ and for all $y \in \mathbb{R}$.

Now, we are now in position to state an existence result for weak solutions:

Theorem 6.1 *Assume the separated-variable nonlinear function $h(x,u) = q(x)g(u)$ is of Carathéodory type with $q \in L^p(\mathbb{R})$ $(1 < p < +\infty)$, and g satisfies the general polynomial growth condition:*

$$\begin{array}{l} \exists\, k, \sigma > 0,\ \ |g(y)| \leq k|y|^\sigma,\ \ \text{for a.e. } x \in \mathbb{R} \text{ and for all } y \in \mathbb{R}. \\ \alpha\colon= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |G(x,y)|^p q^p(y) dy \right)^{\frac{\sigma}{p-1}} dx < \infty, \\ \text{with } (\theta \neq 1) \text{ or } (\theta = 1 \text{ and } k\alpha^{\frac{p-1}{p\sigma}} \leq 1) \text{ with } \theta\colon= \frac{(p-1)^2}{p^2\sigma}\cdot \end{array} \tag{6.1}$$

Then problem (1.1) has a solution in $L^r(\mathbb{R})$ with $r = \frac{p\sigma}{p-1}\cdot$

Remark 6.1 *The sublinear case $\sigma = 1$ was studied in [2] with $p = 2$; the solutions are then found to be $L^2(\mathbb{R})$.*

Proof :
Consider the Banach space $E = L^r(\mathbb{R})$ endowed with the usual norm $\|u\|_r = (\int_{-\infty}^{+\infty} |u(s)|^r\, ds)^{\frac{1}{r}}$; hereafter, the notation $\|u\|_r$ will be shorten to $\|u\|$. The mapping $T\colon E \longrightarrow E$ is as defined in subsection 2.1. We will make use of Hölder inequality $\|fg\|_1 \leq \|f\|_p.\|g\|_{p^*}$ with $p^* = \frac{p}{p-1}$ the conjugate of p, that is $p^* = \frac{r}{\sigma}\cdot$

Claim 1: T is continuous:
Consider some $u_0 \in L^r(\mathbb{R})$ and prove the continuity of T at u_0. By Hölder inequality, we have:

$$|Tu(x) - Tu_0(x)|^r = \left| \int_{-\infty}^{+\infty} G(x,y)q(y)[g(u(y)) - g(u_0(y))]\, dy \right|^r$$
$$\leq \left(\int_{-\infty}^{+\infty} |G(x,y)|^p q^p(y)\, dy \right)^{\frac{r}{p}} . \left(\int_{-\infty}^{+\infty} |g(u(y)) - g(u_0(y))|^{p^*}\, dy \right)^{\frac{r}{p^*}}.$$

Therefore, noting that $\frac{r}{p}=\frac{\sigma}{p-1}$, it holds that

$$\begin{array}{rl} & \|Tu-Tu_0\|^r=\int_{-\infty}^{+\infty}|Tu(x)-Tu_0(x)|^r\,dx \\ = & \int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}G(x,y)\,q(y)[g(u(y))-g(u_0(y))]\,dy\right|^r\,dx \\ \leq & \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|G(x,y)|^pq^p(y)\,dy\right)^{\frac{r}{p}}\left(\int_{-\infty}^{+\infty}|g(u(y))-g(u_0(y))|^{p^*}dy\right)^{\frac{r}{p^*}}\,dx \\ \leq & \alpha\left(\int_{-\infty}^{+\infty}|g(u(y))-g(u_0(y))|^{p^*}\,dy\right)^{\frac{r}{p^*}}. \end{array}$$

Let $\varepsilon>0$. From the growth condition satisfied by the function g in Assumption (6.1), we know that the Nemytskii operator $\mathcal{G}$ defined by $\mathcal{G}u(x)=g(u(x))$ is continuous from $L^r(\mathbb{R})$ to $L^{p^*}(\mathbb{R})$ (see [4], Theorem 12.10, p. 78). Then for the given ε, there exists some $\delta>0$ such that $\|u-u_0\|<\delta\Rightarrow\int_{-\infty}^{+\infty}|g(u(s))-g(u_0(s))|^{p^*}\,ds<\frac{\varepsilon^{p^*}}{\alpha}$, whence $\|Tu-Tu_0\|^r<\varepsilon^r$, and the continuity of T on $L^r(\mathbb{R})$ follows.

Claim 2: The mapping T is completely continuous, that is for any $M>0$, the image $\{T(u),\|u\|\leq M\}$ is relatively compact in E. Using again Hölder inequality, we find that:

$$\begin{array}{rcl} \|Tu\|^r & = & \int_{-\infty}^{+\infty}|Tu(x)|^r\,dx \\ & = & \int_{-\infty}^{+\infty}|\int_{-\infty}^{+\infty}G(x,s)q(s)g(u(s))ds|^r\,dx \\ & \leq & \left(\int_{-\infty}^{+\infty}|g(u(y))|^{p^*}dy\right)^{\frac{r}{p^*}}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|G(x,y)|^pq^p(y)\,dy\right)^{\frac{r}{p}}\,dx \\ & \leq & \alpha k^r\left(\int_{-\infty}^{+\infty}|u(s)|^{\sigma p^*}\,ds\right)^{\frac{r}{p^*}}=\alpha k^r\left(\int_{-\infty}^{+\infty}|u(s)|^r\,ds\right)^{\frac{r}{p^*}} \\ & \leq & \alpha k^r\|u\|^{\frac{1}{p^*}}. \end{array}$$

We deduce that $\|Tu\|\leq k\alpha^{\frac{1}{r}}|u\|^{\frac{1}{rp^*}}$. Putting $S=\{u\in E;\|u\|\leq M\}$, we finally get $\|Tu\|\leq k\alpha^{\frac{1}{r}}M^{\frac{1}{rp^*}}$, for any $u\in S$. Then the image $S'=T(S)$ is bounded in $L^r(\mathbb{R})$. Moreover, for any $u\in S$, we have, again by Hölder Inequality:

$$\begin{array}{rl} & \int_{-\infty}^{+\infty}|Tu(x+h)-Tu(x)|^r\,dx \\ = & \int_{-\infty}^{+\infty}|\int_{-\infty}^{+\infty}(G(x+h,y)-G(x,y))q(y)g(u(y))dy|^r\,dx \\ \leq & \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|G(x+h,y)-G(x,y)|^pq^p(y)\,dy\right)^{\frac{r}{p}}\left(\int_{-\infty}^{+\infty}|g(u(y))|^{p^*}dy\right)^{\frac{r}{p^*}}\,dx \\ \leq & k^rM^{\frac{1}{p^*}}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|G(x+h,y)-G(x,y)|^pq^p(y)\,dydx\right)^{\frac{r}{p}}. \end{array}$$

Since $0<\alpha<\infty$, we infer that $\forall\,\varepsilon>0,\exists\,\delta>0,\,\forall\,u\in S,\,\forall\,h\,(0<h<\delta)$, $\int_{-\infty}^{+\infty}|Tu(x+h)-Tu(x)|^r\,dx<\varepsilon^r$.

In addition, it holds that for any $u\in S$

$$\int_{\mathbb{R}\setminus[-N,N]}|Tu(x)|^r\,dx\leq k^rM^{\frac{1}{p^*}}\int_{\mathbb{R}\setminus[-N,N]}\left(\int_{-\infty}^{+\infty}|G(x,s)|^pq^p(s)\,ds\right)^{\frac{\sigma}{p-1}}\,dx.$$

In conclusion for any $u \in S$, and for any $\varepsilon > 0$, there is some $N = N(\varepsilon)$ such that $\int_{\mathbb{R}\setminus[-N,N]} |Tu(x)|^r \, dx < \varepsilon^r$. Thanks to Theorem 5, we deduce that the image set $T(S)$ is relatively compact in $L^r(\mathbb{R})$.

Claim 3: There exists some $R > 0$ such that T maps the closed ball $B(0, R)$ into itself. Indeed, for all $u \in E$ satisfying $\|u\| \leq R$, we have that $\|Tu\| \leq k\alpha^{\frac{1}{r}} \|u\|^{\frac{(p-1)^2}{p^2\sigma}} \leq k\alpha^{\frac{1}{r}} R^{\frac{(p-1)^2}{p^2\sigma}}$. Now, for any $\sigma \neq 1$, there exists some $R > 0$ such that $k\alpha^{\frac{1}{r}} R^{\sigma} \leq R$ and this still holds true if $\sigma = 1$ and $k\alpha^{\frac{1}{r}} \leq 1$. Then, the following implications hold true

$$\|u\| \leq R \Rightarrow \|Tu\| \leq R,$$

proving that $T(B) \subset B$.

The claim of Theorem 6.1 then follows from Schauder's fixed point theorem.

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