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Oscillation of Second-Order Forced Nonlinear Dynamic Equations on Time Scales

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Abstract

In this paper, we discuss the oscillatory behavior of the second-order forced nonlinear dynamic equation

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + p(t)f(x^{\sigma}) = r(t),$$

on a time scale $\mathbb{T}$ when $a(t) > 0$. We establish some sufficient conditions which ensure that every solution oscillates or satisfies $\liminf_{t\to\infty} |x(t)| = 0$. Our oscillation results when $r(t) = 0$ improve the oscillation results for dynamic equations on time scales that has been established by Erbe and Peterson [Proc. Amer. Math. Soc 132 (2004), 735-744], Bohner, Erbe and Peterson [J. Math. Anal. Appl. 301 (2005), 491–507] since our results do not require $\int_{t_0}^{\infty} q(t)\Delta t > 0$ and $\int_{\pm t_0}^{\pm\infty} \frac{du}{f(u)} < \infty$. Also, as a special case when $\mathbb{T} = \mathbb{R}$, and $r(t) = 0$ our results improve some oscillation results for differential equations. Some examples are given to illustrate the main results.

Keywords. Oscillation, forced second-order nonlinear dynamic equation, time scale, positive solution.
AMS Subject Classification. 34K11, 39A10, 39A99 (34A99, 34C10, 39A11)

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph. D. Thesis in 1988 in order to unify continuous and discrete analysis, see [14]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [6]). Not only the new theory of the so-called "dynamic equations" unify the theories of differential

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equations and difference equations, but also extends these classical cases to cases " in between", e.g., to the so-called $q-$difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0$ for $q > 1\}$ and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. A book on the subject of time scales by Bohner and Peterson [6] summarizes and organizes much of the time scale calculus. The reader is referred to [6, Chapter 1] for the necessary time scale definitions and notation used throughout this paper.

In recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales. The oscillation results not only unify the oscillation results of differential and difference equations but also involve the oscillation conditions for different types of equations on different time scales which the oscillation behavior of the solutions is not known before. The problem of obtaining sufficient conditions for oscillation of the nonlinear dynamic equation

$$(a(t)x^{\Delta})^{\Delta} + p(t)f(x^{\sigma}) = 0, \quad for\ t \in \mathbb{T}, \tag{1.1}$$

with $a(t) > 0$ and $p(t) \geq 0$ has been studied by some authors, such as Saker [17], Bohner and Saker[7], Erbe, Peterson and Saker [13] and Bohner, Erbe and Peterson [5]. For oscillation of (1.1), when no explicit sign assumptions are made with respect to the coefficients $p(t)$, Erbe and Peterson [12] established some sufficient conditions for oscillation when

$$\liminf_{t\to\infty} \int_T^t p(t)\Delta t > 0, \tag{1.2}$$

for large T. One can see that (1.2) implies that either

$$\int_{t_0}^{\infty} p(t)\Delta t = \infty, \tag{1.3}$$

or

$$\int_T^{\infty} p(t)\Delta t = \lim_{t\to\infty} \int_T^t p(s)\Delta s, \tag{1.4}$$

exists and satisfies $\int_T^{\infty} p(t)\Delta t \geq 0$ for large T. Also in [5], the authors established some sufficient conditions for oscillation of Eq.(1.1) when $f(u)$ is superlinear, i.e, when $\int_{t_0}^{\infty} \frac{du}{f(u)} < \infty$.

For qualitative behavior of solutions of forced nonlinear dynamic equations on time scales, the author in [19] considered the equation

$$x^{\Delta\Delta}(t) + p^{\sigma}f(x(t)) = r(t), \quad t \in \mathbb{T},$$

where $p^{\sigma}(t)$ and $r(t)$ are real-valued $rd-$continuous positive functions defined on the time scale $\mathbb{T}$ and established some sufficient conditions for boundedness and continuation. For oscillation of different dynamic equations on time scales, we refer the reader to the papers [1-5, 8-11, 18, 20, 21].

In this paper, we are concerned with the oscillatory behavior of the forced non-linear dynamic equation

$$(a(t)x^{\Delta})^{\Delta}+p(t)f(x^{\sigma})=r(t),\ \text{ for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.5}$$

where $\mathbb{T}$ is a time scale, $a(t)>0$, $p(t)$ and $r(t)$ are real-valued $rd-$continuous function defined on the time scale $\mathbb{T}$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. Our attention is restricted to those solutions $x(t)$ of (1.5) which exist on some half line $[t_x,\infty)$ and satisfy $\sup\{|x(t)|: t>t_0\}>0$ for any $t_0\geq t_x$.

We note that, if $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$, $\mu(t)=0$, $x^{\Delta}(t)=x^{'}(t)$ and (1.5) becomes the second-order nonlinear differential equation

$$(a(t)x^{'}(t))^{'}+p(t)f(x(t))=r(t). \tag{1.6}$$

If $\mathbb{T}=\mathbb{Z}$, then $\sigma(t)=t+1$, $\mu(t)=1$, $x^{\Delta}(t)=\Delta x(t)=x(t+1)-x(t)$, $\int_a^b f(t)\Delta t=\sum_{i=a}^{b-1}f(i)$ and (1.5) becomes the second order nonlinear difference equation

$$\Delta(a(t)\Delta x(t))+p(t)f(x(t+1))=r(t). \tag{1.7}$$

If $\mathbb{T}=h\mathbb{Z}$, for $h>0$, then $\sigma(t)=t+h$, $\mu(t)=h$, $x^{\Delta}(t)=\Delta_h x(t)=\frac{x(t+h)-x(t)}{h}$, $\int_a^b f(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}}f(a+kh)h$ and (1.5) becomes the generalized difference equation

$$\Delta_h(a(t)\Delta_h x(t))+p(t)f(x(t+h))=r(t). \tag{1.8}$$

If $\mathbb{T}=q^{\mathbb{N}}=\{q^k,\ k\in\mathbb{N},\ q>1\}$, then $\sigma(t)=qt$, $\mu(t)=(q-1)t$, $x^{\Delta}(t)=\Delta_q x(t)=\frac{x(qt)-x(t)}{(q-1)t}$, $\int_a^{\infty}f(t)\Delta t=\sum_{k=0}^{\infty}\mu(q^k)f(q^k)$ and (1.5) becomes the $q-$difference equation

$$\Delta_q(a(t)\Delta_q x(t))+p(t)f(x(qt))=r(t). \tag{1.9}$$

If $\mathbb{T}=\mathbb{N}_0^2=\{t^2:t\in\mathbb{N}_0\}$, then $\sigma(t)=(\sqrt{t}+1)^2$, $\mu(t)=1+2\sqrt{t}$, $x^{\Delta}(t)=\Delta_N x(t)=\frac{x((\sqrt{t}+1)^2)-x(t)}{1+2\sqrt{t}}$, and (1.5) becomes the difference equation

$$\Delta_N(a(t)\Delta_N x(t))+p(t)f(x(\sqrt{t}+1)^2)=r(t). \tag{1.10}$$

If $\mathbb{T}=\mathbb{T}_n=\{H_n:n\in\mathbb{N}_0\}$ where $\{H_n\}$ is the set of the harmonic numbers defined by

$$H_0=0,\quad H_n=\sum_{k=1}^{n}\frac{1}{k},\quad n\in\mathbb{N}_0,$$

then $\sigma(H_n)=H_{n+1}$, $\mu(H_n)=\frac{1}{n+1}$, $x^{\Delta}(t)=x^{\Delta}(H_n)=\Delta_{H_n}x(H_n)=(n+1)x(H_n)$, and (1.5) becomes the difference equation

$$\Delta_{H_n}(a(H_n)\Delta_{H_n}x(H_n))+p(H_n)f(x(H_{n+1}))=r(t). \tag{1.11}$$

In this paper, we follow the technique that has been used by Kwong and Wong [15] for differential equations and establish some sufficient conditions which ensure that every solution of (1.5) oscillates or satisfies $\liminf_{t\to\infty}|x(t)| = 0$. Our results do not assume that $p(t)$ and $r(t)$ be of definite sign (it possible that the function $p(t)$ is oscillatory). When $r(t) = 0$, our results improve the oscillation results that has been established by Erbe and Peterson [12] and Bohner, Erbe and Peterson [5] for the equation (1.1), since our results do not require the condition (1.2) and $\int_{\pm t_0}^{\pm\infty} \frac{du}{f(u)} < \infty$. As a special case when $\mathbb{T} = \mathbb{R}$, $a(t) = 1$, and $r(t) = 0$ our results improve the oscillation results that has been established for the differential equation

$$x''(t) + p(t)f(x(t)) = 0,$$

by Kwong and Wong [15] and Li [16], since our results do not require that $\int_{\pm t_0}^{\pm\infty} \frac{du}{f(u)} < \infty$. To the best of our knowledge this approach for investigation the asymptotic behavior of Eq.(1.5) on time scales has not been studied before. We note that our results cover the oscillation behavior of the equations (1.6)-(1.11) and also can be extended on different types of time scales. The paper is ended by some examples to illustrate the main oscillation results.

2 Main Results

In this Section, we study the oscillation and nonoscillation of Eq.(1.5). Before stating our main results we need the following lemmas.

Lemma 2.1 [6]. *Assume that $g : T \to R$ is delta differentiable on T. Assume that $f : R \to R$ is continuously differentiable. Then $f \circ g : T \to R$ is delta differentiable and satisfies*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t). \tag{2.1}$$

Lemma 2.2. *Let $g(t, s, z)$ be real-valued function of t and s in $[T, C]$ and z in $[T_1, C_1]$ such that, for each fixed $t = t_0$ and $s = s_0$, $g(t_0, s_0, z)$ is a nondecreasing function of z. Let $G(t)$ be a differentiable function on $[T, C]$, let u and v be functions on $[T, C]$ such that $u(t)$ and $v(t)$ are in $[T_1, C_1]$ for all t in $[T, C]$; let $g(t, s, u(s))$ and $g(t, s, v(s))$ be are rd-continuous functions in s for fixed t; and for all t in $[T, C]$ let*

$$v(t) = G(t) + \int_T^t g(t, s, v(s))\Delta s, \tag{2.2}$$

and

$$u(t) \geq G(t) + \int_T^t g(t, s, u(s))\Delta s, \tag{2.3}$$

then $u(t) \geq v(t)$ for all t in $[T, C]$.

Proof. For $t = T$, the result is obvious. Assume that there exits $T_1 \in [T, C]$, such that $u(T_1) = v(T_1)$ and $v(t) < u(t)$ for $t \in [T, T_1]$. Since $g(t, s, z)$ is monotone nondecreasing in z for fixed t and s, it follows that $g(T_1, s, v(s)) \leq g(T_1, s, u(s))$. Then from (2.2), we have

$$v(T_1) = G(T_1) + \int_T^{T_1} g(T_1, s, v(s))\Delta s \leq G(T_1) + \int_T^{T_1} g(T_1, s, u(s))\Delta s = u(T_1). \quad (2.4)$$

This is a contradiction to the fact that $u(T_1) = v(T_1)$. Hence $u(t) \geq v(t)$ *for all* t *in* $[T, C]$. The proof is complete.

Throughout the following, we will assume that: a, p, $r : \mathbb{T} \to \mathbb{R}$ are rd-continuous functions, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable such that $a(t) > 0$ and

$$uf(u) > 0 \quad \text{and } f'(u) \geq k > 0 \quad \text{for} \quad u \neq 0, \quad (2.5)$$

Theorem 2.1. *Assume that (2.5) holds. Let $x(t)$ be a positive (negative) solution of (1.5) on $[T_1, C)$ for some positive T_1 satisfying $t_0 \leq T_1 < C \leq \infty$, and define $w(t)$ by the Riccati substitution*

$$w(t) = \frac{a(t)x^\Delta}{f(x(t))}. \quad (2.6)$$

If there exists a T in $[T_1, C)$ and a positive constant A_1 such that

$$-w(T_1) + \int_{T_1}^{t} [p(s) - r(s)/(f \circ x^\sigma)(s)]\Delta s + \int_{T_1}^{T} \left[Q(x(s))w^2(s)/a(s)\right] \Delta s \geq A_1, \quad (2.7)$$

for all t in $[T, C]$, then $a(t)x^\Delta(t) \leq -A_1 f(x(T))$ $(a(t)x^\Delta(t) \geq -A_1 f(x(T)))$ for all $t \in [T, C)$.

Proof. Let $x(t)$ be a solution of (1.5) satisfying the hypotheses of the theorem. From (1.5) and the definition of $w(t)$, we have

$$w^\Delta(t) + Q(x(t))\frac{w^2(t)}{a(t)} = r(t)/(f \circ x^\sigma)(t) - p(t), \quad (2.8)$$

where

$$Q(x) := \frac{f(x)}{(f \circ x^\sigma)} \left\{ \int_0^1 f'(x(t) + h\mu(t)x^\Delta(t))dh \right\} > 0.$$

Integrating (2.8) form T_1 to t $(T_1 \leq t \leq C)$, we get

$$w(t) - w(T_1) + \int_{T_1}^{t} \left[Q(x(s))w^2(s)/a(s)\right] \Delta s = \int_{T_1}^{t} [r(s)/(f \circ x^\sigma)(s) - p(s)]\Delta s.$$

This implies after applying (2.7) that

$$-w(t) \geq A_1 + \int_T^t \left[Q(x(s))w^2(s)/a(s)\right] \Delta s > 0, \ \ for\ T \leq t < C. \tag{2.9}$$

Since $a(t) > 0$, we see that $w(t) < 0$ and

$$x(t)x^{\Delta}(t) < 0 \ \ on \ \ [T, C).$$

Suppose that $x(t) > 0$ and let $u(t) = -a(t)x^{\Delta}(t)$. Then $u(t) > 0$ and (2.9) becomes

$$u(t) \geq A_1 f(x(t)) + \int_T^t \left[f(x(t))Q(x(s))[-x^{\Delta}(s)]u(s)/f^2(x(s))\right] \Delta s. \tag{2.10}$$

Define

$$g(t, s, z) = \left[f(x(t))Q(x(s))[-x^{\Delta}(s)]z/f^2(x(s))\right],$$

for t and s in $[T, C)$ and z in $[0, \infty)$. It is clear that the function g is nondecreasing in the variable z. Letting $G(t) = A_1 f(x(t))$ and applying Lemma 2.2, we have $u(t) \geq v(t)$, where $v(t)$ satisfies the integral equation

$$v(t) = A_1 f(x(t)) + \int_T^t \left[f(x(t))Q(x(s))[-x^{\Delta}(s)]v(s)/f^2(x(s))\right] \Delta s. \tag{2.11}$$

Multiplying (2.11) by $1/f(x(t))$ and differentiating with respect to t, we obtain

$$\left[\frac{v(t)}{f(x(t))}\right]^{\Delta} = \left[A_1 + \int_T^t \left[Q(x(s))[-x^{\Delta}(s)]v(s)/f^2(x(s))\right] \Delta s\right]^{\Delta} = \frac{-Q(x(t))x^{\Delta}(t)v(t)}{f^2(x(t))}.$$

On the other hand, by using Lemma 2.1, we have

$$\begin{aligned}
\left[\frac{v(t)}{f(x(t))}\right]^{\Delta} &= \frac{f(x(t))v^{\Delta}(t) - v(t)\left(f(x(t))\right)^{\Delta}}{f(x(t))f(x^{\sigma})} \\
&= \frac{v^{\Delta}(t)}{f(x^{\sigma})} - \frac{v(t)\left\{\int_0^1 f'(x(t) + h\mu(t)x^{\Delta}(t))dh\right\} x^{\Delta}(t)}{f(x(t))f(x^{\sigma})} \\
&= \frac{v^{\Delta}(t)}{f(x^{\sigma})} - \frac{Q(x(t))[x^{\Delta}(t)]v(t)}{f^2(x(t))}.
\end{aligned}$$

Then, we have

$$\frac{v^{\Delta}(t)}{f(x^{\sigma})} = 0.$$

This implies that $v^{\Delta}(t) = 0$, so that $v(t) = v(T) = A_1 f(x(T)) > 0$ for all t in $[T, C)$. Now, since $u(t) \geq v(t)$, we have

$$a(t)x^{\Delta}(t) \leq -A_1 f(x(T)), \ \ for\ T \leq t < C.$$

The proof for the case when $x(t)$ is negative follows from a similar argument as above by taking $u(t) = a(t)x^{\Delta}(t)$ and $G(t) = -A_1 f(x(t))$ to prove that $a(t)x^{\Delta}(t) \geq -A_1 f(x(T))$. The proof is complete.

Remark 2.1. From Theorem 2.1, we saw that $x^{\Delta}(t) < 0$ for positive solution $x(t)$, this implies that $f(x(t))/(f \circ x^{\sigma}) \geq 1$, so that

$$Q(x) = \frac{f(x)}{f(x^{\sigma})} \left\{ \int_0^1 f'(x + h\mu(t)x^{\Delta})dh \right\} \geq L,$$

for some $L \geq k > 0$.

Theorem 2.2. *Assume that (2.5) holds, and*

$$\int_{t_0}^{\infty} p(s)\Delta s < \infty, \tag{2.12}$$

$$\int_{t_0}^{\infty} |r(s)|\,\Delta s < \infty, \tag{2.13}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\Delta t = \infty. \tag{2.14}$$

If $x(t)$ *is a solution of Eq.(1.5) such that* $\liminf_{t\to\infty} |x(t)| > 0$, *then*

$$L\int_T^{\infty} \left[w^2(s)/a(s)\right]\Delta s \leq \int_T^{\infty} \left[Q(x(s))w^2(s)/a(s)\right]\Delta s < \infty, \tag{2.15}$$

$$\lim_{t\to\infty} w(t) = 0, \tag{2.16}$$

and

$$w(t) = \int_t^{\infty} [Q(x(s))w^2(s)/a(s)]\Delta s + \int_t^{\infty} [p(s) - r(s)/(f \circ x^{\sigma})(s)]\Delta s, \tag{2.17}$$

for all sufficiently large t.

Proof. Let $x(t)$ be a solution of (1.5) such that $\lim_{t\to\infty} |x(t)| > 0$. Then there exist $A_2 > 0$, $m > 0$ and $t_1 > t_0$ such that $|x(t)| \geq m$ and $|f(x(t))| \geq A_2$ for $t \geq t_1$.

This, together with (2.13), since $\int_{t_0}^{\infty} |r(s)|\,\Delta s < \infty$, implies that there exists a positive constant A_3 such that

$$\left| \int_{t_1}^{t} [r(s)/(f \circ x^{\sigma})(s)]\Delta s \right| \le \frac{1}{A_2} \int_{t_1}^{t} |r(s)|\,\Delta s \le A_3 \quad \text{for all } t \ge t_1, \tag{2.18}$$

where A_3 is a positive constant. If (2.15) does not hold, then there exists $t_2 > t_1$ such that $\int_{t_1}^{t_2} [Q(x(s))w^2(s)/a(s)]\Delta s > 0$ and then it follows from (2.12), since $\int_{t_0}^{\infty} p(s)\Delta s < \infty$, that there exist $A_1 > 0$ and $t_2 > t_1$ such that (2.7) holds for $t \ge t_2$. For the case when $x(t) > 0$ on $[t_2, \infty)$ it follows from Theorem 2.1 and its proof that $x^{\Delta}(t) < 0$ and $a(t)x^{\Delta}(t) \le -A_1 f(x(t_2))$ for $t \ge t_2$. Therefore from (2.14), we have

$$x(t) \le x(t_2) - A_1 f(x(t_2)) \int_{t_1}^{t} \frac{1}{a(s)}\Delta s \to -\infty \quad \text{as } t \to \infty,$$

which contradicts the fact that $x(t) > 0$ on $[t_2, \infty)$. The proof when $x(t) < 0$ on $[t_2, \infty)$ is similar and will be omitted. This completes the proof of (2.15). Next, we prove (2.16) and (2.17) hold. From Theorem 2.1, since

$$w^{\Delta}(t) + Q(x(t))\frac{w^2(t)}{a(t)} = r(t)/(f \circ x^{\sigma})(t) - p(t),$$

we have after integration from t to z (for $t > z$) that

$$w(z) + \int_{t}^{z} [Q(x(s))w^2(s)/a(s)]\Delta s = w(t) + \int_{t}^{z} [r(s)/(f \circ x^{\sigma})(s) - p(s)]\Delta s. \tag{2.19}$$

This together with (2.12), (2.15) and (2.18), implies that $\lim_{z\to\infty} w(z)$ exists. Then from (2.19), we have

$$w(t) = \lim_{z\to\infty} w(z) + \int_{t}^{\infty} [Q(x(s))w^2(s)/a(s)]\Delta s + \int_{t}^{\infty} [p(s) - r(s)/(f \circ x^{\sigma})(s)]\Delta s, \quad \text{for } t \ge t_1. \tag{2.20}$$

To prove that (2.16) and (2.17) hold it suffices to prove that $\lim_{z\to\infty} w(z) = 0$. First suppose that $x(t) > 0$ on $[t_1, \infty)$, so that by Theorem 2.1, we have $x(t)x^{\Delta}(t) < 0$ and (2.9) holds. If not assume that $\lim_{t\to\infty} w(z) = A_4$ exists. If $A_4 > 0$, we see that $w(t) = \frac{a(t)x^{\Delta}}{f(x(t))} \ge A_4/2$ for $t \ge T_2 > t_1$ which implies that $x^{\Delta}(t) > 0$ for $t \ge T_2$ and this is a contradiction since $x(t)x^{\Delta}(t) < 0$. If $A_4 < 0$, then (2.12), (2.15) and (2.18) imply that there exists a $T_1 > t_1$ so large such that

$$\left.\begin{array}{c} \left| \int_{t}^{\infty} p(s)\Delta s \right| \le -A_4/6, \quad \left| \int_{t}^{\infty} [r(s)/(f \circ x^{\sigma})(s)]\Delta s \right| \le -A_4/6, \\ \int_{t_1}^{\infty} [Q(x(s))w^2(s)/a(s)]\Delta s \le -A_4/6. \end{array}\right\} \tag{2.21}$$

Letting $t = t_0$ in (2.20), we have

$$w(t_0) = A_4 + \int_{t_0}^{\infty} [Q(x(s))w^2(s)/a(s)]\Delta s + \int_{t_0}^{\infty} [p(s) - r(s)/(f \circ x^\sigma)(s)]\Delta s. \qquad (2.22)$$

Using (2.21) and (2.22) and the fact that $Q(x(s)) > 0$, we have

$$\begin{aligned} & -w(t_0) + \int_{t_0}^{t_1} [Q(x(s))w^2(s)/a(s)]\Delta s + \int_{t_0}^{t} [p(s) - r(s)/(f \circ x^\sigma)(s)]\Delta s \\ = \quad & -A_4 - \int_{t_1}^{\infty} [Q(x(s))w^2(s)/a(s)]\Delta s - \int_{t}^{\infty} [p(s) - r(s)/(f \circ x^\sigma)(s)]\Delta s \\ > \quad & -A_4 + \frac{A_4}{6} + \frac{A_4}{6} + \frac{A_4}{6} = -\frac{A_4}{2} := A_1 > 0. \end{aligned}$$

Thus the condition (2.7) of Theorem 2.1 is satisfied. Then by applying Theorem 2.1, we obtain

$$a(t)x^\Delta(t) \le -A_1 f(x(T)).$$

Then integrating the last inequality leads to a contradiction with the positive nature of $x(t)$. Thus $A_4 = 0$ and this proved (2.16). The proof when $x(t) < 0$ is similar and will be omitted.

As consequence from Theorem 2.2, we have the following property of the nonoscillatory solutions of Eq.(1.5).

Corollary 2.1. *Assume that (2.5), (2.12)-(2.14) hold. If $x(t)$ is a nonoscillatory solution (1.5), then*

$$\lim_{t\to\infty} \frac{a(t)x^\Delta(t)}{f(x(t))} = 0,$$

holds.

We note that if (2.12) and (2.13) hold then the function

$$h_0(t) = \int_{t}^{\infty} [p(s) - \gamma\,|r(s)|]\Delta s/\sqrt{a(t)},$$

is well defined on $[t_0, \infty)_{\mathbb{T}}$ for every positive constant γ and for any arbitrary function g we define $(g(t))_+ = \frac{1}{2}(g(t) + |g(t)|)$.

Theorem 2.3. *Assume that (2.5), (2.12)-(2.14) hold. Then either*

$$\int_t^{\infty} [(h_0(t))_+]^2 \Delta t = \infty, \qquad (2.23)$$

or

$$\int_t^{\infty}\left(\left[h_0(s)+L\int_s^{\infty}(h_0(u))^2\,\Delta u/\sqrt{a(s)}\right]_+\right)\Delta s=\infty, \tag{2.24}$$

implies that every solution $x(t)$ of (1.5) oscillates or satisfies $\liminf_{t\to\infty}|x(t)|=0$.

Proof. Assume to the contrary that Eq.(1.5) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)>0$ such that $\liminf_{t\to\infty}x(t)>0$. It then follows from (2.5), there exist $K>0$, $m>0$ and $t_1>t_0$ such that $x(t)\geq m$ and $f(x^{\sigma})>0$ for $t\geq t_1$. Since the hypothesis of Theorem 2.2 hold, we have from (2.17) that

$$w(t)\geq L\int_t^{\infty}[w^2(s)/a(s)]\Delta s+h_0(t)\sqrt{a(t)}\geq h_0(t)\sqrt{a(t)}, \tag{2.25}$$

for $t\geq t_1$ and some constant $L>0$. Then, we have by (2.23) that

$$\int_t^{\infty}w^2(s)/a(s)\Delta t\geq\int_t^{\infty}[h_0(t)]^2\Delta t=\infty,$$

which contradicts (2.15). Now, suppose that (2.24) holds. From (2.25), and the last inequality, we have

$$w(t)\geq L\int_t^{\infty}[h_0(s)]^2\Delta s+h_0(t)\sqrt{a(t)},$$

so that

$$w(t)/\sqrt{a(t)}\geq h_0(t)+L\int_t^{\infty}[h_0(s)]^2/\sqrt{a(t)}\Delta s.$$

Using the fact that $w^2(t)\geq w(t)$, (noting that $w(t)<0$ since $x(t)>0$), we have

$$w^2(t)/a(t)\geq[h_0(t)+L\int_t^{\infty}[h_0(s)]^2/\sqrt{a(t)}\Delta s].$$

Integrating the last inequality, we get a contradiction with (2.15). The proof is complete.

Define

$$h_1(t)=\int_t^{\infty}[h_0(s)_+]^2\Delta s \text{ and } h_{n+1}(t)=\int_t^{\infty}([h_0(s)+Lh_n(s)/\sqrt{a(t)}]_+)^2\Delta s,$$

for $n=1,2,3,\ldots$.

Theorem 2.4. *Assume that (2.5), (2.12)-(2.14) hold. If there exists a positive integer N such that*

$$h_n \textit{ exists for } n=0,\ 1,\ 2,\ 3,\ldots,\ N-1 \textit{and } h_N \textit{ does not exist}, \tag{2.27}$$

then every solution $x(t)$ of (1.5) oscillates or satisfies $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Assume to the contrary that Eq.(1.5) has a nonoscillatory solution $x(t)$ and proceed as in the proof of Theorem 2.3 to get (2.25). From (2.25) we have

$$w(t) \geq L \int_t^\infty [w^2(s)/a(s)]\Delta s + h_0(t)\sqrt{a(t)}, \tag{2.28}$$

for $t \geq t_1$ and some constant $L > 0$. From (2.15), we have

$$\int_{t_1}^\infty [w^2(s)/a(s)]\Delta s < \infty. \tag{2.29}$$

As in the proof of Theorem 2.3, we have

$$w^2(t)/a(t) \geq [h_0(t)_+]^2. \tag{2.30}$$

If $N = 1$, then (2.29) and (2.30) imply that

$$h_1(t) = \int_t^\infty [h_0(s)_+]^2 \Delta s < \infty,$$

which contradicts the nonexistence of $h_N(t) = h_1(t)$. If $N = 2$, then from (2.28) and (2.30), we get

$$w(t) \geq L \int_t^\infty [h_0(s)_+]^2 \Delta s + h_0(t)\sqrt{a(t)} = h_0(t)\sqrt{a(t)} + Lh_1(t),$$

so

$$w(t)/\sqrt{a(t)} \geq h_0(t) + Lh_1(t)/\sqrt{a(t)},$$

and this implies that

$$w^2(t)/a(t) \geq ([h_0(t) + Lh_1(t)/\sqrt{a(t)}]_+)^2.$$

Then in view of (2.27) and (2.29), an integration of the last inequality leads to a contradiction of the nonexistence of $h_N = h_2$. A similar arguments lead to a contradiction for any $N > 2$. The proof is complete.

In the following theorems, we establish some sufficient conditions for oscillation of (1.1). We start with the following theorem.

Theorem 2.5. *Assume that (2.5) holds, and*

$$\int_{t_0}^\infty p(s)\Delta s < \infty, \tag{2.31}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty. \tag{2.32}$$

If $x(t)$ is a solution of Eq.(1.1) such that $\liminf_{t\to\infty} |x(t)| > 0$, *then*

$$\int_{T}^{\infty} \left[Q(x(s))w^2(s)/a(s)\right] \Delta s < \infty, \tag{2.33}$$

$$\lim_{t\to\infty} w(t) = 0, \tag{2.34}$$

and

$$w(t) = \int_{t}^{\infty} [Q(x(s))w^2(s)/a(s)]\Delta s + \int_{t}^{\infty} p(s)\Delta s, \tag{2.35}$$

for all sufficiently large t.

Proof. Assume to the contrary that Eq.(1.1) has a nonoscillatory solution $x(t)$ such that $\liminf_{t\to\infty} x(t) > 0$. Without loss of generality, we assume there exits $t_1 \geq t_0$ be such that $x(t) > 0$ and $f(x) > 0$ for $t \geq t_1$. (the proof when $x(t)$ is negative is similar since $xf(x) > 0$ for all $x \neq 0$). From Theorem 2.1, since $r(t) \equiv 0$, we have from (2.7) that

$$-w(T_1) + \int_{T_1}^{t} p(s)\Delta s + \int_{T_1}^{T} [Q(x(s))w^2(s)/\sqrt{a(s)}]\Delta s \geq A_1.$$

Proceeding as in Theorem 2.2, we see that $w(t) \to 0$ as $t \to \infty$, and (2.35) holds. The proof is complete.

Remark 2.2. The same arguments lead to the following conclusion for Eq.(1.1) under the weaker condition on the function $p(t)$. Suppose that the assumptions on the function f be as defined above are satisfied and $\int_{t_0}^{t} p(s)\Delta s$ is bonded below. If (1.1) has a nonoscillatory solution $x(t)$, then

$$w(t) = \beta - \int_{t_0}^{t} p(s)\Delta s + \int_{t}^{\infty} [Q(x(s))w^2(s)/a(s)]\Delta s,$$

is satisfied for some constant β such that

$$\liminf_{t\to\infty} \int_{t_0}^{t} p(s)\Delta s \leq \beta \leq \limsup_{t\to\infty} \int_{t_0}^{t} p(s)\Delta s.$$

From Theorems 2.3 and 2.4, we have the following oscillation results for Eq.(1.1).

Theorem 2.6. *Assume that (2.5), (2.31) and (2.33) hold. Then either*

$$\int_t^{\infty} [(h_0^*(t))_+]^2 \Delta t = \infty, \text{ where } h_0^*(t) = \int_t^{\infty} p(s)\Delta s/\sqrt{a(t)},$$

or

$$\int_t^{\infty} \left(\left[h_0^*(s) + L \int_t^{\infty} \left(h_0^*(s)/\sqrt{a(t)} \right)^2 \right]_+ \right) \Delta s = \infty,$$

implies that every solution $x(t)$ of (1.1) oscillates or satisfies $\liminf_{t\to\infty} |x(t)| = 0$.

Define

$$h_1^*(t) = \int_t^{\infty} [h_0^*(s)_+]^2 \Delta s \text{ and } h_{n+1}^*(t) = \int_t^{\infty} ([h_0^*(s) + Lh_n^*(s)/\sqrt{a(t)}]_+)^2 \Delta s,$$

for $n = 1, 2, 3, ...$, where L is a positive constant.

Theorem 2.7. *Assume that (2.5), (2.31) and (2.33) hold. If there exists a positive integer N such that*

$$h_n^* \text{ exists for } n = 0,\ 1,\ 2,\ 3, ...,\ N-1 \text{ and } h_N \text{ does not exist.}$$

Then every solution $x(t)$ of (1.1) oscillates or satisfies $\liminf_{t\to\infty} |x(t)| = 0$.

Remark 2.3. We note that our results in Theorem 2.5 and 2.6 improve the results that has been established for Eq.(1.1) by Saker [16], Bohner and Saker [7], Erbe, Peterson and Saker [13] since our results do not require that condition $p(t) > 0$. The results also improve the results that has been established by Erbe and Peterson [12], since our results do not require the condition (1.2) and improve the results that has been established by Bohner, Erbe and Peterson [5, Theorem 3.3], since our results do not require that the condition $\int_{\pm t_0}^{\pm\infty} \frac{du}{f(u)} < \infty$.

In the case when $\mathbb{T} = \mathbb{R}$, we note that Theorem 2.5 improve the results that has been established by Li [16, Theorem 3.1] and when $a(t) = 1$, Theorem 2.5 improve the results by Kwong and Wong [15, Theorem 3] for differential equations, where our results do not require that the condition $\int_{\pm t_0}^{\pm\infty} \frac{du}{f(u)} < \infty$ to be satisfied.

3 Examples

In this Section, we give some examples which demonstrate how the theory of previous section may be applied to specific problems.

Example 3.1. Consider the second-order nonlinear forced dynamic equation

$$x^{\Delta\Delta}(t) + \frac{1}{t\sigma(t)} x^{\sigma}(1 + (x^{\sigma})^2) = \frac{1}{t\sigma(t)}, \ \textit{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.1}$$

Here $p(t) = r(t) = \frac{1}{t\sigma(t)}$ and $f(u) = u(1+u^2)$. It is easy to see that $f(u)$ satisfies (2.5) with $k = 1$, and

$$\int_{t_0}^{\infty} p(t)\Delta t = \int_{t_0}^{\infty} |r(t)|\,\Delta t = \int_{t_0}^{\infty} \frac{1}{t\sigma(t)}\Delta t = \int_{t_0}^{\infty} \left(-\frac{1}{t}\right)^{\Delta} \Delta t = 1/t_0 < \infty.$$

On the other hand,

$$h_0(t) = \int_t^{\infty} (p(s) - \gamma r(s))\Delta s = \frac{(1-\gamma)}{t} > 0, \text{ for each } 0 < \gamma < 1,\ t \geq t_0 > 0,$$

$$h_1(t) = \int_t^{\infty} [h_0(s)]^2\,\Delta s = (1-\gamma)\int_t^{\infty} \frac{1}{s^2}\Delta s < \infty, \text{ for each } 0 < \gamma < 1,$$

and by putting $L = 1$, we have

$$\begin{aligned}
\int_t^{\infty} [h_0(s) + h_1(s)]\,\Delta s &= \int_t^{\infty} \left[\frac{(1-\gamma)}{s} + (1-\gamma)^2 \left(\int_s^{\infty} \frac{1}{u^2}\Delta u\right)^2\right] \Delta s \\
&\geq \int_t^{\infty} \left[\frac{(1-\gamma)}{s} + (1-\gamma)^2 \left(\int_s^{\infty} \frac{1}{u\sigma(u)}\Delta u\right)^2\right] \Delta s \\
&= \int_t^{\infty} \left[\frac{(1-\gamma)}{s} + (1-\gamma)^2 \left(\int_s^{\infty} \left(\frac{1}{u}\right)^{\Delta} \Delta u\right)^2\right] \Delta s \\
&= \int_t^{\infty} \left[\frac{(1-\gamma)}{s} + (1-\gamma)^2 \frac{1}{s^2}\right] \Delta s = \infty.
\end{aligned}$$

Hence, it follows from Theorem 2.3 that every solution $x(t)$ of Eq.(3.1) is either oscillates or satisfies $\liminf_{t\to\infty} |x(t)| = 0$.

Example 3.2. Consider the second-order nonlinear forced dynamic equation

$$x^{\Delta\Delta}(t) + \frac{1}{t\sigma(t)} x^{\sigma}(1 + (x^{\sigma})^2) = -\frac{1}{t\sigma(t)},\ \textit{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.2}$$

Here $p(t) = \frac{1}{t\sigma(t)}$, $r(t) = -\frac{1}{t\sigma(t)}$ and $f(u) = u(1+u^2)$. It is easy to see as above that $f(u)$ satisfies the conditions of Theorem 2.3,

$$\int_{t_0}^{\infty} p(t)\Delta t = \int_{t_0}^{\infty} |r(t)|\,\Delta t = \int_{t_0}^{\infty} \frac{1}{t\sigma(t)}\Delta t = \int_{t_0}^{\infty} \left(-\frac{1}{t}\right)^{\Delta} \Delta t = 1/t_0 < \infty.$$

On the other hand,

$$h_0(t) = \int_t^{\infty} (p(s) - r(s))\Delta s = \frac{2}{t} > 0,\ \int_t^{\infty} [h_0(s)]^2\,\Delta s = \int_t^{\infty} \frac{4}{s^2}\Delta s < \infty, \text{ for } t \geq t_0.$$

and by putting $L=1$, we have

$$\begin{aligned}
\int_t^\infty [h_0(s)+Lh_1(s)]\,\Delta s &= \int_t^\infty \left[\frac{2}{s}+16\left(\int_s^\infty \frac{1}{u^2}\Delta u\right)^2\right]\Delta s \\
&\geq \int_t^\infty \left[\frac{2}{s}+16\left(\int_s^\infty \frac{1}{u\sigma(u)}\right)^2\Delta u\right]\Delta s \\
&= \int_t^\infty \left[\frac{2}{s}+16\left(\int_s^\infty \left(\frac{1}{u}\right)^\Delta \Delta u\right)^2\right]\Delta s \\
&= \int_t^\infty \left[\frac{2}{s}+\frac{16}{s^2}\right]\Delta s=\infty.
\end{aligned}$$

Hence, it follows from Theorem 2.3 that every solution $x(t)$ of Eq.(3.2) is either oscillates or satisfies $\liminf_{t\to\infty}|x(t)|=0$.

Example 3.3. Consider the second-order nonlinear forced dynamic equation

$$x^{\Delta\Delta}(t)+\frac{1}{t\sigma(t)}x^\sigma(1+(x^\sigma)^{\frac{1}{3}})=\frac{1}{t\sigma(t)},\quad for\ t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.3}$$

Here $p(t)=\frac{1}{t\sigma(t)}$, $r(t)=\frac{1}{t\sigma(t)}$ and $f(u)=u(1+u^{\frac{1}{3}})$. By Theorem 2.3, we can easily see that every solution $x(t)$ of Eq.(3.3) is either oscillates or satisfies $\liminf_{t\to\infty}|x(t)|=0$.

Note that the results that has been established by Bohner, Erbe and Peterson [5, Theorem 3.3] can not be applied for Eq.(3.3), since the condition $\int_{\pm t_0}^{\pm\infty}\frac{du}{f(u)}<\infty$ is not satisfied. Also a special case when $\mathbb{T}=\mathbb{R}$, the results by Kwong and Wong [15] and Li [16] can not be applied for the equation

$$x''(t)+\frac{1}{t^2}x(1+(x)^{\frac{1}{3}})=\frac{1}{t^2},\ \text{for } t>0,$$

since

$$\int^\infty \frac{du}{u(1+u^{\frac{1}{3}})}=\infty.$$

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