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Attractors for a Class of Doubly Nonlinear Parabolic Systems*

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Abstract

In this paper, we establish the existence and boundedness of solutions of a doubly nonlinear parabolic system. We also obtain the existence of a global attractor and the regularity property for this attractor in $[L^\infty(\Omega)]^2$ and $\prod_{i=1}^{2} B_\infty^{1+\sigma_i,p_i}(\Omega)$.

1 Introduction

This paper deals with the doubly nonlinear parabolic system of the form

$$
(S)\begin{cases}
\frac{\partial b_1(u_1)}{\partial t} - \Delta_{p_1} u_1 + f_1(x,t,u_1,u_2) = 0 & \text{in } \Omega\times(0,\infty),\\
\frac{\partial b_2(u_2)}{\partial t} - \Delta_{p_2} u_2 + f_2(x,t,u_1,u_2) = 0 & \text{in } \Omega\times(0,\infty),\\
u_1 = u_2 = 0 & \text{in } \partial\Omega\times(0,\infty),\\
b_1(u_1(.,0)) = b_1(\varphi_1) & \text{on } \Omega,\\
b_2(u_2(.,0)) = b_2(\psi_1) & \text{on } \Omega.
\end{cases}
$$

Where Ω is a bounded and open subset in $\mathbb{R}^N$, $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $T > 0$. The operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

Monotone operators, in particular the ones that are subdifferentials of convex functions, like p-Laplacian, appear frequently in equations modeling the behaviour of viscoelastic materials (see [16] for instance), reaction-diffusion (see [17], and references therein) and in mathematical glaciology.

Here, we study the existence of solutions for a class of doubly nonlinear systems including the p-Laplacian as the principal part of the operator, and we use the general setting of attractors (see [19]) to prove that all the solutions converge to a set $\mathcal{A}$, which is called the global attractor. In fact, few papers consider the question in such situations. For instance, Marion [17] considered the problem of solutions of reaction-diffusion systems in which $b_i(s) = s$ and $p_1 = p_2 = 2$. L.Dung [13, 14] treated a system involving the p-Laplacian and

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$b_i(s) = s$, and proved that weak L^q dissipativity implies strong L^∞ one for solutions of degenerate nonlinear diffusion systems and gives the existence of global attractors to which all solutions converge in uniform norm. We mention that to our knowledge, the doubly nonlinear parabolic system for the p-Laplacian operator has never been studied, not even in the case $b_i(s) \neq s$. In the classical setting, i.e with $p_1 = p_2 = 2$, the system with b_i has been previously considered, for example in [9] and [10]. We follow the approach of [10], generalizing some results to the case $p_i > 1$ and we extend the results of [11] to nonlinear system (S). In the first section of this paper, we give some assumptions and preliminaries, in section 2 and section 3, we prove the existence of an absorbing set and the existence of the attractor, in section 4, we present the regularity of the attractor and obtain the asymptotic behaviour of the solutions in the framework of dynamical systems associated to the system (S).

2 Preliminaries, Existence and Uniqueness

2.1 Notations and Assumptions

Let b_i, $(i = 1, 2)$ be a continuous function with $b_i(0) = 0$. For $t \in \mathbb{R}$,define, $\Psi_i(t) = \int_0^t b_i(s)ds$. The Legendre transform Ψ_i^* of Ψ_i is defined as $\Psi_i^*(\tau) = \sup\limits_{s\in\mathbb{R}}\{\tau s - \Psi_i(s)\}$. We shall assume throughout the paper that Ω is a regular open bounded subset of $\mathbb{R}^N$ and for any $T > 0$, we set $Q_T = \Omega \times (0, T)$ and $S_T = \partial\Omega \times (0, T)$, with $\partial\Omega$ the boundary of Ω. The norm in a space X will be denoted by : $\|.\|_r$ if $X = L^r(\Omega)$ for all r : $1 \leq r \leq +\infty$. $\|.\|_{1,q}$ if $X = W^{1,q}(\Omega)$ for all q : $1 \leq q \leq +\infty$, $\|.\|_X$ otherwise and $\langle ., . \rangle_{X,X'}$ will denote the duality product between X and its dual X'. We use the standard notation for Sobolev spaces $W_0^{1,r}(\Omega), 1 < r < +\infty$, and their duals $W^{-1,r'}(\Omega)$, where $r' = r/(r-1)$. The following lemma are useful and frequently used :

Lemma 2.1 *(Ghidaghia lemma, cf* [19])
Let y be a positive absolutely continuous function on $(0,\infty)$ which satisfies

$$y' + \mu y^q \leq \lambda,$$

with $q > 1, \mu > 0, \lambda \geq 0$. Then for $t > 0$

$$y(t) \leq \left(\frac{\lambda}{\mu}\right)^{\frac{1}{q}} + [\mu(q-1)t]^{\frac{-1}{q-1}} .$$

Lemma 2.2 *(Uniform Gronwall's lemma, cf* [19]) *Let y and h be locally integrable functions such that :*
$\exists r > 0,\ a_1 > 0,\ a_2 > 0,\ \tau > 0,\ \forall t \geq \tau$

$$\int_t^{t+r} y(s)ds \leq a_1, \quad \int_t^{t+r} |h(s)|\, ds \leq a_2, \quad y' \leq h.$$

Then

$$y(t+r) \le \frac{a_1}{r} + a_2, \quad \forall t \ge \tau.$$

We start by introducing our assumptions and making precise the meaning of solution of (**S**).

We shall assume that the following hypotheses are satisfied :

(H1) $(\varphi_1, \psi_1) \in \left[L^2(\Omega)\right]^2$.

(H2) $b_i \in C^1(\mathbb{R})$, $b_i(0) = 0$, and there exist positive constants γ_i and M_i such that

$$|b_i| \le \gamma_i |s| + M_i, \ \forall s \in \mathbb{R}, \quad i = 1, 2.$$

(H3) $f_i \in C^1(\Omega \times \mathbb{R} \times \mathbb{R})$.

(H4) a) There exists positive constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ and $\alpha_1 > \sup(2, p_1)$ such that for any $\xi \in \mathbb{R}$ any $N > 0$ we have for any u_2 : $|u_2| < N$

$$\begin{cases} sign(\xi) f_1(x,t,\xi,u_2) \ge c_1 \left|b_1(\xi)\right|^{\alpha_1 - 1} - c_2\,, \\ \lim_{t\to 0^+} \sup \left|f_1(x,t,\xi,u_2)\right| \le c_3 \left(|\xi|^{\alpha_1-1} + 1\right) \\ |f_1(x,t,\xi,u_2)| \le a_1(|\xi|) \quad \text{almost everywhere in } \Omega \times \mathbb{R}^+ \\ \text{where} \quad a_1 : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is an increasing function.} \end{cases}$$

b) There exists positive constants $c_4 > 0$, $c_5 > 0$, $c_6 > 0$ and $\alpha_2 > \sup(2, p_2)$ such that for any $\xi \in \mathbb{R}$ any $M > 0$ we have for any u_1 : $|u_1| < M$

$$\begin{cases} sign(\xi) f_2(x,t,u_1,\xi) \ge c_4 \left|b_2(\xi)\right|^{\alpha_2 - 1} - c_5\,, \\ \lim_{t\to 0^+} \sup \left|f_2(x,t,u_1,\xi)\right| \le c_6 \left(|\xi|^{\alpha_2-1} + 1\right) \\ |f_2(x,t,u_1,\xi)| \le a_2(|\xi|) \quad \text{almost everywhere in } \Omega \times \mathbb{R}^+ \\ \text{where} \quad a_2 : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is an increasing function.} \end{cases}$$

(H5) $\frac{\partial f_i}{\partial t}(x,t,\eta,\zeta)$ exist and for all $L > 0$, there exists $C_L > 0$ such that : if $|\eta| + |\zeta| \le L$ then $\left|\frac{\partial f_i}{\partial t}(x,t,\eta,\zeta)\right| \le C_L$, for almost every $(x,t) \in \Omega \times \mathbb{R}^+$.

(H6) a) There exist $\delta_1 > 0$ such that for almost every $(x,t) \in \Omega \times \mathbb{R}^+$ and for any $N > 0$ and any u_2 : $|u_2| < N$ then

$$\xi \to f_1(x,t,\xi,u_2) + \delta_1 b_1(\xi) \quad \textit{is increasing.}$$

b) There exist $\delta_2 > 0$ such that for almost every $(x,t) \in \Omega \times R^+$ and for any $M > 0$ and any u_1 : $|u_1| < M$ then

$$\xi \to f_2(x,t,u_1,\xi) + \delta_2 b_2(\xi) \quad \textit{is increasing.}$$

(H7) $\exists \varepsilon > 0 : b_i'(s) \geq \varepsilon$, for all $s \in \mathbb{R}$.

Definition 2.1 *By a weak solution of* $(\mathbf{S})$*, we mean an element* $w = (u_1, u_2)$:

$$u_i \in L^{p_i}(0,T; W_0^{1,p_i}(\Omega)) \cap L^{\alpha_i}(Q_T) \cap L^\infty(\tau, T; L^\infty(\Omega)) \quad for \quad all\ \tau > 0,$$

$$\frac{\partial b_i(u_i)}{\partial t} \in L^{p_i'}(0,T; W^{-1,p_i'}(\Omega)) + L^{\alpha_i'}(Q_T),$$

and for all $\phi_i \in L^{p_i}(0,T; W_0^{1,p_i}(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega))$

$$\int_0^T \left\langle \frac{\partial b_i(u_i)}{\partial t}, \phi_i \right\rangle_{X_i, X_i'} dt + \int_0^T \int_\Omega F_i(\nabla u_i) \nabla \phi_i dxdt = -\int_0^T \int_\Omega f_i(x,w)\phi_i dxdt$$

and if $\frac{\partial \phi_i}{\partial t} \in L^2(0,T; L^2(\Omega)), \phi_i(T) = 0$ *then*

$$\int_0^T \left\langle \frac{\partial b_i(u_i)}{\partial t}, \phi_i \right\rangle_{X_i, X_i'} dt = -\int_0^T \int_\Omega (b_i(u_i) - b_i(u_i(.,0))) \frac{\partial \phi_i}{\partial t} dxdt,$$

where $X_i = L^\infty(\Omega) \cap W_0^{1,p_i}(\Omega)$, $X_i' = L^1(\Omega) + W^{-1,p_i'}(\Omega)$ and $F_i(\xi) = |\xi|^{p_i - 2} \xi$ for any $\xi \in \mathbb{R}^N$.

2.2 Existence

2.2.1 Existence

We have the following.

Theorem 2.1 *Let the general assumptions (H1)-(H7) be satisfied, then for any* $\tau > 0$*, the problem* $(\mathbf{S})$ *has a weak solution* (u_1, u_2) *such that*

$$u_i \in L^{p_i}(0,T; W_0^{1,p_i}(\Omega)) \cap L^\infty(\tau, T; W_0^{1,p_i}(\Omega) \cap L^\infty(\Omega)),$$
$$and \quad b_i(u_i) \in L^{\alpha_i}(Q_T) \cap L^\infty(0,T; L^2(\Omega)).$$

Proof. By the existence of theorem $[11, theorem\ 3.1, p.3]$, there exists two functions u_1^0, u_2^0 solutions of the problem

$$(P_{1,0}) \begin{cases} \frac{\partial b_1(u_1^0)}{\partial t} - \Delta_{p_1} u_1^0 + f_1(x,t,u_1^0,0) = 0 & in \quad Q_T \\ u_1^0 = 0 & \text{on} \quad S_T \\ b_1(u_1^0(.,0)) = b_1(\varphi_1) & in\ \Omega \end{cases}$$

$$(P_{2,0}) \begin{cases} \frac{\partial b_2(u_2^0)}{\partial t} - \Delta_{p_2} u_2^0 + f_2(x,t,0,u_2^0) = 0 & in \quad Q_T \\ u_2^0 = 0 & \text{on} \quad S_T \\ b_2(u_2^0(.,0)) = b_2(\psi_1) & in\ \Omega \end{cases}$$

and $u_i^0 \in L^{p_i}(0,T; W_0^{1,p_i}(\Omega)) \cap L^\infty(\tau, T; W_0^{1,p_i}(\Omega) \cap L^\infty(\Omega))$ $, \forall \tau > 0$. By (u_1^0, u_2^0) we construct two sequences of functions $(u_1^n), (u_2^n)$ such that

$$(P_{1,n}) \begin{cases} \frac{\partial b_1(u_1^n)}{\partial t} - \Delta_{p_1} u_1^n + f_1(x,t,u_1^n,u_2^{n-1}) = 0 & \text{in } Q_T, & (2.1) \\ u_1^n = 0 & \text{in } S_T, & (2.2) \\ b_1(u_1^n(.,0)) = b_1(\varphi_1) & \text{on } \Omega. & (2.3) \end{cases}$$

And

$$
(P_{2,n})\begin{cases} \frac{\partial b_2(u_2^n)}{\partial t} - \Delta_{p_2} u_2^n + f_2(x,t,u_1^{n-1},u_2^n) = 0 & \text{in } Q_T, \quad (2.4)\\ u_2^n = 0 & \text{in } S_T, \quad (2.5)\\ b_2(u_2^n(.,0)) = b_2(\psi_1) & \text{on } \Omega. \quad (2.6) \end{cases}
$$

The existence of solutions will be shown via some a-priori L^∞ estimates on (u_1^n, u_2^n) and lemma 2.3 and lemma 2.4. In all this paper, we denote by c_i different constants independent of n and depending on p_i, Ω, T. Sometimes we shall refer to a constant depending on specific parameters : $c(\tau)$, $c(T)$, $c(\tau,T)$.

Lemma 2.3 *Under the hypothesis (H1)-(H7), there exist $c_i > 0$ such that for any $n \in \mathbb{N}$ and any $\tau > 0$, the following estimate holds*

$$
\|u_i^n\|_{L^\infty(\tau,T;L^\infty(\Omega))} \leq c_7(\tau,T). \tag{2.7}
$$

Proof. The case $n = 0$ has been observed. Assume that (2.7) is valid for $(n-1)$ and let us derive the estimate for n. Now multiplying (2.1) by $|b_1(u_1^n)|^k b_1(u_1)$ and using the growth condition on b_1, and (H4) a) we deduce for all $\tau > 0$

$$
\frac{1}{k+2}\frac{d}{dt}\left\{\int_\Omega |b_1(u_1^n)|^{k+2}\,dx\right\} + c_8\int_\Omega |b_1(u_1^n)|^{k+\alpha_1}\,dx \leq
$$

$$
c_9\int_\Omega |b_1(u_1^n)|^{k+1}\,dx. \tag{2.8}
$$

Setting $y_{k,n}(t) = \|b_1(u_1^n)\|_{L^{k+2}(\Omega)}$ and using Hölder inequality on both sides, there exists two constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that

$$
\frac{dy_{k,n}(t)}{dt} + \mu_0 y_{k,n}^{\alpha_1-1}(t) \leq \lambda_0, \tag{2.9}
$$

which implies from a lemma 2.1 that

$$
y_k(t) \leq \left(\frac{\lambda_0}{\mu_0}\right)^{\frac{1}{q_1-1}} + \frac{1}{[\mu_0(\alpha_1-2)t]^{\frac{1}{\alpha_1-2}}} = c_{10}(t) \qquad \forall\, t > 0. \tag{2.10}
$$

As $k \to +\infty$, and for any all $t \geq \tau > 0$, we have by (2.10) and (H2)

$$
\|\, u_1^n(t)\|_{L^\infty(\Omega)} \leq c_{11}(\tau). \tag{2.11}
$$

The same holds for u_2^n

$$
\|\, u_2^n(t)\|_{L^\infty(\Omega)} \leq c_{12}(\tau). \tag{2.12}
$$

Lemma 2.4 *Under the hypothesis (H1)-(H7), for all $\tau > 0$, there exists constants c_j, c_τ such that the following estimates hold*

$$
\|u_i^n\|_{L^{p_i}(0,T;W_0^{1,p_i}(\Omega))} \leq c_{13}(T), \tag{2.13}
$$

$$\|u_i^n\|_{L^\infty(\tau,T;W_0^{1,p_i}(\Omega))} \leq c_{14}(\tau,T), \tag{2.14}$$

$$\int_\tau^T \int_\Omega b_i'(u_i^n)(\frac{\partial u_i^n}{\partial t})^2 dxds \leq c_{15}(\tau,T), \tag{2.15}$$

$$\textit{and} \quad \int_t^{t+\tau} \int_\Omega b_i'(u_i^n)(\frac{\partial u_i^n}{\partial t})^2 dxds \leq c_{16}(\tau), \ \textit{for any } t \geq \tau > 0. \tag{2.16}$$

Proof. Taking the scalar product of equation (2.1) by u_1^n and (2.4) by u_2^n, integrating on Ω and using hypothesis (H4), we get

$$\frac{d}{dt}\left[\sum_{i=1}^2 \left\{\int_\Omega \Psi_i^*(b_i(u_i^n))dx\right\}\right] + \sum_{i=1}^2 \int_\Omega |\nabla u_i^n|^{p_i}\, dx$$

$$+c_1 \sum_{i=1}^2 \int_\Omega |\, u_i^n|^{\alpha_i}\, dx \leq c_2. \tag{2.17}$$

But $\|\varphi_1\|_{L^2(\Omega)}+\|\psi_1\|_{L^2(\Omega)} \leq c \Longrightarrow \int_\Omega (\Psi_1^*(b_1(\varphi_1)) + \Psi_2^*(b_2(\psi_1)))\, dx \leq c,$ where Ψ_i^* is the Legendre transform of Ψ_i, $\Psi_i(t) = \int_0^t b_i(s)ds$. So, integrating (2.17) from 0 to T we obtain

$$\sum_{i=1}^2 \left(\int_0^T \int_\Omega |\nabla u_i^n|^{p_i}\right) dxds + c_{17}\sum_{i=1}^2 \left(\int_0^T \int_\Omega |\, u_i^n|^{\alpha_i}\right) dxds \leq c_{17}(T). \tag{2.18}$$

Hence (2.13) follows.

Taking the scalar product of equation (2.1) by $\frac{\partial u_1^n}{\partial t}$ and (2.4) by $\frac{\partial u_2^n}{\partial t}$ integrating on Ω, it follows by (H2),(H7) and lemma 2.1 that for any all $t \geq \tau > 0$,

$$\sum_{i=1}^2 \int_\Omega b_i'(u_i^n)(\frac{\partial u_i^n}{\partial t})^2 dx + \frac{d}{dt}\sum_{i=1}^2 \left[\frac{1}{p_i}\int_\Omega |\nabla u_i^n|^{p_i}\, dx\right] =$$

$$-\int_\Omega f_1(x,t,u_1^n,u_2^{n-1})\frac{\partial u_1^n}{\partial t}dx - \int_\Omega f_2(x,t,u_1^{n-1},u_2^n)\frac{\partial u_2^n}{\partial t}dx$$

$$\leq \frac{1}{2}\sum_{i=1}^2 \int_\Omega b_i'(u_i^n)(\frac{\partial u_i^n}{\partial t})^2 dx + c_{18}(\tau). \tag{2.19}$$

Then, we have

$$\sum_{i=1}^2 \int_\Omega b_i'(u_i^n)(\frac{\partial u_i^n}{\partial t})^2 dx + \frac{d}{dt}\sum_{i=1}^2 \left[\frac{2}{p_i}\int_\Omega |\nabla u_i^n|^{p_i}\, dx\right] \leq c_{19}(\tau). \tag{2.20}$$

Integrating (2.20) on $(t, t+\tau)$, then yields

$$\sum_{i=1}^2 \int_t^{t+\tau} \int_\Omega b_i'(u_i^n)(\frac{\partial u_i^n}{\partial t})^2 dx + \sum_{i=1}^2 \left[\frac{2}{p_i}\int_\Omega |\nabla u_i^n(t+\tau)|^{p_i}\, dx\right] =$$

$$\sum_{i=1}^{2}\left[\frac{2}{p_i}\int_\Omega |\nabla u_i^n(\tau)|^{p_i}\,dx\right]+c_\tau. \tag{2.21}$$

Integrating (2.17) on $(t,t+\tau)$ and using lemma 2.3, we get

$$\sum_{i=1}^{2}\int_t^{t+\tau}\left[\frac{1}{p_i}\int_\Omega |\nabla u_i^n(s)|^{p_i}\,dxds\right]\le c_\tau,\ \forall t\ge\tau>0. \tag{2.22}$$

By the uniform Gronwall's lemma 2.1, we obtain

$$\sum_{i=1}^{2}\left[\int_\Omega |\nabla u_i^n(t)|^{p_i}\,dx\right]\le c_\tau,\ \forall t\ge\tau>0,\forall n\in\mathbb{N}^*.$$

Integrating now (2.20) on $(t,t+\tau)$, we have

$$\sum_{i=1}^{2}\int_t^{t+\tau}\int_\Omega b_i'(u_i^n)(\frac{\partial u_i^n}{\partial t})^2dxds\ \le c_{20}(\tau),$$

which gives by (H2)

$$\sum_{i=1}^{2}\int_t^{t+\tau}\int_\Omega (\frac{\partial b_i u_i^n}{\partial t})^2dxds\ \le c_{21}(\tau).$$

Passage to the limit in in the process $(P_{1,n})$ **and** $(P_{2,n})$
By lemma 2.3 and lemma 2.4, there exist a subsequence (denoted again by u_i^n, $i=1,2$) such that as $n\to+\infty$: $u_i^n\to u_i$ weak in $L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$ and in $L^{\alpha_i}(Q_T)$, $u_i^n\to u_i^n$ weak star in $L^\infty(\tau,T;W_0^{1,p_i}(\Omega),\forall\ \tau>0$, $b_i(u_i^n)\to\eta_i$ in $L^2(Q_T)$, $\frac{\partial b_i(u_i^n)}{\partial t}$ is bounded in $L^2(\tau,T;W^{-1,p_i'}(\Omega))$ for any $\tau>0$, $divF_i(\nabla u_i^n)$ $\to\chi_i$ in weak $L^{p_i'}(0,T;W^{-1,p_i'}(\Omega))$. Moreover standard monotonicity argument gives $\chi_i=divF_i(\nabla u)$, $\eta_i=b_i(u_i)$. To conclude that $w=(u_1,u_2)$ is a weak solution of (**S**) it is enough to observe that $f_1(x,t,u_1^n,u_2^{n-1})$ converges to $f_1(x,t,u_1,u_2)$ and $f_2(x,t,u_1^{n-1},u_2^n)$ converges to $f_2(x,t,u_1,u_2)$ strongly in $L^1(Q_T)$ and in $L^s(\tau,T;L^s(\Omega))$ for all $\tau>0$; and for all $s\ge1$, thanks to Vitali's theorem. Whence $w=(u_1,u_2)$ is a solution of (**S**).

2.2.2 Uniqueness

Proposition 2.1 *The solution of (S) is unique. Moreover, if* (u_1,u_2) *and* (v_1,v_2) *are two solutions, corresponding respectively to initial data* (φ_1,ψ_1) *and* (φ_2,ψ_2) *such that* $\varphi_1\le\psi_1$ *and* $\varphi_2\le\psi_2$ *then* $u_i\le v_i$.

Proof. Suppose that (u_1,u_2) and (v_1,v_2) are two solutions, corresponding respectively to initial data (φ_1,ψ_1) and (φ_2,ψ_2) such that $\varphi_1\le\psi_1$ and $\varphi_2\le\psi_2$. Following Diaz [5, p.269], we consider the following test function : $w_i=H_n(u_i-v_i)$, $n\ge1$, $(i=1,2)$ by

$$H_n(s) = \begin{cases} 0 & \text{if } s \le 0, \\ \frac{n^2 s^2}{2} & 0 < s \le \frac{1}{n}, \\ 2ns - \frac{n^2 s^2}{2} - 1 & \frac{1}{n} < s \le \frac{2}{n}, \\ 1 & s > \frac{2}{n}. \end{cases}$$

It is easy to see that

$$\begin{cases} 0 \le (H_n)'(s) \le n, \quad \lim_{n\to+\infty} s(H_n)'(s) = 0. \\ |H_n(s)| \le 1, \quad \lim_{n\to+\infty} H_n(s) = sign_+(s) \\ and \quad \lim_{n\to+\infty} s(H_n)(s) = s_+ = \begin{cases} 0 & s \le 0 \\ s & s > 0 \end{cases} \end{cases}$$

Considering the systems (S) verified by $u = (u_1, u_2)$ and v=(v_1,v_2) , we get

$$\sum_{i=1}^{2} \int_0^t \int_\Omega [b_i(u_i) - b_i(v_i)]_t \, H_n(u_i - v_i) +$$
$$\sum_{i=1}^{2} \int_0^t \int_\Omega \left[|\nabla u_i|^{p_i - 2} \nabla u_i - |\nabla v_i|^{p_i-2} \nabla v_i \right] (\nabla u_i - \nabla v_i)(H_n)'(u_i - v_i) +$$
$$\sum_{i=1}^{2} \int_0^t \int_\Omega [f_i(x, u_1, u_2) - f_i(x, v_1, v_2)] \, H_n(u_i - v_i). \tag{2.23}$$

Since $(H_n)'(s) \ge 0$, we deduce that

$$\lim_{n\to+\infty} \sum_{i=1}^{2} \int_0^t \int_\Omega \left[|\nabla u_i|^{p_i-2} \nabla u_i - |\nabla v_i|^{p_i-2} \nabla v_i \right] (\nabla u_i - \nabla v_i)(H_n)'(u_i - v_i) \ge 0. \tag{2.24}$$

By (H7) and (2.24), (2.23) becomes

$$\sum_{i=1}^{2} \int_0^t \int_\Omega [b_i(u_i) - b_i(v_i)]_t \, sign_+(u_i - v_i) \le k_1 \sum_{i=1}^{2} \int_0^t \int_\Omega [b_i(u_i(.,s)) - b_i(v_i(.,s)]_+ \,, \tag{2.25}$$

by Gronwall's lemma, we get

$$\sum_{i=1}^{2} \int_0^t \int_\Omega [b_i(u_i(.,t)) - b_i(v_i(.,t))]_+ \le e^{k_1 t} \sum_{i=1}^{2} \int_\Omega [b_i(\varphi_i) - b_i(\psi_i)]_+ \,, \forall\, t \in [0, T] \,. \tag{2.26}$$

Since the second term vanishes and recalling that $\varphi_1 \le \psi_1$ and $\varphi_2 \le \psi_2$, this means that $b_i(u_i) \le b_i(v_i)$, and by monotonicity of b_i, we obtain $u_i \le v_i$. Uniqueness is now an obvious consequence.

Remark. i) Our calculations above are formal. We may assume that the solutions are smooth enough to have all estimates we need. Such assumptions can be justified by working with regularized problem

$$\frac{\partial b_i(u_i)}{\partial t} - div\left[\left\{|\nabla u_i|^2+\varepsilon\right\}^{\frac{p_i-2}{2}}\nabla u_i\right] + f_i(x,u) = 0$$

whose solutions are smooth so that the following argument can be carried out rigorously. One can see that the estimates obtained are independent of the parameter ε, so that, by taking the limit, they also hold for (**S**).
ii) Assume that hypothesis (H1) to (H7) are satisfied and f_i does not depend on t : $f_i(x,t,u_1,u_2) = f_i(x,u_1,u_2)$, then theorem 2.1 establishes the existence of dynamical system $\{S(t)\}_{t\geq 0}$ which maps $\left[L^2(\Omega)\right]^2$into $\left[L^2(\Omega)\right]^2$ such that $S(t)(\varphi_1,\psi_2) = (u_1(t),u_2(t))$.

3 Global attractor

Proposition 3.1 *Assume that (H1)-(H7) hold and f_i does not depend on t, the semi-group $S(t)$ associated with problem (S) is such that*

(i) There exist absorbing sets in $L^{\sigma_i}(\Omega)$, for $1 \leq \sigma_i \leq +\infty$.

(ii) There exist absorbing sets in $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$.

Proof. Let u_i be solution of (S) and u_i^n solution of $(P_{i,n})$ such that $u_i^n \to u_i$. Then for fixed $t \geq \tau > 0$, lemma 2.3, lemma 2.4 and Sobolev's injection theorem imply

$$\|u_i^n(t)\|_{L^{\sigma_i}(\Omega)} \leq c_\tau,$$

$$\text{and} \quad \|u_i^n(t)\|_{W_0^{1,p_i}(\Omega))} \leq c_\tau, \qquad \forall\ t \geq \tau.$$

As $n \to +\infty$, we get

$$\|u_i(t)\|_{L^\infty(\Omega)} \leq c_\tau,$$

$$\text{and} \quad \|u_i(t)\|_{W_0^{1,p_i}(\Omega))} \leq c_\tau, \qquad \forall\ t \geq \tau.$$

Remark. By proposition 3.1 we deduce that the assumptions (1.1),(1.4) and (1.12) in theorem 1.1 [19] p23 are satisfied with $U = \left[L^2(\Omega)\right]^2$. So, by means of the uniform compactness lemma in [7, p. 111], we get the following result.

Theorem 3.1 *Assume that (H1)-(H7) are satisfied and that f_i does not depend on time. Then the semi-group $S(t)$ associated with the boundary value problem* (**S**) *possesses a maximal attractor* A *which is bounded in* $\left[W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)\right]\cap \left[L^\infty(\Omega)\right]^2$, *compact and connected in* $\left[L^2(\Omega)\right]^2$. *Its domain of attraction is the whole space* $\left[L^2(\Omega)\right]^2$.

4 A regularity property of the attractor

In this section we shall show supplementary regularity estimates on the solution of problem (**S**) and by use of them, we shall obtain more regularity on the attractor obtained in section 3. We shall assume that there exist positive constants $\delta_i > 0$ and a function H from $\mathbb{R}^{N+2}$ to $\mathbb{R}$ such that :

$$(H8)\begin{cases} f_i(x,u) = g_i(u) - h_i(x) = \delta_i \frac{\partial H}{\partial u_i}; \\ f_i \text{ satisfy (H3),(H4),(H5) and (H6),} \\ \quad and \quad h_i \in L^\infty(\Omega) \end{cases}$$

(H9) $b_i \in C^2(\mathbb{R})$ and $\exists \gamma_i,\ M_i > 0$ such that $\gamma_i \le b_i'(s) \le M_i$, $\forall s \in \mathbb{R}$. We shall denote : $E_i(\xi) = |\xi|^{(p_i-2)/2}\xi$, for all $\xi \in \mathbb{R}^N$. The following lemmas are used in the proof of the main results of this section.

Lemma 4.1 *Assume that (H1)-(H9) are satisfied, there exist constants $C = C(\varphi_1, \psi_1)$, such that for any $T > 0$*

$$\|u_i\|_{L^\infty(0,T,W_0^{1,p_i}(\Omega))} \le C < \infty, \tag{4.1}$$

$$and \quad \left\|\frac{\partial u_i}{\partial t}\right\|_{L^2(Q_T)} \le C < \infty. \tag{4.2}$$

Proof. Multiplying the equation $\frac{\partial b_i(u_i)}{\partial t} - div\left[|\nabla u_i|^{p_i-2}\nabla u_i\right] + \delta_i \frac{\partial H}{\partial u_i} = 0$ by $\frac{1}{\delta_i}(u_i)_t$ and we obtain

$$\sum_{i=1}^{2} \frac{1}{\delta_i}\int_{Q_T} b_i'(u_i)(\frac{\partial u_i}{\partial t})^2 dxdt + \sum_{i=1}^{2}\frac{1}{p_i\delta_i}\int_\Omega |\nabla u_i(.,T)|^{p_i}\,dx = \tag{4.3}$$

$$\int_\Omega [-H(.,u_1(T),u_2(T)) + H(.,\varphi_1,\psi_1]\,dx = \frac{1}{p_1\delta_1}\int_\Omega |\nabla\varphi_1|^{p_1}\,dx + \frac{1}{p_2\delta_2}\int_\Omega |\nabla\psi_1|^{p_2}\,dx.$$

H is continuous and (u_1, u_2) is bounded, we then obtain

$$\sum_{i=1}^{2}\frac{\gamma_i}{\delta_i}\int_{Q_T}(\frac{\partial u_i}{\partial t})^2 dxdt + \sum_{i=1}^{2}\frac{1}{p_i\delta_i}\int_\Omega |\nabla u_i(.,T)|^{p_i}\,dx \le C(\varphi_1,\psi_1), \tag{4.4}$$

hence (4.1) and (4.2). ■

Lemma 4.2 *Let $p_i \in]1,2[$, then we have the following estimate*

$$\sum_{i=1}^{2}\int_\Omega |\nabla u_i'|^{p_i}\,dx \le c_{22}\sum_{i=1}^{2}\int_\Omega |\nabla u_i|^{p_i}\,dx + \sum_{i=1}^{2}\frac{2(p_i-1)}{p_i^2}\int_\Omega \left|(E_i(\nabla u_i))'\right|^2 dx, \tag{4.5}$$

with a constant $c_{22} > 0$.

Proof. Straigthforward calculations see [9] give

$$\int_\Omega (F_i(\nabla w_i))' . \nabla w_i' dx \geq (p_i - 1) \left(\frac{2}{p_i}\right)^2 \int_\Omega \left|(E_i(\nabla w_i))'\right|^2 dx.$$

As $E_i(\nabla w_i) = |\nabla w_i|^{\frac{p_i-2}{2}} \nabla w_i$, we get $|\nabla w_i| = |E_i(\nabla w_i)|^{\frac{2}{p_i}}$ and $\nabla w_i = |E_i(\nabla w_i|^{\frac{2-p_i}{2}} E(\nabla w_i)$
Hence

$$(\nabla w_i)' = \frac{2}{p_i} |E_i(\nabla w_i)|^{\frac{2-p_i}{p_i}} (E_i(\nabla w_i))',$$

which yields

$$|\nabla w_i'|^{p_i} = (\frac{2}{p_i})^{p_i} |E_i(\nabla w_i)|^{2-p_i} \left|(E_i(\nabla w_i))'\right|^{p_i}.$$

So that, the Hölder inequality can be applied to give

$$\int_\Omega |\nabla w_i'|^{p_i} dx \leq c_{23} \int_\Omega |E_i(\nabla w_i)|^{2-p_i} \left|(E_i(\nabla w_i))'\right|^{p_i} dx$$

$$\leq \frac{c_{24}}{2} \int_\Omega |E_i(\nabla w_i)|^2 dx + \frac{2(p_i-1)}{p_i^2} (\int_\Omega \left|(E_i(\nabla w_i))'\right|^2 dx,$$

then yields (4.5). For stating the next theorem we introduce the hypothesis

(H10) $N = 1$ and $1 < p_i < 2$ or $N \geq 2$ and $\frac{3N}{N+2} \leq p_i < 2$.

Theorem 4.1 *Let f_i, b_i and p_i satisfies hypothesis (H1) to (H10).*
Let $r(t) = \sum_{i=1}^2 \int_\Omega b_i'(u_i) \left(u_i'\right)^2 dx$. Then

$$r(t) \leq c_{25}(\tau), \quad \forall\ t \geq \tau > 0. \tag{4.6}$$

where c_{25} is a positive constant depending on τ.

Proof. Differentiating equation $\frac{\partial b_i(u_i)}{\partial t} - div \left[|\nabla u_i|^{p_i-2} \nabla u_i\right] + g_i(x,u) = h_i(x)$ with respect to t, we get

$$b_i'(u_i)u_i'' + b_i''(u_i)(u_i')^2 - div \left((F_i(\nabla u_i))'\right) + \sum_{j=1}^2 \frac{\partial g_i(u)}{\partial u_j} u_j' = 0. \tag{4.7}$$

Now multiplying (4.7) by u_i', and integrating over Ω gives

$$\frac{1}{2} r'(t) + \frac{1}{2} \sum_{i=1}^2 \int_\Omega b_i''(u_i)(u_i')^3 dx + \sum_{i=1}^2 \int_\Omega (F_i(\nabla u_i))' \nabla u_i' dx + \int_\Omega \left(\sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial g_i(u)}{\partial u_j} u_j' \right) u_i' dx = 0, \tag{4.8}$$

the L^∞ estimate and hypothesis (H9) imply successively

$$\int_\Omega \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial g_i(u)}{\partial u_j}u_j'\right)u_i'dx \le M\ \sum_{i=1}^{2}\int_\Omega (u_i')^2\,dx. \tag{4.9}$$

$$\gamma\sum_{i=1}^{2}\int_\Omega (u_i')^2\,d \le r(t). \tag{4.10}$$

On the other hand, by Gagliardo-Nirenberg's inequality, Young's inquality and (4.5), we obtain

$$-\frac{1}{2}\sum_{i=1}^{2}\int_\Omega b''(u_i)(u_i')^3dx \le c_{25}\sum_{i=1}^{2}||u_i'||_2^{3(1+q_i)} + c_{26}\sum_{i=1}^{2}||\nabla u_i||_{p_i}^{p_i}\ +$$

$$\sum_{i=1}^{2}\frac{4(p_i-1)}{p_i^2}\int_\Omega \left|(E_i(\nabla u_i))'\right|^2dx, \tag{4.11}$$

where $q_i=\frac{N(3-p_i)}{3Np_i+6p_i-9N}$.
By (4.9),(4.10),(4.11), (4.7) becomes

$$\frac{1}{2}r'(t)+\sum_{i=1}^{2}\frac{(p_i-1)}{2}\left(\frac{2}{p_i}\right)^2\int_\Omega\left|(E_i(\nabla u_i))'\right|^2dx \le c_{27}\sum_{i=1}^{2}||u_i'||_2^{3(1+q_i)}+$$

$$c_{128}\sum_{i=1}^{2}||\nabla u_i||_{p_i}^{p_i}\ + M\sum_{i=1}^{2}\|u_i'\|_2^2\,. \tag{4.12}$$

Now (4.11) and estimate (2.13) give

$$\frac{1}{2}r'(t)+\sum_{i=1}^{2}\frac{2(p_i-1)}{p_i^2}\int_\Omega\left|(E_i(\nabla u_i))'\right|^2dx \le c_{29}\left(r(t)\right)^2+c_{30}\ \text{for any } t\ge\tau>0. \tag{4.13}$$

Using estimate (2.14) now gives

$$\sum_{i=1}^{2}\left[\frac{1}{p_i}\int_\Omega|\nabla u_i^n|^{p_i}\,dx\right]\le c_{31}(\tau)\ ,\forall t\ge\frac{\tau}{2}\ \text{for any } \tau>0,$$

integrating (2.20) on $\left[t,t+\frac{\tau}{2}\right]$, then yields

$$\sum_{i=1}^{2}\int_t^{t+\frac{\tau}{2}}\int_\Omega b_i'(u_i)\left(u_i'\right)^2dxdt\le c_{32}(\tau),\ \text{for any } t\ge\frac{\tau}{2}>0. \tag{4.14}$$

That is

$$\int_t^{t+\frac{\tau}{2}} r(s)ds\le c_{33}(\tau),\ \text{for any}\ \ t\ge\frac{\tau}{2}>0. \tag{4.15}$$

Coming back to (4.13) and using the uniform Gronwall's lemma 2.2 gives

$$r(t+\frac{\tau}{2}) \leq c_{34}(\tau), \text{ for any } t \geq \frac{\tau}{2} > 0.$$

Hence

$$r(t) \leq c_\tau, \text{ for any } t \geq \tau > 0.$$

By use of theorem 4.1, we shall now arrive to the aim result of this section.

Theorem 4.2 *Let f_i, b_i and p_i satisfies hypothesis (H1) to (H10). Then, for any $\tau > 0$, the solution of system (S) satisfies the following regularity estimates*

$$\frac{\partial\, b_i(u_i)}{\partial t} \in L^2(\tau, +\infty; L^2(\Omega)), \tag{4.16}$$

$$\text{and} \quad u_i \in L^\infty(\tau, +\infty; B_\infty^{1+\sigma_i, p_i}(\Omega)). \tag{4.17}$$

Moreover, there exists a constant $c_\tau > 0$ such that

$$\lim_{t\to+\infty} \| |\nabla u_i|^{(p_i-2)/2} \frac{\partial \nabla u_i}{\partial t} \|_{L^2(t,t+1;L^2(\Omega))} \leq c(\tau). \tag{4.18}$$

Proof. By theorem 4.1 and hypothesis (H2), we get :

$$\sum_{i=1}^{2} \int_\Omega \left(\frac{\partial b_i(u_i)}{\partial t} \right)^2 dx \leq Mr(t) \leq c(\tau) \text{ for any } t \geq \tau > 0, \text{then yields (4.16)}.$$

Integrating (4.13) on $[t, t+1]$, for any $t \geq \tau$ and using theorem 4.1 then yields:

$$\sum_{i=1}^{2} \int_t^{t+1} \int_\Omega \left| (E_i(\nabla u_i))' \right|^2 dx ds \leq c(\tau), \text{ for any } \tau > 0, \tag{4.19}$$

whence the estimate (4.18). On the other hand by (H10) there is some σ_i', $0 < \sigma_i' < 1$, such that : $L^2(\Omega) \subset W^{-\sigma_i', p_i'}(\Omega)$ where p_i' is the conjugate of p_i : that is , $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ Simon's regularity results [18], concerning the problem

$$-\triangle_{p_i} u_i = -f_i(.,u) - b_i(u_i)_t \in L^\infty(\tau, +\infty; B_\infty^{-\sigma_i', p_i'}(\Omega)).$$

Then give for any $t \geq \tau$,

$$\|u_i(.,t)\|_{B_\infty^{1+(1-\sigma_i')(1-p_i)^2, p_i}(\Omega)} \leq c_{35} \left\| f_i(.,w) - b_i'(u_i)\,(u_i)_t \right\|_{B_\infty^{-\sigma_i', p_i'}(\Omega)} + c_{36}(\tau).$$

Hence estimate (4.17) follows.

For a solution (u_1, u_2) of $(\mathbf{S})$, we define the $\omega -$ limit set by : $\omega(\varphi_1, \psi_1) =$

$$\left\{ \begin{array}{c} w = (w_1, w_2) \in \left(W_0^{1,p_1}(\Omega) \times L^\infty(\Omega) \right) \cap \left(W_0^{1,p_2}(\Omega) \times L^\infty(\Omega) \right) \\ \exists t_n \to +\infty \left| \begin{array}{l} u_1(.,t_n) \to w_1 \text{ in } L^{p_1}(\Omega) \\ u_2(.,t_n) \to w_2 \text{ in } L^{p_2}(\Omega) \end{array} \right. \end{array} \right\}$$

Corollary 4.1 *Under the assumptions (H1) to (H10), we have* $\omega(\varphi_1, \psi_1) \neq \emptyset$ *and any* $(w_1, w_2) \in \omega(\varphi_1, \psi_1)$ *is a bounded weak solution of the stationary problem*

$$\begin{cases} -\Delta_{p_i} w_i + f_i(x, w) = 0 & \text{in } \Omega \\ w_i = 0 & \text{on } \partial\Omega \end{cases}$$

Proof. From (4.19) we obtain $\omega(\varphi_1, \psi_1) \neq \emptyset$, letting $w_i = \lim\limits_{n \to +\infty} u_i(., t_n)$ and $w = (w_1, w_2) \in \omega(\varphi_1, \psi_1)$, we get that w is a solution of the Dirichlet problem for elliptic system. The proof is analogous to DIAZ and DE THELIN [4] and is omitted.

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