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COMPARISON OF EIGENVALUES FOR A FOURTH-ORDER FOUR-POINT BOUNDARY VALUE PROBLEM

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Abstract. We establish the existence of a smallest eigenvalue for the fourth-order four-point boundary value problem $(\phi_p(u''(t)))'' = \lambda h(t)u(t)$, $u'(0) = 0$, $\beta_0 u(\eta_0) = u(1)$, $\phi_p'(u''(0)) = 0$, $\beta_1\phi_p(u''(\eta_1)) = \phi_p(u''(1))$, $p > 2$, $0 < \eta_1, \eta_0 < 1, 0 < \beta_1, \beta_0 < 1$, using the theory of u_0-positive operators with respect to a cone in a Banach space. We then obtain a comparison theorem for the smallest positive eigenvalues, λ_1 and λ_2, for the differential equations $(\phi_p(u''(t)))'' = \lambda_1 f(t)u(t)$ and $(\phi_p(u''(t)))'' = \lambda_2 g(t)u(t)$ where $0 \le f(t) \le g(t), t \in [0,1]$.

1. Introduction

In this paper, we will compare the smallest eigenvalues for the eigenvalue problems,

$$(\phi_p(u''(t)))'' = \lambda_1 f(t)u(t), \tag{1.1}$$

$$(\phi_p(u''(t)))'' = \lambda_2 g(t)u(t), \tag{1.2}$$

$t \in [0,1]$, with eigenvectors satisfying the nonlocal boundary conditions,

$$u'(0) = 0,\ \beta_0 u(\eta_0) = u(1), \tag{1.3}$$

$$\phi_p'(u''(0)) = 0,\ \beta_1\phi_p(u''(\eta_1)) = \phi_p(u''(1)). \tag{1.4}$$

Throughout this paper we assume that $0 < \eta_0, \eta_1 < 1$, $0 < \beta_0, \beta_1 < 1$, $p > 2$, $\phi_p(z) = z|z|^{p-2}$, and $f, g : [0,1] \to [0,+\infty)$ are continuous and do not vanish on any nontrivial compact subsets of $[0,1]$.

We use sign properties of Green's functions and the theory of u_0-positive operators with respect to a cone in a Banach space to establish our results. The theory of u_0-positive operators is developed in the books by Krasnosel'skiĭ [9] and Deimling [2] as well as in the manuscript by Keener and Travis [8]. Many authors have used cone theoretic techniques to compare smallest eigenvalues for a pair of differential equations; see, for example [1, 3, 4, 5, 6, 7, 8, 10] and references therein. In particular, Eloe and Henderson [3] compared smallest eigenvalues for a class of multi-point boundary value problems while Karna [5, 6] considered the comparison of smallest eigenvalues for nonlocal three-point and m-point boundary value problems. Finally, we mention the paper by Lui and Ge [12] who considered the p-Laplacian differential equation

$$(\phi_p(u''(t)))'' = a(t)f(u(t)), \quad t \in (0,1),$$

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with solutions satisfying one of the following two sets of boundary conditions

$$u(0) - \lambda u'(\eta) = u'(1) = 0,\ u'''(0) = \alpha_1 u'''(\xi),\ u''(1) = \beta_1 u''(\xi),$$

or

$$u(1) + \lambda u'(\eta) = u'(0) = 0,\ u'''(0) = \alpha_1 u'''(\xi),\ u''(1) = \beta_1 u''(\xi).$$

In section 2, we present preliminary definitions and fundamental results from the theory of u_0-positive operators with respect to a cone in a Banach space. In section 3, we apply the theorems in section 2 to obtain a comparison theorem for the smallest eigenvalues, λ_1 andλ_2 of (1.1), (1.3), (1.4) and (1.2), (1.3), (1.4), when $0 \leq f(t) \leq g(t)$. In section 4, we compare eigenvalues for the $2m+2$ order problem.

2. Banach Spaces, Cones and Preliminary Results

In this section, we state some definitions and theorems from the theory of u_0-positive operators that we will apply in the next sections to obtain our comparison theorems. Most of the discussion of this section, involving the theory of cones in a Banach space, can be found in [9].

Let $\mathcal{B}$ be a Banach space over the reals. A closed, nonempty set $\mathcal{P} \subset \mathcal{B}$ is said to be a *cone* provided, (i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and, (ii) $u, -u \in \mathcal{P}$ implies $u \equiv 0$. A cone, $\mathcal{P}$, is said to be *reproducing*, if, for each $w \in \mathcal{B}$, there exists $u, v \in \mathcal{P}$ such that $w = u - v$. A cone, $\mathcal{P}$, is said to be *solid*, if $\mathcal{P}^\circ \neq \emptyset$, where $\mathcal{P}^\circ$ is the interior of $\mathcal{P}$.

Remark: Krasnosel'skiĭ [9] proved that every solid cone is reproducing.

A Banach space $\mathcal{B}$ is called a *partially ordered Banach space*, if there exists a partial ordering, $\preceq$, on $\mathcal{B}$ such that, (i) $u \preceq v$, for all $u, v \in \mathcal{B}$, implies $tu \preceq tv$, for all $t \geq 0$, and $tv \preceq tu$, for all $t < 0$, where $tv \prec tu$ means $tv \preceq tu$ and, $tv \neq tu$, and (ii) $u_1 \preceq v_1$ and $u_2 \preceq v_2$, for all $u_1, u_2, v_1, v_2 \in \mathcal{B}$, imply that $u_1 + u_2 \preceq v_1 + v_2$.

Let $\mathcal{P} \subset \mathcal{B}$ be a cone and define $u \preceq v$ if, and only if, $v - u \in \mathcal{P}$. Then $\preceq$ is a partial ordering on $\mathcal{B}$, and we say that $\preceq$ is the partial ordering induced by $\mathcal{P}$. Moreover, $\mathcal{B}$ is a partially ordered Banach space with respect to $\preceq$.

Let $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators. We say that $M \preceq N$ *with respect to* $\mathcal{P}$, if $Mu \preceq Nu$ for all $u \in \mathcal{P}$. A bounded, linear operator $M : \mathcal{B} \to \mathcal{B}$, is said to be u_0-*positive with respect to* $\mathcal{P}$, if there exists a $u_0 \in \mathcal{P}, u_0 \neq 0$, such that for every nonzero $u \in \mathcal{P}$, there exist positive constants, $k_1(u), k_2(u) \in \mathbb{R}$, such that $k_1 u_0 \preceq Mu \preceq k_2 u_0$.

Of the next two results, the first can be found in Krasnosel'skiĭ [9] and the second was proved by Keener and Travis [8] as an extension of results from [9].

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space over the reals and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $M : \mathcal{B} \to \mathcal{B}$ be a compact, linear operator which is u_0-positive with respect to $\mathcal{P}$. Then M has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.2. *Let $\mathcal{B}$ be a Banach space over the reals and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Let $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators, and assume that at least one of the operators is u_0-positive with respect to $\mathcal{P}$. If $M \preceq N$ with respect to $\mathcal{P}$, and if there exists nonzero $u_1, u_2 \in \mathcal{P}$ and positive real numbers λ_1 and λ_2, such that*

$\lambda_1 u_1 \preceq Mu_1$ and $Nu_2 \preceq \lambda_2 u_2$, then $\lambda_1 \le \lambda_2$. Moreover, if $\lambda_1 = \lambda_2$, then u_1 is a scalar multiple of u_2.

Remark: It is well known that the function ϕ_p is invertible and that its inverse is ϕ_q where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore both ϕ_p and ϕ_q are increasing function.

3. Comparison of Eigenvalues

In this section, we apply the results in section 2 to compare the smallest positive eigenvalues of (1.1), (1.3), (1.4) and (1.2), (1.3), (1.4). We do so by defining Hammerstien integral operators, M and N, associated with (1.1), (1.3), (1.4) and (1.2), (1.3), (1.4). Let $\alpha \neq 1$ and consider the second order linear eigenvalue problem,

$$-y'' = \lambda h(t), \tag{3.1}$$

$$y'(0) = 0,\ \alpha y(\xi) = y(1). \tag{3.2}$$

It is well known that y is a solution of (3.1), (3.2) if, and only if, y is a solution of

$$y(t) = \lambda \int_0^1 G(t,s;\alpha,\xi) h(s)\, ds, \tag{3.3}$$

where $G(t,s;\alpha,\xi)$ is the Green's function for $-y''=0$, (3.2) and is given by

$$G(t,s;\alpha,\xi) = \frac{1-s}{1-\alpha} - \begin{cases} \frac{\alpha(\xi-s)}{1-\alpha}, & s \le \xi \\ 0, & s > \xi \end{cases} - \begin{cases} t-s, & s \le t \\ 0, & s > t. \end{cases} \tag{3.4}$$

Note that if $0 \le \alpha < 1$ then

$$G(t,s;\alpha,\xi) > 0$$

for all $(t,s) \in (0,1) \times (0,1)$. As noted in Karna [5],

$$\frac{\partial}{\partial t} G(t,s;\alpha,\xi) = -1 < 0 \text{ for } s < t, \text{ and}$$
$$\frac{\partial}{\partial t} G(t,s;\alpha,\xi) = 0 \text{ for } s > t.$$

Let $y = -\phi_p(u''(t))$ in (3.3) and (3.2), and set $\alpha = \beta_1, \xi = \eta_1$ to obtain,

$$-\phi_p(u''(t)) = \lambda \int_0^1 G(t,s;\beta_1,\eta_1) h(s)\, ds,$$
$$\phi_p'(u''(0)) = 0,\ \beta_1 \phi_p(u''(\eta_1)) = \phi_p(u''(1)).$$

Rewrite the differential equation as

$$-u''(t) = \phi_q\left(\lambda \int_0^1 G(t,s;\beta_1,\eta_1) h(s)\, ds\right). \tag{3.5}$$

We now consider the second order linear boundary value problem (3.5), (1.3). Again, we see that u is a solution of (3.5), (1.3) if, and only if, u satisfies

$$u(t) = \phi_q(\lambda) \int_0^1 G(t,s;\beta_0,\eta_0)\, \phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1) h(s)\, d\tau\right) ds.$$

We define, for our Banach space,

$$\mathcal{B} = \{u \in C^3[0,1] : u \text{ satisfies the boundary conditions (1.3), (1.4) }\}$$

with norm

$$\|u\| = \max_{0\le i\le 3}\left\{\sup_{t\in[0,1]} |u^{(i)}(t)|\right\}.$$

Define $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) \ge 0 \text{ and } u'(t) \le 0 \text{ for } t \in [0,1]\}.$$

Then $\mathcal{P}$ is a cone in $\mathcal{B}$. To prove that $\mathcal{P}$ is solid we employ an auxiliary set, Θ, defined as follows,

$$\Theta = \{u \in \mathcal{B} : u(t) > 0 \text{ for } t \in [0,1] \text{ and } u'(t) < 0 \text{ for } t \in (0,1]\}.$$

Lemma 3.1. *The cone $\mathcal{P}$ is solid and hence reproducing.*

Proof. We show that $\Theta \subset \mathcal{P}^\circ$ from which we have $\mathcal{P}^\circ \ne \emptyset$.

Clearly, $\Theta \subset \mathcal{P}$. Let $u \in \Theta$. Then $u(t) > 0$ on $[0,1]$ and $u'(t) < 0$ on $(0,1]$. Consider the open ball $B_\varepsilon = \{v \in \mathcal{B} : \|v-u\| < \varepsilon\}$. Let $v \in B_\varepsilon$. Since $\|v-u\| < \varepsilon$ then $|v'(t) - u'(t)| < \varepsilon$ for all $t \in (0,1]$. Hence $u'(t) - \varepsilon < v'(t) < u'(t) + \varepsilon < 0$ for ε sufficiently small. Likewise $\|v-u\| < \varepsilon$ implies $u(t) + \varepsilon > v(t) > u(t) - \varepsilon > 0$ for ε sufficiently small. Consequently, for ε sufficiently small $B_\varepsilon \subset \Theta$. Since $u \in \Theta$ was arbitrary, Θ is open in $\mathcal{P}$. Hence $\mathcal{P}^\circ \ne \emptyset$ and the proof is complete. □

Define the integral operators $M, N : \mathcal{B} \to \mathcal{B}$ as follows,

$$Mu(t) = \int_0^1 G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1) f(\tau)u(\tau)\,d\tau\right)ds, \quad t \in [0,1],$$

$$Nu(t) = \int_0^1 G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1) g(\tau)u(\tau)\,d\tau\right)ds, \quad t \in [0,1].$$

Standard arguments are used to show that M and N are completely continuous. Our first theorem states that the operators M and N are u_0-positive with respect to $\mathcal{P}$.

Theorem 3.2. *The operators M and N are u_0-positive with respect to the cone $\mathcal{P}$.*

Proof. We will prove the theorem for the operator M. The proof for the operator N is similar. We first show that $M : \mathcal{P} \to \mathcal{P}$. Next we show that $M : \mathcal{P} \setminus \{0\} \to \Theta$. Finally, given a $u \in \mathcal{P}\setminus\{0\}$, we determine constants k_1, k_2 such that the appropriate inequalities hold.

Let $u \in \mathcal{P}$. Then $u(t) \ge 0$ and $u'(t) \le 0$ for all $t \in [0,1]$. Since $f(t) \ge 0$ and since $G(t,s;\alpha,\xi) \ge 0$ for all $(t,s) \in [0,1]\times[0,1]$, then $Mu(t) \ge 0$ for all $t \in [0,1]$. Also, since $\frac{\partial}{\partial t}G(t,s;\beta_0,\eta_0) = -1$ for $s < t$, then

$$\int_0^t \frac{\partial}{\partial t} G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1) f(\tau)u(\tau)d\tau\right)ds$$
$$= -\int_0^t \phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1) f(\tau)u(\tau)\,d\tau\right)ds \le 0.$$

Since $\frac{\partial}{\partial t}G(t,s;\beta_0,\eta_0) = 0$ for $s > t$, then

$$\int_t^1 \frac{\partial}{\partial t}G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1)f(\tau)u(\tau)\,d\tau\right)ds = 0.$$

Consequently, $(Mu)'(t) \le 0$ for all $t \in [0,1]$ and so, $M(\mathcal{P}) \subseteq \mathcal{P}$.

Now consider $u \in \mathcal{P}\setminus\{0\}$. Since $u'(t) \le 0$ for all $t \in [0,1]$ then u is non-increasing over $[0,1]$. Suppose that $u(0) = 0$ then either $u \equiv 0$ or $u(t) \le 0$ for all $t \in [0,1]$. In either case, $u \notin \mathcal{P}\setminus\{0\}$. Hence $u(0) > 0$. By continuity, there exists $c \in (0,1]$ such that $u(t) > 0$ for all $t \in [0,c)$. Since f does not vanish on any nontrivial compact subsets of $[0,1]$, there exists $[\alpha,\beta] \subset [0,c)$ such that $f(t) > 0$ for all $t \in [\alpha,\beta]$. So, for $t \in [0,1]$

$$\begin{aligned}
Mu(t) &= \int_0^1 G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1)f(\tau)u(\tau)\,d\tau\right)ds \\
&\ge \int_\alpha^\beta G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_\alpha^\beta G(s,\tau;\beta_1,\eta_1)f(\tau)u(\tau)\,d\tau\right)ds \\
&> 0.
\end{aligned}$$

Also, if $t \in (0,1]$ then

$$\begin{aligned}
(Mu)'(t) &= \int_0^1 \frac{\partial}{\partial t}G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1)f(\tau)u(\tau)\,d\tau\right)ds \\
&\le -\int_0^t \phi_q\left(\int_\alpha^\beta G(s,\tau;\beta_1,\eta_1)f(\tau)u(\tau)\,d\tau\right)ds \\
&< 0.
\end{aligned}$$

Consequently, if $u \in \mathcal{P}\setminus\{0\}$, then $Mu \in \Theta \subset \mathcal{P}^\circ$. That is, $M : \mathcal{P}\setminus\{0\} \to \mathcal{P}^\circ$.

To complete the proof, fix $u_0 \in \mathcal{P}\setminus\{0\}$ and let $u \in \mathcal{P}\setminus\{0\}$. From the above we know that $Mu \in \Theta \subset \mathcal{P}^\circ$ and so, there exists k_1 sufficiently small so that $Mu - k_1u_0 \in \mathcal{P}$. Similarly, there exists k_2 sufficiently large so that $u_0 - \frac{1}{k_2}Mu \in \mathcal{P}$. Thus,

$$\begin{aligned}
Mu - k_1u_0 \in \mathcal{P} &\Rightarrow k_1u_0 \preceq Mu, \\
u_0 - \frac{1}{k_2}Mu \in \mathcal{P} &\Rightarrow Mu \preceq k_2u_0.
\end{aligned}$$

That is, given $u_0 \in \mathcal{P}\setminus\{0\}$, for each $u \in \mathcal{P}\setminus\{0\}$, there exists k_1, k_2 such that

$$k_1u_0 \preceq Mu \preceq k_2u_0.$$

The operator M is u_0-positive with respect to the cone $\mathcal{P}$ and the proof is complete. □

Now we apply Theorems 2.1 and 2.2 to obtain results concerning the eigenvectors and eigenvalues of M and N.

Theorem 3.3. *The operator M (N) has an essentially unique eigenvector, $u \in \mathcal{P}^\circ$, and the corresponding eigenvalue, $\Lambda_1, (\Lambda_2)$, is simple, positive, and larger than the absolute value of any other eigenvalue.*

Proof. From Theorem 3.2, we have that the compact, linear operator M is u_0-positive with respect to $\mathcal{P}$. By Theorem 2.1, M has an essentially unique eigenvector, $u_1 \in \mathcal{P}$, and the corresponding eigenvalue, Λ_1 is simple, positive, and larger than the absolute value of any other eigenvalue. Since $u_1 \neq 0$ then, $Mu_1 \in \Theta \subset \mathcal{P}^\circ$. Now $Mu_1 = \Lambda_1 u_1$ and so $u_1 = \frac{1}{\Lambda_1} Mu_1 \in \mathcal{P}^\circ$ and the proof is complete. $\square$

Theorem 3.4. *Assume that $0 \leq f(t) \leq g(t)$ for all $t \in [0,1]$. Let Λ_1 and Λ_2 be the largest positive eigenvalues of M and N respectively with corresponding essentially unique eigenvectors u_1 and u_2. Then $\Lambda_1 \leq \Lambda_2$. Furthermore, $\Lambda_1 = \Lambda_2$, if, and only if, $f(t) = g(t)$ for all $t \in [0,1]$.*

Proof. Since $0 \leq f(t) \leq g(t)$ for all $t \in [0,1]$ and since $G(t,s;\alpha,\xi) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$, then

$$\begin{aligned}(Nu - Mu)(t) &= \int_0^1 G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1)\big(g(\tau) - f(\tau)\big)u(\tau)\,d\tau\right)ds\\ &\geq 0.\end{aligned}$$

Since $\frac{\partial}{\partial t}G(t,s;\alpha,\xi) = -1$ if $s < t$ and $\frac{\partial}{\partial t}G(t,s;\alpha,\xi) = 0$ if $s > t$, then

$$\begin{aligned}&(Nu - Mu)'(t)\\ &\quad= \int_0^1 \frac{\partial}{\partial t}G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1)\big(g(\tau) - f(\tau)\big)u(\tau)\,d\tau\right)ds\\ &\quad= -\int_0^t \phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1)\big(g(\tau) - f(\tau)\big)u(\tau)\,d\tau\right)ds\\ &\quad\leq 0.\end{aligned}$$

Thus, $Nu - Mu \in \mathcal{P}$ and so, $M \preceq N$ with respect to $\mathcal{P}$. From Theorem 2.2, if u_1 and u_2 are eigenvectors of M and N respectively, with corresponding eigenvalues Λ_1 and Λ_2 then $\Lambda_1 \leq \Lambda_2$.

Note if $f(t) = g(t)$ then by Theorem 2.2 $\Lambda_1 \leq \Lambda_2$ and $\Lambda_1 \geq \Lambda_2$. In this case $\Lambda_1 = \Lambda_2$.

To finish the proof, we need to show that $\Lambda_1 = \Lambda_2$ implies $f(t) = g(t)$ for all $t \in [0,1]$. Suppose that $f(t) < g(t)$ on some interval $[\alpha,\beta] \subset [0,1]$. As in Theorem 3.2, $(N - M)u_1 \in \Theta \subseteq \mathcal{P}^\circ$. Since $u_1 \in \mathcal{P}^\circ$, there exists an $\varepsilon > 0$ sufficiently small so that $\varepsilon u_1 \preceq (N - M)u_1 = Nu_1 - Mu_1 = Nu_1 - \Lambda u_1$. Thus $(\Lambda_1 + \varepsilon)u_1 \preceq Nu_1$. Since $N \preceq N, (\Lambda_1 + \varepsilon)u_1 \preceq Nu_1$, and $Nu_2 \preceq \Lambda_2 u_2$, then by Theorem 2.2 we have $\Lambda_1 + \varepsilon \leq \Lambda_2$. Hence $\Lambda_1 < \Lambda_2$. By the contrapositive, $\Lambda_1 = \Lambda_2$ implies $f(t) = g(t)$ for all $t \in [0,1]$ and the proof is complete. $\square$

Let Λ_1 be an eigenvector of M with corresponding eigenvector u_1 and let $\lambda_1 = \phi_p\left(\frac{1}{\Lambda_1}\right)$. Then for all $t \in [0,1]$,

$$\Lambda_1 u_1(t) = Mu_1(t) = \int_0^1 G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1)f(\tau)u_1(\tau)\,d\tau\right)ds.$$

Since $\frac{1}{\Lambda_1} = \phi_q(\lambda)$, then

$$\begin{aligned} u_1(t) &= \frac{1}{\Lambda_1} \int_0^1 G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1) f(\tau) u_1(\tau)\,d\tau\right) ds \\ &= \phi_q(\lambda_1) \int_0^1 G(t,s;\beta_0,\eta_0)\,\phi_q\left(\int_0^1 G(s,\tau;\beta_1,\eta_1) f(\tau) u_1(\tau)\,d\tau\right) ds. \end{aligned}$$

That is, λ_1 is an eigenvalue corresponding to (1.1), (1.3), (1.4). The converse also holds. Thus, $\Lambda_1(\Lambda_2)$ is an eigenvalue of $M(N)$ if, and only if $\lambda_1 = \phi\left(\frac{1}{\Lambda_1}\right)$ $\left(\lambda_2 = \phi\left(\frac{1}{\Lambda_2}\right)\right)$ is an eigenvalue of (1.1), (1.3), (1.4), $\big($(1.2), (1.3), (1.4)$\big)$. Furthermore, since ϕ_p is an increasing one-to-one function, then $\frac{1}{\phi(\lambda_1)} = \Lambda_1 \le \Lambda_2 = \frac{1}{\phi(\lambda_2)}$ implies that $\lambda_2 \le \lambda_1$ and $\Lambda_1 = \Lambda_2$ if, and only if $\lambda_1 = \lambda_2$.

Theorem 3.5. *Let $0 \le f(t) \le g(t)$ for all $t \in [0,1]$. Then there exist smallest positive eigenvalues of* (1.1), (1.3), (1.4) *and* (1.2), (1.3), (1.4), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, the corresponding essentially unique eigenvectors may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_2 \le \lambda_1$, and $\lambda_1 = \lambda_2$ if and only if $f(t) = g(t), 0 \le t \le 1$.*

4. The $2m+2$ Order Problem

Let $m > 1$ be a fixed integer. Define $L_0 u(t) \equiv u''(t)$ and for $k = 1, 2, \ldots, m$

$$L_k u(t) \equiv \phi_{p_k}\Big(\big(L_{k-1}u\big)''(t)\Big).$$

In this section we compare eigenvalues for the $2m+2$ order problems,

$$(-1)^{m+1}\big(L_m u\big)''(t) = \lambda_1 f(t) u(t), \tag{4.1}$$

$$(-1)^{m+1}\big(L_m u\big)''(t) = \lambda_2 g(t) u(t), \tag{4.2}$$

$t \in [0,1]$, with eigenvectors satisfying the nonlocal boundary conditions,

$$u'(0) = 0,\ \big(L_k u\big)'(0) = 0, \tag{4.3}$$

$$\beta_0 u(\eta_0) = u(1),\ \beta_k L_k u(\eta_k) = L_k u(1), \tag{4.4}$$

$k = 1, \ldots, m$, where $0 < \eta_i < 1$, $0 < \beta_i < 1$, and $p_i > 2$ for $1 \le i \le m$.

For each $k = 1, 2, \ldots, m$ define the operator $\mathbb{G}_k$ by

$$\mathbb{G}_k h(s_{k+1}) \equiv \phi_{q_{m+1-k}}\left(\int_0^1 G(s_{k+1}, s_k; \beta_{m+1-k}, \eta_{m+1-k}) h(s_k)\,ds_k\right),$$

where $G(t,s;\alpha,\xi)$ is defined in (3.4). Using the technique outlined in the beginning of section 3, we see that $u(t)$ is a solution of $(-1)^{m+1}\left(L_m u\right)''(t) = \lambda h(t)$, (4.3), (4.4), if, and only if,

$$\begin{aligned} u(t) =& \phi_{q_1}\big(\phi_{q_2}\big(\cdots\big(\phi_{q_m}(\lambda)\big)\cdots\big)\big) \times \\ & \int_0^1 G(t, s_{m+1}; \beta_0, \eta_0)\left[\big(\mathbb{G}_m \circ \mathbb{G}_{m-1} \circ \cdots \circ \mathbb{G}_1\big)(h)\right](s_{m+1})\,ds_{m+1}. \end{aligned} \tag{4.5}$$

Our Banach space is

$$\mathcal{B} = \{u \in C^{2m+1}[0,1] : u \text{ satisfies the boundary conditions (4.3), (4.4) }\}$$

with norm

$$\|u\| = \max\left\{|u(t)|_0, |u'(t)|_0, \left|L_k u(t)\right|_0, \left|\left(L_k u\right)'(t)\right|_0, k = 1, 2, \ldots, m\right\},$$

where $|z|_0 = \sup_{t\in[0,1]} |z(t)|$. Define the cone $\mathcal{P}_2 \subset \mathcal{B}$ by

$$\mathcal{P}_2 = \Big\{u \in \mathcal{B} : u(t) \geq 0,\ (-1)u'(t) \geq 0,\ (-1)^k L_k u(t) \geq 0, k = 1, 2, \ldots, m,$$
$$\text{and } (-1)^{k+1}\left(L_k u\right)'(t) \geq 0, k = 1, 2, \ldots, m, \text{ for } t \in [0,1]\Big\}$$

and the auxiliary set, Θ_2, as follows,

$$\Theta_2 = \Big\{u \in \mathcal{B} : u(t) > 0,\ (-1)^k L_k u(t) > 0, k = 1, 2, \ldots, m, \text{ for } t \in [0,1]$$
$$\text{and } -u'(t) > 0,\ (-1)^{k+1}\left(L_k u\right)'(t) > 0, k = 1, 2, \ldots, m, \text{ for } t \in (0,1]\Big\}.$$

A modification of the proof of Lemma 3.1 yields that $\Theta_2 \subset \mathcal{P}_2^\circ$. Hence the cone $\mathcal{P}_2$ is solid and reproducing. We define operators $\mathcal{M}, \mathcal{N} : \mathcal{B} \to \mathcal{B}$ as follows,

$$\mathcal{M}u(t) = \int_0^1 G(t, s_{m+1}; \beta_0, \eta_0) \left[\left(\mathbb{G}_m \circ \mathbb{G}_{m-1} \circ \cdots \circ \mathbb{G}_1\right)(f)\right](s_{m+1})\, ds_{m+1},$$
$$\mathcal{N}u(t) = \int_0^1 G(t, s_{m+1}; \beta_0, \eta_0) \left[\left(\mathbb{G}_m \circ \mathbb{G}_{m-1} \circ \cdots \circ \mathbb{G}_1\right)(g)\right](s_{m+1})\, ds_{m+1},$$

for all $t \in [0,1]$. The proofs of the following theorems are similar to those of their counterparts in section 3 and are omitted.

Theorem 4.1. *The operators $\mathcal{M}, \mathcal{N} : \mathcal{P}_2 \to \mathcal{P}_2$ are completely continuous and u_0-positive with respect to the cone $\mathcal{P}_2$.*

Theorem 4.2. *The operator $\mathcal{M}$ ($\mathcal{N}$) has an essentially unique eigenvector, $u \in \mathcal{P}_2^\circ$, and the corresponding eigenvalue, $\Lambda_1, (\Lambda_2)$, is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 4.3. *Assume that $0 \leq f(t) \leq g(t)$ for all $t \in [0,1]$. Let Λ_1 and Λ_2 be the largest positive eigenvalues of $\mathcal{M}$ and $\mathcal{N}$ respectively with corresponding essentially unique eigenvectors u_1 and u_2. Then $\Lambda_1 \leq \Lambda_2$. Furthermore, $\Lambda_1 = \Lambda_2$, if, and only if, $f(t) = g(t)$ for all $t \in [0,1]$.*

Let Λ_1 be an eigenvector of M with corresponding eigenvector u_1 and let $\lambda_1 = \phi_{p_m}\left(\cdots \phi_{p_1}\left(\frac{1}{\Lambda_1}\right)\cdots\right)$. Then λ_1 is an eigenvalue corresponding to (4.1), (4.3), (4.4). The converse also holds. Thus, $\Lambda_1(\Lambda_2)$ is an eigenvalue of $\mathcal{M}$ ($\mathcal{N}$) if, and only if $\lambda_1 = \phi_{p_m}\left(\cdots \phi_{p_1}\left(\frac{1}{\Lambda_1}\right)\cdots\right)$ $\left(\phi_{p_m}\left(\cdots \phi_{p_1}\left(\frac{1}{\Lambda_2}\right)\cdots\right)\right)$ is an eigenvalue of (4.1), (4.3), (4.4), $\big($(4.2), (4.3), (4.4)$\big)$. Furthermore, since each $\phi_{p_k}, k = 1, \ldots, m$, is an increasing one-to-one function, and since $\Lambda_1 \leq \Lambda_2$, then $\lambda_2 \leq \lambda_1$ and $\Lambda_1 = \Lambda_2$ if, and only if, $\lambda_1 = \lambda_2$.

Theorem 4.4. *Let* $0 \leq f(t) \leq g(t)$ *for all* $t \in [0,1]$. *Then there exist smallest positive eigenvalues of* (4.1), (4.3), (4.4) *and* (4.2), (4.3), (4.4), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, the corresponding essentially unique eigenvectors may be chosen to belong to* $\mathcal{P}_2^\circ$. *Finally,* $\lambda_2 \leq \lambda_1$, *and* $\lambda_1 = \lambda_2$ *if and only if* $f(t) = g(t), 0 \leq t \leq 1$.

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