Electronic Journal of Qualitative Theory of Differential Equations
2005, No. **21**, 1-16; **http://www.math.u-szeged.hu/ejqtde/**

FIRST ORDER INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH ALGEBRAS INVOLVING CARATHEODORY AND DISCONTINUOUS NONLINEARITIES

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Abstract

In this paper some existence theorems for the first order differential equations in Banach algebras is proved under the mixed generalized Lipschitz, Carathéodory and monotonicity conditions.

Key words and phrases : Banach algebra, second order differential equation, existence theorem.
AMS (MOS) Subject Classifications: 34K10.

1 Introduction

Let $\mathbb{R}$ denote the real line. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, consider the first order functional integro-differential equation (in short IGDE)

$$\left.\begin{array}{l} \dfrac{d}{dt}\Big[\dfrac{x(t)}{f(t, x(t))}\Big] = \displaystyle\int_0^t g(s, x(s))\,ds, \quad \text{a.e. } \ t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{array}\right\} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$, and $g : J \times \mathbb{R} \to \mathbb{R}$.

By a solution of IGDE (1.1) we mean a function $x \in AC^1(J, \mathbb{R})$ such that

(i) the function $t \to \Big(\dfrac{x}{f(t, x)}\Big)$ is continuous for each $x \in \mathbb{R}$, and

(ii) x satisfies the equations in (1.1),

where, $AC^1(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions on J.

In recent years, the topic of nonlinear differential equations in Banach algebras is received the attention of several authors and at present, there is a considerable literature available in this direction. See Dhage and O'Regan [5] Dhage et. al. [6] and and the references therein. In this paper we deal with the second order ordinary differential equations in Banach algebras and discuss the existence results under mixed Lipschitz and Carathéodory conditions. We will employ the fixed point theorems of Dhage [2, 3, 4] for proving our main existence results. The nonlinear differential equation as well as the existence results of this are new to the literature on the theory of ordinary differential equations.

Our method of study is to convert the IGDE (1.1) into equivalent integral equation and apply the fixed point theorems of Dhage [2, 3, 4] under suitable conditions on the nonlinearities f and g. In the following section we shall give some preliminaries needed in the sequel.

2 Auxiliary Results

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called $\mathcal{D}$-**Lipschitz** if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ $(\alpha > 0)$, A is called a Lipchitz with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with a contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear contraction on X. Sometimes we call the function ψ a $\mathcal{D}$-**function** for convenience.

An operator $T : X \to X$ is called **compact** if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. Similarly $T : X \to X$ is called **totally bounded** if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \to X$ is called **completely continuous** operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [4] is embodied in the following theorem.

Theorem 2.1 (Dhage[4]) *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) A is a $\mathcal{D}$-Lipschitz with a $\mathcal{D}$-function ψ,

(b) B is compact and continuous, and

(c) $M\psi(r) < r$ whenever $r > 0$, where $M = \|B(X)\| = \sup\{\|Bx\| : x \in X\}$.

Then either

(i) the equation $\lambda Ax\, Bx = x$ has a solution for $\lambda = 1$, or

(ii) the set $\mathcal{E} = \{u \in X \mid \lambda Au\, Bu = u, 0 < \lambda < 1\}$ is unbounded.

It is known that Theorem 2.1 is useful for proving the existence theorems for the integral equations of mixed type. See [2] and the references therein. The method is commonly known as *priori bound method* for the nonlinear equations. See, for example, Dugundji and Granas [7], Zeidler [12] and the references therein.

An interesting corollary to Theorem 2.1 in its applicable form is

Corollary 2.1 *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) A is Lipschitz with a Lipschitz constant α,

(b) B is compact and continuous, and

(c) $\alpha M < 1$, where $M = \|B(X)\| := \sup\{\|Bx\| : x \in X\}$.

Then either

(i) the equation $\lambda Ax\, Bx = x$ has a solution for $\lambda = 1$, or

(ii) the set $\mathcal{E} = \{u \in X \mid \lambda Au\, Bu = u, 0 < \lambda < 1\}$ is unbounded.

A non-empty closed set K in a Banach algebra X is called a **cone** if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called to be **positive** if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leq$ in X as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called to be **normal** if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [9].

We equip the space $AC^1(J, \mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}. \tag{2.2}$$

It is well known that the cone K is positive and normal in $AC^1(J, \mathbb{R})$. As a result of positivity of the cone K in $AC^1(J, \mathbb{R})$ we have:

Lemma 2.1 (Dhage [3]). Let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$.

For any $a, b \in X = AC^1(J, \mathbb{R}), a \leq b$, the order interval $[a, b]$ is a set in X defined by

$$[a, b] = \{x \in X : a \leq x \leq b\}. \tag{2.3}$$

We use the following fixed point theorem of Dhage [3] for proving the existence of extremal solutions of the IGDE (1.1) under certain monotonicity conditions.

Theorem 2.2 (Dhage [3]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) A is Lipschitz with a Lipschitz constant α,

(b) B is completely continuous,

(c) $Ax\, Bx \in [a, b]$ for each $x \in [a, b]$, and

(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| :=$ $\sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 2.3 (Dhage [4]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) *A is completely continuous,*

(b) *B is totally bounded,*

(c) *$Ax\,By \in [a, b]$ for each $x, y \in [a, b]$, and*

(d) *B is nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a, b]$.

Theorem 2.4 (Dhage [4]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*

(b) *B is totally bounded,*

(c) *$Ax\,By \in [a, b]$ for each $x, y \in [a, b]$, and*

(d) *B is nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has least and a greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| :=$ $\sup\{\|Bx\| : x \in [a, b]\}$.

Remark 2.1 *Note that hypothesis (c) of Theorems 2.2, 2.3, and 2.4 holds if the operators A and B are monotone increasing and there exist elements a and b in X such that $a \le Aa\,Ba$ and $Ab\,Bb \le b$.*

3 Existence Theory

Let $M(J, \mathbb{R})$ and $B(J, \mathbb{R})$ respectively denote the spaces of measurable and bounded real-valued functions on J. Let $C(J, \mathbb{R})$, be the space of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with this norm and the multiplication "$\cdot$" defined by $(xy)(t) = x(t)y(t)$ for all $t \in J$. By $L^1(J,\mathbb{R})$ we denote the set of Lebesgue integrable functions on J and the norm $\|\cdot\|$ in $L^1(J,\mathbb{R})$ is defined by

$$\|x\|_{L1} = \int_0^1 |x(t)|\,ds.$$

We need the following lemma in the sequel.

Lemma 3.1 *If $h \in L^1(J,\mathbb{R})$, then x is a solution of the IGDE*

$$\left.\begin{aligned} &\frac{d}{dt}\Big[\frac{x(t)}{f(t,x(t))}\Big] = \int_0^t h(s)\,ds, \quad a.e.\ \ t \in J \\ &x(0) = x_0, \end{aligned}\right\} \tag{3.1}$$

if and only if is solution of the integral equation (in short IE)

$$x(t) = \big[f(t,x(t))\big]\Big(\frac{x_0}{f(0,x_0)} + \int_0^t (t-s)h(s)\,ds\Big),\ t \in J. \tag{3.2}$$

Proof. The proof is simple and omit the details.

We need the following definition in the sequel.

Definition 3.1 *A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be* **Carathéodory** *if*

(i) $t \to \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \to \beta(t,x)$ is continuous almost everywhere for $t \in J$.

Again a Carathódory function $\beta(t,x)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h_r(t), \quad a.e.\ \ t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally a Carathéodory function $\beta(t,x)$ is called L^1_X-Carathéodory if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h(t), \quad a.e.\ \ t \in J$$

for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a **bound function** *of β.*

We will need the following hypotheses in the sequel.

(H_1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $k \in B(J, \mathbb{R})$ such that $k(t) > 0$, a.e. $t \in J$ and

$$|f(t,x) - f(t,y)| \leq k(t)\,|x-y|\,, \text{ a.e. } t \in J$$

for all $x, y \in \mathbb{R}$.

(H_2) The function g is L^1_X-Carathéodory with bound function h.

(H_3) There exists a continuous and nondecreasing function $\Omega : [0,\infty) \to (0,\infty)$ and a function $\gamma \in L^1(J,\mathbb{R})$ such that $\gamma(t) > 0$, a.e. $t \in J$ and

$$|g(t,x)| \leq \gamma(t)\Omega\big(|x|\big), \quad \text{a.e. } t \in J,$$

for all $x \in \mathbb{R}$.

Theorem 3.1 *Assume that the hypotheses (H_1)-(H_3) hold. Suppose that*

$$\int_{C_1}^{\infty} \frac{ds}{\Omega(s)} > C_2 \|\gamma\|_{L^1}, \tag{3.3}$$

where

$$C_1 = \frac{F\left|\frac{x_0}{f(0,x_0)}\right|}{1 - \|k\|\big[\big|\frac{x_0}{f(0,x_0)}\big| + T\|h\|_{L^1}\big]}, \quad C_2 = \frac{FT}{1 - \|k\|\big[\big|\frac{x_0}{f(0,x_0)}\big| + T\|h\|_{L^1}\big]},$$

$\|k\|\big[\big|\frac{x_0}{f(0,x_0)}\big| + T\|h\|_{L^1}\big] < 1$, $F = \max_{t\in J}|f(t,0)|$, *and* $\|k\| = \max_{t \in J}|k(t)|$. *Then the IGDE (1.1) has a solution on J.*

Proof. By Lemma 3.1, the IGDE (1.1) is equivalent to integral equation

$$x(t) = \big[f(t,x(t))\big]\Big(\frac{x_0}{f(0,x_0)} + \int_0^t (t-s) g(s,x(s))\,ds\Big), \quad t \in J. \tag{3.4}$$

Set $X = C(J,\mathbb{R})$. Define the two mappings A and B on X by

$$Ax(t) = f(t,x(t)), \quad t \in J, \tag{3.5}$$

and

$$Bx(t) = \frac{x_0}{f(0,x_0)} + \int_0^t (t-s) g(s,x(s))\,ds, \quad t \in J. \tag{3.6}$$

Obviously A and B define the operators $A, B : X \to X$. Then the IGDE (1.1) is equivalent to the operator equation

$$x(t) = Ax(t)\,Bx(t), \quad t \in J. \tag{3.7}$$

We shall show that the operators A and B satisfy all the hypotheses of Corollary 2.1.

We first show that A is a Lipschitz on X. Let $x, y \in X$. Then by (H_1),

$$\begin{aligned} |Ax(t) - Ay(t)| &\le |f(t, x(t)) - f(t, y(t))| \\ &\le k(t)\,|x(t) - y(t)| \\ &\le \|k\|\,\|x - y\| \end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \le \|k\|\|x - y\|$$

for all $x, y \in X$. So A is a Lipschitz on X with a Lipschitz constant $\|k\|$. Next we show that B is completely continuous on X. Using the standard arguments as in Granas et al. [8], it is shown that B is a continuous operator on X. Let S be a bounded set in X. We shall show that $B(X)$ is a uniformly bounded and equicontinuous set in X. Since $g(t, x(t))$ is L^1_X-Carathéodory, we have

$$\begin{aligned} |Bx(t)| &\le \left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t |t - s||g(s, x(s))|\,ds \\ &\le \left|\frac{x_0}{f(0, x_0)}\right| + T\int_0^t h(s)\,ds \\ &\le \left|\frac{x_0}{f(0, x_0)}\right| + T\,\|h\|_{L^1}. \end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \le M$ for all $x \in S$, where $M = \left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}T$. This shows that $B(X)$ is a uniformly bounded set in X. Now we show that $B(X)$ is an equicontinuous set. Let $t, \tau \in J$. Then for any $x \in X$ we have by (3.6),

$$\begin{aligned} |Bx(t) - Bx(\tau)| &\le \left|\int_0^t (t - s)g(s, x(s))\,ds - \int_0^\tau (\tau - s)g(s, x(s))\,ds\right| \\ &\le \left|\int_0^t (t - s)g(s, x(s))\,ds - \int_0^t (\tau - s)g(s, x(s))\,ds\right| \\ &\quad + \left|\int_0^t (\tau - s)g(s, x(s))\,ds - \int_0^\tau (\tau - s)g(s, x(s))\,ds\right| \\ &\le \left|\int_0^t (t - \tau)g(s, x(s))\,ds\right| + \left|\int_\tau^t (\tau - s)g(s, x(s))\,ds\right| \\ &\le \int_0^T |t - \tau||g(s, x(s))|\,ds + T\left|\int_\tau^t |g(s, x(s))|\,ds\right| \\ &\le \int_0^T |t - \tau|h(s)\,ds + T\left|\int_\tau^t h(s)\,ds\right| \\ &\le |t - \tau|\,\|h\|_{L^1} + |p(t) - p(\tau)| \end{aligned}$$

where $p(t) = T\displaystyle\int_0^t h(s)\,ds$. Therefore,

$$|Bx(t) - Bx(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

Hence $B(X)$ is an equi-continuous set and consequently $B(X)$ is relatively compact by Arzelà-Ascoli theorem. As a result B is a compact and continuous operator on X. Thus all the conditions of Theorem 2.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $x \in X$ be any solution to IGDE (1.1). Then we have, for any $\lambda \in (0, 1)$,

$$x(t) = \lambda\left[f(t, x(t))\right]\left(\left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t (t-s)g(s, x(s))\,ds\right)$$

for $t \in J$. Therefore,

$$\begin{aligned}
|x(t)| &\le \lambda\,|f(s, x(t)|\left(\left|\frac{x_0}{f(0, x_0)}\right| + \left|\int_0^t (t-s)g(s, x(s))\,ds\right|\right)\\
&\le \lambda\Big(|f(s, x(t)) - f(t, 0)| + |f(t, 0)|\Big)\\
&\qquad \times\left(\left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t |t-s||g(s, x(s))|\,ds\right)\\
&\le [k(t)\,|x(t)| + F]\left(\left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t |t-s||g(s, x(s))|\,ds\right)\\
&\le k(t)\,|x(t)|\left(\left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t |t-s||g(s, x(s))|\,ds\right)\\
&\qquad + F\left(\left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t |t-s||g(s, x(s))|\,ds\right)\\
&\le \|k\|\,|x(t)|(\left|\frac{x_0}{f(0, x_0)}\right| + \|h\|_{L^1}) + F\left|\frac{x_0}{f(0, x_0)}\right|\\
&\qquad + FT\int_0^t \gamma(s)\Omega(|x(t)|)\,ds. \qquad (3.8)
\end{aligned}$$

Put $u(t) = \sup_{s\in[0,t]}|x(s)|$, for $t \in J$. Then we have $|x(t)| \le u(t)$ for all $t \in J$, and so, there is a point $t^* \in [0, t]$ such that $u(t) = |x(t^*)|$. From (3.9) it follows that

$$\begin{aligned}
u(t) &= |x(t^*)|\\
&\le \|k\|\,|x(t^*)|(\left|\frac{x_0}{f(0, x_0)}\right| + T\|h\|_{L^1})\\
&\qquad + F\left(\left|\frac{x_0}{f(0, x_0)}\right| + T\int_0^{t^*} \gamma(s)\Omega(|x(t)|)\,ds\right)\\
&\le \|k\|\,u(t)(\left|\frac{x_0}{f(0, x_0)}\right| + T\|h\|_{L^1}) + F\left(\left|\frac{x_0}{f(0, x_0)}\right| + T\int_0^t \gamma(s)\Omega(u(s))\,ds\right)\\
&= C_1 + C_2\int_0^t \gamma(s)\Omega(u(s)))\,ds \qquad (3.9)
\end{aligned}$$

where,

$$C_1 = \frac{F\left|\frac{x_0}{f(0,x_0)}\right|}{1-\|k\|\left[\left|\frac{x_0}{f(0,x_0)}\right|+T\|h\|_L^1\right]} \quad \text{and} \quad C_2 = \frac{FT}{1-\|k\|\left[\left|\frac{x_0}{f(0,x_0)}\right|+T\|h\|_{L^1}\right]}.$$

Let

$$w(t) = C_1 + C_2\int_0^t \gamma(s)\Omega(u(s)))\,ds.$$

Then $u(t) \leq w(t)$ and a direct differentiation of $w(t)$ yields

$$\left.\begin{array}{rcl} w'(t) & \leq & C_2\gamma(t)\Omega(w(t)) \\ w(0) & = & C_1, \end{array}\right\} \tag{3.10}$$

that is

$$\int_0^t \frac{w'(s)}{\Omega(w(s))}\,ds \leq C_2\int_0^t \gamma(s)\,ds \leq C_2\|\gamma\|_{L^1}$$

A change of variables in the above integral gives that

$$\int_{C_1}^{w(t)} \frac{ds}{\Omega(s)} \leq C_2\|\gamma\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\Omega(s)}.$$

Now an application of mean value theorem yields that there is a constant $M > 0$ such that $w(t) \leq M$ for all $t \in J$.This further implies that

$$|x(t)| \leq u(t) \leq w(t) \leq M.$$

for all $t \in J$. Thus the conclusion (ii) of Corollary 2.1 does not hold. Therefore the operator equation $AxBx = x$ and consequently the IGDE (1.1) has a solution on J. This completes the proof. □

4 Existence of Extremal Solutions

We need the following definitions in the sequel.

Definition 4.1 *A function $u \in AC^1(J,\mathbb{R})$ is called a lower solution of the IGDE (1.1) on J if*

$$\frac{d}{dt}\Big[\frac{u(t)}{f(t,u(t))}\Big] \leq \int_0^t g(s,u(s))\,ds, \quad a.e. \quad t \in J, \ \ and \ \ u(0) \leq x_0.$$

Again a function $v \in AC^1(J,\mathbb{R})$ is called an upper solution of the IGDE (1.1) on J if

$$\frac{d}{dt}\Big[\frac{v(t)}{f(t,v(t))}\Big] \geq \int_0^t g(s,u(s))\,ds, \quad a.e. \quad t \in J, \ \ and \ \ v(0) \geq x_0.$$

Definition 4.2 *A solution x_M of the IGDE (1.1) is said to be maximal if for any other solution x to IGDE(1.1) one has $x(t) \leq x_M(t)$, for all $t \in J$. Again a solution x_m of the IGDE (1.1) is said to be minimal if $x_m(t) \leq x(t)$, for all $t \in J$, where x is any solution of the IGDE (1.1) on J.*

4.1 Carathéodory case

We consider the following set of assumptions:

(B$_0$) $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$, $g : J \times \mathbb{R} \to \mathbb{R}^+$ and $\dfrac{x_0}{f(0, x_0)} \geq 0$.

(B$_1$) g is Carathéodory.

(B$_2$) The functions $f(t, x)$ and $g(t, x)$ are nondecreasing in x and y almost everywhere for $t \in J$.

(B$_3$) The IGDE (1.1) has a lower solution u and an upper solution v on J with $u \leq v$.

(B$_4$) The function $\ell : J \to \mathbb{R}$ defined by

$$\ell(t) = |g(t, u(t))| + |g(t, v(t))|, \ t \in J$$

is Lebesgue measurable.

Remark 4.1 *Assume that (B$_2$)-(B$_4$) hold. Then*

$$|g(t, x(t))| \leq \ell(t), \quad a.e. \ t \in J,$$

for all $x \in [u, v]$.

Theorem 4.1 *Suppose that the assumptions (H$_1$)-(H$_3$) and (B$_0$)-(B$_4$) hold. Further if $\|k\|\big(\big|\frac{x_0}{f(0,x_0)}\big| + T\|\ell\|_{L^1}\big) < 1$, and ℓ is given in Remark 4.1, then IGDE (1.1) has a minimal and a maximal positive solution on J.*

Proof. Now IGDE (1.1) is equivalent to IE (3.4) on J. Let $X = C(J, \mathbb{R})$ and define an order relation " $\leq$ " by the cone K given by (2.2). Clearly K is a normal cone in X. Define two operators A and B on X by (3.5) and (3.5) respectively. Then IE (1.1) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach algebra X. Notice that (B$_1$) implies $A, B : [u, v] \to K$. Since the cone K in X is normal, $[u, v]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 3.1, that A is a Lipschitz with a Lipschitz constant $\|\alpha\|$ and B is completely continuous operator on $[u, v]$. Again the hypothesis (B$_2$) implies that A and B are nondecreasing on $[u, v]$. To see this, let $x, y \in [u, v]$ be such that $x \leq y$. Then by (B$_2$),

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t), \ \forall t \in J.$$

Similarly,

$$\begin{aligned} Bx(t) &= \frac{x_0}{f(0, x_0)} + \int_0^t (t-s) g(s, x(s))\, ds \\ &\leq \frac{x_0}{f(0, x_0)} + \int_0^t (t-s) g(s, x(s)) ds \\ &= By(t) \end{aligned}$$

for all $t \in J$. So A and B are nondecreasing operators on $[u, v]$. Again Lemma 4.1 and hypothesis (B_3) implies that

$$\begin{aligned}
u(t) &\leq [f(t, u(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t (t-s) g(s, u(s)))\, ds\right) \\
&\leq [f(t, x(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t (t-s) g(s, x(s))\, ds\right) \\
&\leq [f(t, v(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t (t-s) g(s, v(s)))\, ds\right) \\
&\leq v(t),
\end{aligned}$$

for all $t \in J$ and $x \in [u, v]$. As a result $u(t) \leq Ax(t)Bx(t) \leq v(t)$, $\forall t \in J$ and $x \in [u, v]$. Hence $Ax\,Bx \in [u, v]$ for all $x \in [u, v]$.

Again

$$\begin{aligned}
M &= \|B([u, v])\| \\
&= \sup\{\|Bx\| : x \in [u, v]\} \\
&\leq \sup\left\{\left|\frac{x_0}{f(0, x_0)}\right| + T \sup_{t \in J} \int_0^t |g(s, x(s))|\, ds \;\Big|\; x \in [u, v]\right\} \\
&\leq \left|\frac{x_0}{f(0, x_0)}\right| + T \int_0^T \ell(s)\, ds \\
&= \left|\frac{x_0}{f(0, x_0)}\right| + T\|\ell\|_{L^1}.
\end{aligned}$$

Since $\alpha M \leq \|k\|(\left|\frac{x_0}{f(0,x_0)}\right| + T\|\ell\|_{L^1}) < 1$, we apply Theorem 4.1 to the operator equation $Ax\,Bx = x$ to yield that the IGDE (1.1) has a minimal and a maximal positive solution on J. This completes the proof. □

4.2 Discontinuous case

We need the following definition in the sequel.

Definition 4.3 *A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be* **Chandrabhan** *if*

(i) $t \to \beta(t, x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \to \beta(t, x)$ is nondecreasing almost everywhere for $t \in J$.

Again a Chandrabhan function $\beta(t, x)$ is called L^1-Chandrabhan if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x)| \leq h_r(t), \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Finally a Chandrabhan function $\beta(t,x)$ is called L^1_X-Chandrabhan if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \leq h(t), \quad a.e.\ t \in I$$

for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a **bound function** *of β.*

We consider the following hypotheses in the sequel.

(C_1) The function f is continuous on $J \times \mathbb{R}$.

(C_2) There is a function $k \in B(J,\mathbb{R})$ such that $k(t) > 0$, a.e. $t \in I$ and

$$|f(t,x) - f(t,y)| \leq k(t)\,|x-y|\,, \text{ a.e. } t \in J$$

for all $x, y \in \mathbb{R}$.

(C_3) The function $f(t,x)$ is nondecreasing in x almost everywhere for $t \in J$.

(C_4) The function g is Chandrabhan.

Theorem 4.2 *Suppose that the assumptions (B_0),(B_3)-(B_4) and (C_1)-(C_4) hold. Then IGDE (1.1) has a minimal and a maximal positive solution on J.*

Proof. Now IGDE (1.1) is equivalent to IE (3.4) on J. Let $X = C(J,\mathbb{R})$ and define an order relation "$\leq$" by the cone K given by (2.2). Clearly K is a normal cone in X. Define two operators A and B on X by (3.5) and (3.6) respectively. Then FIE (1.1) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [u,v] \to K$. Since the cone K in X is normal, $[u,v]$ is a norm bounded set in X.

Step I : Next we show that A is completely continuous on $[a,b]$. Now the cone K in X is normal, so the order interval $[a,b]$ is norm-bounded. Hence there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in [a,b]$. As f is continuous on compact $J \times [-r,r]$, it attains its maximum, say M. Therefore for any subset S of $[a,b]$ we have

$$\begin{aligned}
\|A(S)\|_{\mathcal{P}} &= \sup\{\|Ax\| : x \in S\} \\
&= \sup\Big\{\sup_{t\in J}|f(t,x(t))| : x \in S\Big\} \\
&\leq \sup\Big\{\sup_{t\in J}|f(t,x)| : x \in [-r,r]\Big\} \\
&\leq M.
\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that the function $f(t,x)$ is uniformly continuous on $[0,T]\times[-r,r]$. Therefore for any $t,\tau\in[0,T]$ we have

$$|f(t,x)-f(\tau,x)|\to 0 \text{ as } t\to\tau$$

for all $x\in[-r,r]$. Similarly for any $x,y\in[-r,r]$

$$|f(t,x)-f(t,y)|\to 0 \text{ as } x\to y$$

for all $t\in[0,T]$. Hence any $t,\tau\in[0,T]$ and for any $x\in S$ one has

$$\begin{aligned}|Ax(t)-Ax(\tau)| &= |f(t,x(t))-f(\tau,x(\tau))|\\ &\le |f(t,x(t))-f(\tau,x(t)|+|f(\tau,x(t))-f(\tau,x(\tau))|\\ &\to 0 \quad \text{as} \quad t\to\tau.\end{aligned}$$

This shows that $A(S)$ is an equi-continuous set in X. Now an application of Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[a,b]$.

Step II : Next we show that B is totally bounded operator on $[a,b]$. To finish, we shall show that $B(S)$ is uniformly bounded ad equi-continuous set in X for any subset S of $[a,b]$. Since the cone K in X is normal, the order interval $[a,b]$ is norm-bounded. Hence there is a real number $r>0$ such that $\|x\|\le r$ for all $x\in[a,b]$. Let $y\in B(S)$ be arbitrary. Then,

$$y(t)=\frac{x_0}{f(0,x_0)}+\int_0^t (t-s)g(s,x(s))\,ds$$

for some $x\in S$. By hypothesis (B_2) one has

$$\begin{aligned}|y(t)| &= \left|\frac{x_0}{f(0,x_0)}\right|+T\int_0^t |g(s,x(s))|\,ds\\ &\le \left|\frac{x_0}{f(0,x_0)}\right|+T\int_0^t \ell(s)\,ds\\ &\le \left|\frac{x_0}{f(0,x_0)}\right|+T\|\ell\|_{L^1}.\end{aligned}$$

Taking the supremum over t,

$$\|y\|\le\left|\frac{x_0}{f(0,x_0)}\right|+T\|\ell\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Similarly let $t, \tau \in J$. Then for any $y \in B(S)$,

$$
\begin{aligned}
|y(t) - y(\tau)| &\leq \left| \int_0^t (t-s) g(s, x(s))\, ds - \int_0^\tau (\tau - s) g(s, x(s))\, ds \right| \\
&\leq \left| \int_0^t (t-s) g(s, x(s))\, ds - \int_0^t (\tau - s) g(s, x(s))\, ds \right| \\
&\quad + \left| \int_0^t (\tau-s) g(s, x(s))\, ds - \int_0^\tau (\tau - s) g(s, x(s))\, ds \right| \\
&\leq \left| \int_0^t (t-\tau) g(s, x(s))\, ds \right| + \left| \int_\tau^t (\tau - s) g(s, x(s))\, ds \right| \\
&\leq \int_0^T |t-\tau| |g(s, x(s))|\, ds + T \left| \int_\tau^t |g(s, x(s))|\, ds \right| \\
&\leq \int_0^T |t-\tau| \ell(s)\, ds + T \left| \int_\tau^t \ell(s)\, ds \right| \\
&\leq |t-\tau|\, \|\ell\|_{L^1} + |p(t) - p(\tau)|
\end{aligned}
$$

where $p(t) = T \displaystyle\int_0^t \ell(s)\, ds$. Since the function p is continuous on compact interval J, it is uniformly continuous , and therefore we have

$$
|y(t) - y(\tau)| \to 0 \quad \text{as} \quad t \to \tau,
$$

for all $y \in B(S)$. Hence $B(S)$ is an equi-continuous set in X. Thus B is totally bounded in view of Arzelà-Ascoli theorem.

Thus all the conditions of Theorem 2.3 are satisfied and hence an application of it yields that the IGDE (1.1) has a maximal and a minimal solution on J. $\square$

Theorem 4.3 *Suppose that the assumptions* (B_0),(B_3) *and* (C_1)-(C_4) *hold. Further if* $\|k\|\big(\big|\frac{x_0}{f(0,x_0)}\big| + T\|\ell\|_{L^1}\big) < 1$, *and* ℓ *is given in Remark 4.1, then IGDE (1.1) has a minimal and a maximal positive solution on* J.

Proof. Now IGDE (1.1) is equivalent to IE (3.4) on J. Let $X = C(J, \mathbb{R})$ and define an order relation " $\leq$ " by the cone K given by (2.2). Clearly K is a normal cone in X. Define two operators A and B on X by (3.5) and (3.6) respectively. Then FIE (1.1) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [u, v] \to K$. Since the cone K in X is normal, $[u, v]$ is a norm bounded set in X. Now it can be shown as in the proofs of Theorem 3.1 and Theorem 2.4 that the operator A is a Lipschitz with a Lipschitz constant $\alpha = \|k\|$ and B is totally bounded with $M = \|B([u, v])\| = \big|\frac{x_0}{f(0,x_0)}\big| + T\|\ell\|_{L^1}$. respectively. Since $\alpha M = \|k\|\big(\big|\frac{x_0}{f(0,x_0)}\big| + T\|\ell\|_{L^1}\big) < 1$, the desired conclusion follows by an application of Theorem 2.4. This completes the proof. $\square$

5 An Example

Given the closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the nonlinear IGDE

$$\frac{d}{dt}\Big[\frac{x(t)}{f(t,x(t))}\Big] = \int_0^t \Big(\frac{p(s)\,x(s)}{1+|x(s)|}\Big)\,ds, \quad \text{a.e. } t \in J \tag{5.1}$$

where $p \in L^1(J,\mathbb{R})$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t,x) = \frac{1}{2}\big[1 + \alpha\,|x|\big], \ \alpha > 0$$

for all $t \in J$. Obviously $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$. Define a function $g : J \times \mathbb{R} \to \mathbb{R}$ by

$$g(t,x) = \frac{p(t)x}{1+|x|}.$$

It is easy to verify that f is continuous and Lipschitz on $J \times \mathbb{R}$ with a Lipschitz constant α. Further $g(t,x)$ is L^1_X-Carathéodory with the bound function $h(t) = p(t)$ on J. Therefore if $\alpha(1 + \|p\|_{L^1}) < 1$, then by Theorem 3.1, the IGDE (5.1) has a solution on J, because the function Ω satisfies condition (3.3) with $\gamma(t) = p(t)$ for all $t \in J$ and $\Omega(r) = 1$ for all $r \in \mathbb{R}^+$.

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(Received June 30, 2005)