Electronic Journal of Qualitative Theory of Differential Equations
2005, No. **14**, 1-23, **http://www.math.u-szeged.hu/ejqtde/**

On the unique continuation property for a nonlinear dispersive system

Alice Kozakevicius* & Octavio Vera †

Abstract

We solve the following problem: If $(u,\, v) = (u(x,\, t),\, v(x,\, t))$ is a solution of the Dispersive Coupled System with $t_1 < t_2$ which are sufficiently smooth and such that: $\operatorname{supp} u(\,.\,,\, t_j) \subset (a,\, b)$ and $\operatorname{supp} v(\,.\,,\, t_j) \subset (a,\, b)$, $-\infty < a < b < \infty$, $j = 1,\, 2$. Then $u \equiv 0$ and $v \equiv 0$.

Keywords and phrases: Dispersive coupled system, evolution equations, unique continuation property.

1 Introduction

This paper is concerned with unique continuation results for some system of nonlinear evolution equation. Indeed, a partial differential equation $\mathcal{L}u = 0$ in some open, connected domain Ω of $\mathbb{R}^n$ is said to have the weak unique continuation property (UCP) if every solution u of $\mathcal{L}u = 0$ (in a suitable function space), which vanishes on some nonempty open subset of Ω vanishes in Ω. We study the UCP of the system of nonlinear evolution equations

$$(1.1)\left\{\begin{array}{ll} \partial_t u + \partial_x^3 u + \partial_x(u\, v^2) = 0 & (P_1) \\ \partial_t v + \partial_x^3 v + \partial_x(u^2\, v) + \partial_x v = 0 & (P_2) \end{array}\right.$$

with $0 \le x \le 1$, $t \ge 0$ and where $u = u(x,\, t)$, $v = v(x,\, t)$ are real-valued functions of the variables x and t. The general system is

$$(1.2)\left\{\begin{array}{l} \partial_t u + \partial_x^3 u + \partial_x(u^p\, v^{p+1}) = 0 \\ \partial_t v + \partial_x^3 v + \partial_x(u^{p+1}\, v^p) = 0 \end{array}\right.$$

with domain $-\infty < x < \infty$, $t \ge 0$ and where $u = u(x,\, t)$, $v = v(x,\, t)$ are real-valued functions of the variables x and t. The power p is an integer greater than or equal to one. This system appears as a special case of a broad class of nonlinear evolution equations studied by Ablowitz *et al.* [1] which can be solved by the inverse scattering method. It has the structure of a pair of Korteweg - de Vries(KdV) equations coupled through both dispersive and nonlinear effects. A system of the form (1.2) is of interest because it models the physical problem of describing the strong interaction of two-dimensional long internal gravity waves propagating on neighboring pynoclines in a stratified fluid, as in the derived model by Gear and Grimshaw [8]. Indeed,

$$(1.3)\left\{\begin{array}{l} \partial_t u + \partial_x^3 u + a_3\, \partial_x^3 v + u\, \partial_x u + a_1\, v\, \partial_x v + a_2\, \partial_x(u\, v) = 0 \quad \text{in} \;\; x \in \mathbb{R},\; t \ge 0 \\ b_1\, \partial_t v + \partial_x^3 v + b_2\, a_3\, \partial_x^3 u + v\, \partial_x v + b_2\, a_2\, u\, \partial_x u + b_2\, a_1\, \partial_x(u\, v) = 0 \\ u(x,\, 0) = \varphi(x) \\ v(x,\, 0) = \psi(x) \end{array}\right.$$

where $u = u(x,\, t)$, $v = v(x,\, t)$ are real-valued functions of the variables x and t and a_1, a_2, a_3, b_1, b_2 are real constants with $b_1 > 0$ and $b_2 > 0$. Mathematical results on (1.3) were given by J. Bona *et al.* [4].

*Departamento de Matemática, CCNE, Universidade Federal de Santa Maria, Faixa de Camobi, Km 9, Santa Maria, RS, Brasil, CEP 97105-900. E-mail: alicek@smail.ufsm.br: This research was partially supported by CONICYT-Chile through the FONDAP Program in Applied Mathematics (Project No. 15000001).

†Departamento de Matemática, Universidad del Bío-Bío, Collao 1202, Casilla 5-C, Concepción, Chile. E-mail: overa@ubiobio.cl ; octavipaulov@yahoo.com

They proved that the coupled system is globally well posed in $H^m(\mathbb{R}) \times H^m(\mathbb{R})$, for any $m \geq 1$ provided $|a_3| < 1/\sqrt{b_2}$. Recently, this result was improved by F. Linares and M. Panthee [19]. Indeed, they proved the following:

Theorem 1.1. *For any* $(\varphi, \psi) \in H^m(\mathbb{R}) \times H^m(\mathbb{R})$, *with* $m \geq -3/4$ *and any* $b \in (1/2, 1)$, *there exist* $T = T(||\varphi||_{H^m}, ||\psi||_{H^m})$ *and a unique solution of* (1.3) *in the time interval* $[-T, T]$ *satisfying*

$$\begin{gathered} u, v \in C([-T, T]; H^m(\mathbb{R})), \\ u, v \in X_{m, b} \subset L^p_{x\, Loc}(\mathbb{R}; L^2_t(\mathbb{R})) \quad \text{for} \quad 1 \leq p \leq \infty, \\ (u^2)_x, (v^2)_x \in X_{m, b-1}, \\ u_t, v_t \in X_{m-3, b-1} \end{gathered}$$

Moreover, given $t \in (0, T)$, *the map* $(\varphi, \psi) \mapsto (u(t), v(t))$ *is smooth from* $H^m(\mathbb{R}) \times H^m(\mathbb{R})$ *to* $C([-T, T]; H^m(\mathbb{R})) \times C([-T, T]; H^m(\mathbb{R}))$.

Similar results in weighted Sobolev spaces were given by [29, 30] and references therein. In 1999, Alarcón-Angulo-Montenegro [2] showed that the system (1.2) is global well-posedness in the classical Sobolev space $H^m(\mathbb{R}) \times H^m(\mathbb{R})$, $m \geq 1$. For the UCP the first results are due to Saut and Scheurer [24]. They considered some dispersive operators in one space dimension of the type $L = i\, D_t + \alpha\, i^{2k+1}\, D^{2k+1} + R(x, t, D)$ where $\alpha \neq 0$, $D = \frac{1}{i}\frac{\partial}{\partial x}$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ and $R(x, t, D) = \sum_{j=0}^{2k} r_j(x, t)\, D^j$, $r_j \in L^\infty_{loc}(\mathbb{R}; L^2_{loc}(\mathbb{R}))$. They proved that if $u \in L^2_{loc}(\mathbb{R}; H^{2k+1}_{loc}(\mathbb{R}))$ is a solution of $Lu = 0$, which vanishes in some open set Ω_1 of $\mathbb{R}_x \times \mathbb{R}_t$, then u vanishes in the horizontal component of Ω_1. As a consequence of the uniqueness of the solutions of the KdV equation in $L^\infty_{loc}(\mathbb{R}; H^3(\mathbb{R}))$, their result immediately yields the following:

Theorem 1.2. *If* $u \in L^\infty_{loc}(\mathbb{R}; H^3(\mathbb{R}))$ *is a solution of the KdV equation*

$$u_t + u_{xxx} + u\, u_x = 0 \qquad (1.4)$$

and vanishes on an open set of $\mathbb{R}_x \times \mathbb{R}_t$, *then* $u(x, t) = 0$ *for* $x \in \mathbb{R}$, $t \in \mathbb{R}$.

In 1992, B. Zhang [32] proved using inverse scattering transform and some results from Hardy function theory that if $u \in L^\infty_{Loc}(\mathbb{R}; H^m(\mathbb{R}))$, $m > 3/2$ is a solution of the KdV equation (1.4), then it cannot have compact support at two different moments unless it vanishes identically. As a consequence of the Miura transformation, the above results for the KdV equation (1.4) are also true for the modified Korteweg-de Vries equation

$$u_t + u_{xxx} - u^2\, u_x = 0. \qquad (1.5)$$

A variety of techniques such as spherical harmonics [26], singular integral operators [20], inverse scattering [31], and others have been used. However the Carleman methods which consists in establishing a priori estimates containing a weight has influenced a lot the development on the subject.
This paper is organized as follows: In section 2, we prove two conserved integral quantities and local existence theorem. In section 3, we prove the Carleman estimate and Unique Continuation Property. In section 4, we prove the main theorem.

2 Preliminaries

We consider the following dispersive coupled system

$$(P) \begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u\, v^2) = 0 & (P_1) \\ \partial_t v + \partial_x^3 v + \partial_x(u^2\, v) + \partial_x v = 0 & (P_2) \\ u(x, 0) = u_0(x) \quad ; \quad v(x, 0) = v_0(x) & (P_3) \\ \partial_x^k u(0, t) = \partial_x^k u(1, t), \quad k = 0, 1, 2. & (P_4) \\ \partial_x^k v(0, t) = \partial_x^k v(1, t), \quad k = 0, 1, 2. & (P_5) \end{cases}$$

with $0 \leq x \leq 1$, $t \geq 0$ and where $u = u(x,\, t)$, $v = v(x,\, t)$ are real-valued functions of the variables x and t.

Notation. We write time derivative by $u_t = \frac{\partial u}{\partial t} = \partial_t u$. Spatial derivatives are denoted by $u_x = \frac{\partial u}{\partial x} = \partial_x u$, $u_{xx} = \frac{\partial^2 u}{\partial x^2} = \partial_x^2 u$, $u_{xxx} = \frac{\partial^3 u}{\partial x^3} = \partial_x^3 u$.

If $\mathbb{E}$ is any Banach space, its norm is written as $|| \cdot ||_{\mathbb{E}}$. For $1 \leq p \leq +\infty$, the usual class of p^{th}-power Lebesgue-integrable (essentially bounded if $p = +\infty$) real-valued functions defined on the open set Ω in $\mathbb{R}^n$ is written by $L^p(\Omega)$ and its norm is abbreviated as $|| \cdot ||_p$. The Sobolev space of L^2-functions whose derivatives up to order m also lie in L^2 is denoted by H^m. We denote $[H^{m_1}(\Omega),\, H^{m_2}(\Omega)] = H^{(1-\theta)m_1+\theta m_2}(\Omega)$ for all $m_i > 0 (i = 1,\, 2)$, $m_2 < m_1$, $0 < \theta < 1$ (with equivalent norms) the interpolation of $H^m(\Omega)$-spaces. If a function belongs, locally, to L^p or H^m we write $f \in L^p_{loc}$ or $f \in H^m_{loc}$. $C(0,\, T;\, \mathbb{E})$ denote the class of all continuous maps $u : [0,\, T] \to \mathbb{E}$ equipped with the norm $||u||_{C(0,\, T;\, \mathbb{E})} = \sup_{0 \leq t \leq T} ||u||_{\mathbb{E}}$. $u(x,\, t) \in C^{3,\,1}(\mathbb{R}^2)$ if $\partial_x u$, $\partial_x^2 u$, $\partial_x^3 u$, $\partial_t u \in C(\mathbb{R}^2)$. $u(x,\, t) \in C_0^{3,\,1}(\mathbb{R}^2)$ if $u \in C^{3,\,1}(\mathbb{R}^2)$ and u with compact support.
Throughout this paper c is a generic constant, not necessarily the same at each occasion(will change from line to line), which depends in an increasing way on the indicated quantities. The next proposition is well known and it will be used frequently

Proposition 2.1. *Let K be a non empty compact set and F a close subset of $\mathbb{R}$ such that $K \cap F = \emptyset$. Then there is $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ in K, $\psi = 0$ in F and $0 \leq \psi(x) \leq 1$, $\forall\, x \in \mathbb{R}$.*

Definition 2.2. *Let $\mathcal{L}$ be an evolution operator acting on functions defined on some connected open set Ω of $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_t$. $\mathcal{L}$ is said to have the horizontal unique continuation property if every solution u of $\mathcal{L}u = 0$ that vanishes on some nonempty open set $\Omega_1 \subset \Omega$ vanishes in the horizontal component of Ω_1 in Ω, i. e., in $\Omega_h = \{(x,\, t) \in \Omega /\, \exists\, x_1,\, (x_1,\, t) \in \Omega_1\}$.*

Lemma 2.3. *The equation (P) has the following conserved integral quantities, i. e.,*

$$\frac{d}{dt}\int_0^1 (u^2 + v^2)\, dx = 0, \tag{2.1}$$

$$\frac{d}{dt}\int_0^1 \left[u^2\, v^2 - (u_x^2 + v_x^2) + v^2\right]\, dx = 0. \tag{2.2}$$

Proof. (2.1) Straightforward. We show (2.2). Multiplying (P_1) by $(u\, v^2 + u_{xx})$ and integrating over $x \in (0,\, 1)$ we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 (u^2)_t\, v^2\, dx + \int_0^1 u_{xx}\, u_t\, dx + \int_0^1 (u\, v^2)\, u_{xxx}\, dx \\ &+\frac{1}{2}\int_0^1 (u_{xx}^2)_x\, dx + \frac{1}{2}\int_0^1 [(u\, v^2)^2]_x\, dx + \int_0^1 (u\, v^2)_x\, u_{xx}\, dx = 0.\end{aligned}$$

Each term is treated separately integrating by parts

$$\begin{aligned}&\frac{1}{2}\int_0^1 (u^2)_t\, v^2\, dx - \int_0^1 u_x\, u_{xt}\, dx - \int_0^1 (u\, v^2)_x\, u_{xx}\, dx \\ &+\frac{1}{2}\int_0^1 (u_{xx}^2)_x\, dx + \frac{1}{2}\int_0^1 [(u\, v^2)^2]_x\, dx + \int_0^1 (u\, v^2)_x\, u_{xx}\, dx = 0\end{aligned}$$

where

$$\frac{1}{2}\int_0^1 (u^2)_t v^2 dx - \frac{1}{2}\int_0^1 (u_x^2)_t dx = 0. \tag{2.3}$$

Similarly, multiplying (P_2) by $(u^2\, v + v_{xx} + v)$ and integrating over $x \in (0,\, 1)$ we have

$$\frac{1}{2}\int_0^1 u^2\,(v^2)_t\,dx + \int_0^1 v_{xx}\,v_t\,dx + \frac{1}{2}\int_0^1 (v^2)_t\,dx + \int_0^1 (u^2\,v)\,v_{xxx}\,dx$$
$$+\frac{1}{2}\int_0^1 (v_{xx}^2)_x\,dx + \int_0^1 v\,v_{xxx}\,dx + \frac{1}{2}\int_0^1 [(u^2\,v)^2]_x\,dx + \int_0^1 (u^2\,v)_x\,v_{xx}\,dx$$
$$+\int_0^1 (u^2\,v)_x\,v\,dx + \int_0^1 (u^2\,v)_x\,v_x\,dx + \frac{1}{2}\int_0^1 (v_x^2)_x\,dx + \frac{1}{2}\int_0^1 (v^2)_x\,dx = 0.$$

Each term is treated separately, integrating by parts

$$\frac{1}{2}\int_0^1 u^2\,(v^2)_t\,dx - \int_0^1 v_x\,v_{xt}\,dx + \frac{1}{2}\int_0^1 (v^2)_t\,dx - \int_0^1 (u^2\,v)_x\,v_{xx}\,dx + \frac{1}{2}\int_0^1 (v_{xx}^2)_x\,dx$$
$$+\int_0^1 v\,v_{xxx}\,dx + \frac{1}{2}\int_0^1 [(u^2\,v)^2]_x\,dx + \int_0^1 (u^2\,v)_x\,v_{xx}\,dx + \int_0^1 (u^2\,v)_x\,v\,dx - \int_0^1 (u^2\,v)_x\,v\,dx$$
$$+\frac{1}{2}\int_0^1 (v_x^2)_x\,dx + \frac{1}{2}\int_0^1 (v^2)_x\,dx = 0$$

where

$$\frac{1}{2}\int_0^1 u^2\,(v^2)_t\,dx - \frac{1}{2}\int_0^1 (v_x^2)_t\,dx + \frac{1}{2}\int_0^1 (v^2)_t\,dx = 0 \tag{2.4}$$

adding (2.3) and (2.4), we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[u^2\,v^2 - (u_x^2 + v_x^2) + v^2\right]\,dx = 0$$

where

$$\frac{d}{dt}\int_0^1 \left[u^2 v^2 - (u_x^2 + v_x^2) + v^2\right]dx = 0. \tag{2.5}$$

Lemma 2.4. *For all $u \in H^1(\Omega)$*

$$||u||_{L^\infty(\Omega)} \le c\,||u||_{L^2(\Omega)}^{1/2}\left(||u||_{L^2(\Omega)} + \left|\left|\frac{du}{dx}\right|\right|_{L^2(\Omega)}\right)^{1/2} \tag{2.6}$$

and for all $u \in H^3(\Omega)$

$$||u||_{L^4(\Omega)} \le c\,||u||_{L^2(\Omega)}^{11/12}\left(||u||_{L^2(\Omega)} + \left|\left|\frac{d^3u}{dx^3}\right|\right|_{L^2(\Omega)}\right)^{1/12} \tag{2.7}$$

$$\left|\left|\frac{du}{dx}\right|\right|_{L^4(\Omega)} \le c\,||u||_{L^2(\Omega)}^{7/12}\left(||u||_{L^2(\Omega)} + \left|\left|\frac{d^3u}{dx^3}\right|\right|_{L^2(\Omega)}\right)^{5/12} \tag{2.8}$$

Proof. See [28].

Theorem 2.5 (Local Existence). *Let $(u_0,\, v_0) \in H^1(0,\,1) \times H^1(0,\,1)$ with $u_0(0) = u_0(1)$ and $v_0(0) = v_0(1)$. Then there exist $T > 0$ and $(u,\, v)$ such that $(u,\, v)$ is a solution of (P). $(u,\, v) \in L^\infty(0,\,T;\, H^1(0,\,1)) \times L^\infty(0,\,T;\, H^1(0,\,1))$ and the initial data $u(x,\,0) = u_0(x)$, $v(x,\,0) = v_0(x)$ are satisfied.*

Proof. For $\epsilon > 0$, we approximate the system (P) by the parabolic system

$$(R)\begin{cases} \partial_t u_\epsilon + \partial_x^3 u_\epsilon + \partial_x(u_\epsilon\, v_\epsilon^2) + \epsilon\,\partial_x^4 u_\epsilon = 0 & (R_1)\\ \partial_t v_\epsilon + \partial_x^3 v_\epsilon + \partial_x(u_\epsilon^2\, v_\epsilon) + \partial_x v_\epsilon + \epsilon\,\partial_x^4 v_\epsilon = 0 & (R_2)\\ u_\epsilon(x,\,0) = u_0(x)\ ;\ v_\epsilon(x,\,0) = v_0(x) & (R_3)\\ \partial_x^k u_\epsilon(0,\,t) = \partial_x^k u_\epsilon(1,\,t)\ ;\ \partial_x^k v_\epsilon(0,\,t) = \partial_x^k v_\epsilon(1,\,t), \quad k = 0,\,1,\,2,\,3. & (R_4) \end{cases}$$

We rewrite the above equations in a more friendly way as

$$u_t + u_{xxx} + (u\,v^2)_x + \epsilon\, u_{xxxx} = 0 \tag{2.9}$$
$$v_t + v_{xxx} + (u^2\,v)_x + v_x + \epsilon\, v_{xxxx} = 0 \tag{2.10}$$

We multiply (2.9) by u, integrate over $x \in \Omega = (0,\,1)$, to have

$$\frac{1}{2}\,\frac{d}{dt}\int_0^1 u^2\,dx - \frac{1}{2}\int_0^1 (u^2)_x\,v^2\,dx + \epsilon\int_0^1 u_{xx}^2\,dx = 0. \tag{2.11}$$

Similarly, we multiply (2.10) by v, integrate over $x \in \Omega = (0,\,1)$, and we have

$$\frac{1}{2}\,\frac{d}{dt}\int_0^1 v^2\,dx - \frac{1}{2}\int_0^1 u^2(v^2)_x\,dx + \epsilon\int_0^1 v_{xx}^2\,dx = 0. \tag{2.12}$$

Adding (2.11) and (2.12) we obtain

$$\frac{1}{2}\,\frac{d}{dt}\int_0^1 (u^2+v^2)\,dx + \epsilon\int_0^1 (u_{xx}^2+v_{xx}^2)\,dx = 0$$

where

$$\frac{d}{dt}\int_0^1 (u^2+v^2)\,dx + 2\,\epsilon\int_0^1 (u_{xx}^2+v_{xx}^2)\,dx = 0 \tag{2.13}$$

then

$$||u_\epsilon||^2_{L^2(0,\,1)} + ||v_\epsilon||^2_{L^2(0,\,1)} + 2\,\epsilon\,||u_\epsilon||^2_{L^2(0,\,T;\,H^2(0,\,1))} + 2\,\epsilon\,||v_\epsilon||^2_{L^2(0,\,T;\,H^2(0,\,1))} = c.$$

Where in particular

$$||u_\epsilon||_{L^\infty(0,\,T;\,L^2(0,\,1))} \le c \quad ; \quad ||v_\epsilon||_{L^\infty(0,\,T;\,L^2(0,\,1))} \le c$$

$$\sqrt{\epsilon}\,\left|\left|\frac{\partial^2 u_\epsilon}{\partial x^2}\right|\right|_{L^2(0,\,T;\,L^2(0,\,1))} \le c \quad ; \quad \sqrt{\epsilon}\,\left|\left|\frac{\partial^2 v_\epsilon}{\partial x^2}\right|\right|_{L^2(0,\,T;\,L^2(0,\,1))} \le c$$

if and only if

$$\sqrt{\epsilon}\,\left|\left|\frac{\partial^2 u_\epsilon}{\partial x^2}\right|\right|_{L^2(Q)} \le c \quad ; \quad \sqrt{\epsilon}\,\left|\left|\frac{\partial^2 v_\epsilon}{\partial x^2}\right|\right|_{L^2(Q)} \le c$$

or

$$u_\epsilon,\; v_\epsilon \in L^\infty(0,\,T;\,L^2(0,\,1)) \tag{2.14}$$
$$\sqrt{\epsilon}\,u_\epsilon,\; \sqrt{\epsilon}\,v_\epsilon \in L^\infty(0,\,T;\,H^2(0,\,1)) \tag{2.15}$$

On the other hand, we multiply the equations in (R) by $(u\,v^2 + u_{xx})$ and $(u^2\,v + v_{xx} + v)$, respectively, and integrating over $x \in \Omega = (0,\,1)$ and using Lemma 2.3, we obtain

$$\frac{d}{dt}\int_0^1 \left[\frac{1}{2}\,u^2\,v^2 - \frac{1}{2}\,(u_x^2+v_x^2) + \frac{1}{2}\,v^2\right]\,dx + \frac{1}{2}\int_0^1 u^2\,(v^2)_x\,dx$$
$$-\epsilon\int_0^1 (u\,v^2)_x\,u_{xxx}\,dx - \epsilon\int_0^1 (u^2\,v)_x\,v_{xxx}\,dx - \epsilon\int_0^1 u_{xxx}^2\,dx - \epsilon\int_0^1 u_{xxx}^2\,dx - \int_0^1 v_{xx}^2\,dx = 0$$

then

$$\frac{1}{2}\,\frac{d}{dt}\int_0^1 u_x^2\,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 v_x^2\,dx + \epsilon\int_0^1 u_{xxx}\,dx + \epsilon\int_0^1 v_{xxx}\,dx + \int_0^1 v_{xx}^2\,dx = \frac{1}{2}\,\frac{d}{dt}\int_0^1 u^2\,v^2\,dx$$
$$+\frac{1}{2}\int_0^1 v^2\,dx + \frac{1}{2}\,\frac{d}{dt}\int_0^1 u^2\,(v^2)_x\,dx - \epsilon\int_0^1 (u\,v^2)_x\,u_{xxx}\,dx - \epsilon\int_0^1 (u^2\,v)_x\,v_{xxx}\,dx$$

hence

$$
\begin{aligned}
& \frac{d}{dt}||u_x||^2_{L^2(0,\,1)} + \frac{d}{dt}||v_x||^2_{L^2(0,\,1)} + 2\,\epsilon\,||u_{xxx}||^2_{L^2(0,\,1)} + 2\,\epsilon\,||v_{xxx}||^2_{L^2(0,\,1)} + 2\,||v_{xx}||^2_{L^2(0,\,1)} \\
= \; & \frac{d}{dt}\int_0^1 u^2\,v^2\,dx + \frac{d}{dt}||v||^2_{L^2(0,\,1)} + \int_0^1 u^2 (v^2)_x\,dx - \;2\,\epsilon\int_0^1 (u\,v^2)_x\,u_{xxx}\,dx - 2\,\epsilon\int_0^1 (u^2\,v)_x\,v_{xxx}\,dx.
\end{aligned}
$$

Integrating over $t \in [0,\,T]$ we have

$$
\begin{aligned}
& ||u_x||^2_{L^2(0,\,1)} + ||v_x||^2_{L^2(0,\,1)} + 2\,\epsilon\int_0^t ||u_{xxx}||^2_{L^2(0,\,1)}\,d\sigma + 2\,\epsilon\int_0^t ||v_{xxx}||^2_{L^2(0,\,1)}\,d\sigma + 2\int_0^t ||v_{xx}||^2_{L^2(0,\,1)}\,d\sigma \\
= \; & \left|\left|\frac{du_0}{dx}\right|\right|^2_{L^2(0,\,1)} + \left|\left|\frac{dv_0}{dx}\right|\right|^2_{L^2(0,\,1)} + \int_0^1 u^2 v^2 dx - \int_0^1 u_0^2 v_0^2\,dx + ||v||^2_{L^2(0,\,1)} - ||v_0||^2_{L^2(0,\,1)} \\
& -\,2\,\epsilon\int_0^t\int_0^1 (u\,v^2)_x\,u_{xxx}\,dx\,d\sigma - 2\,\epsilon\int_0^t\int_0^1 (u^2\,v)_x\,v_{xxx}dxd\sigma + \int_0^t\int_0^1 u^2\,(v^2)_x\,dx\,d\sigma
\end{aligned}
$$

or

$$
\begin{aligned}
& ||u_x||^2_{L^2(0,\,1)} + ||v_x||^2_{L^2(0,\,1)} + 2\,\epsilon\int_0^t ||u_{xxx}||^2_{L^2(0,\,1)}\,d\sigma + 2\,\epsilon\int_0^t ||v_{xxx}||^2_{L^2(0,\,1)}\,d\sigma + 2\int_0^t ||v_{xx}||^2_{L^2(0,\,1)}\,d\sigma \\
= \; & \left|\left|\frac{du_0}{dx}\right|\right|^2_{L^2(0,\,1)} + \left|\left|\frac{dv_0}{dx}\right|\right|^2_{L^2(0,\,1)} + \int_0^1 u^2\,v^2\,dx - \int_0^1 u_0^2\,v_0^2\,dx + ||v||^2_{L^2(0,\,1)} - ||v_0||^2_{L^2(0,\,1)} \\
& -\,2\,\epsilon\int_0^t\int_0^1 v^2\,u_x\,u_{xxx}\,dx\,d\sigma - 4\,\epsilon\int_0^t\int_0^1 u\,v\,v_x\,u_{xxx}\,dx\,d\sigma - 4\,\epsilon\int_0^t\int_0^1 u\,v\,u_x\,v_{xxx}\,dx\,d\sigma \\
& -\,2\,\epsilon\int_0^t\int_0^1 u^2\,v_x\,v_{xxx}\,dx\,d\sigma + 2\int_0^t\int_0^1 u^2 v\,v_x\,dx\,d\sigma. \qquad (2.16)
\end{aligned}
$$

On the other hand, using the Lemma 2.4 and performing appropriate calculations we obtain

$$
\begin{aligned}
\left|\int_0^1 u^2\,v^2\,dx\right| \; & \leq \; ||u||_{L^\infty(0,\,1)}\,||v||_{L^\infty(0,\,1)}\int_0^1 |u|\,|v|\,dx \\
& \leq \; ||u||_{L^\infty(0,\,1)}\,||v||_{L^\infty(0,\,1)}\,||u||_{L^2(0,\,1)}\,||v||_{L^2(0,\,1)} \\
& \leq \; c\,||u||_{L^\infty(0,\,1)}\,||v||_{L^\infty(0,\,1)} \\
& \leq \; c\,||u||^{1/2}_{L^2(0,\,1)}\left(||u||_{L^2(0,\,1)} + \left|\left|\frac{du}{dx}\right|\right|_{L^2(0,\,1)}\right)^{1/2} ||v||^{1/2}_{L^2(0,\,1)}\left(||v||_{L^2(0,\,1)} + \left|\left|\frac{dv}{dx}\right|\right|_{L^2(0,\,1)}\right)^{1/2} \\
& \leq \; c + \frac{1}{2}\left|\left|\frac{du}{dx}\right|\right|^2_{L^2(0,\,1)} + \frac{1}{2}\left|\left|\frac{du}{dx}\right|\right|^2_{L^2(0,\,1)} \\
& = \; c + \frac{1}{2}\,||u_x||^2_{L^2(0,\,1)} + \frac{1}{2}\,||v_x||^2_{L^2(0,\,1)}
\end{aligned}
$$

hence

$$
\left|\int_0^1 u^2\,v^2 dx\right| \;\leq\; c + \frac{1}{2}\,||u_x||^2_{L^2(0,\,1)} + \frac{1}{2}\,||v_x||^2_{L^2(0,\,1)}. \qquad (2.17)
$$

We also have

$$
\begin{aligned}
\left|\int_0^1 v^2\, u_x\, u_{xxx}\, dx\right| &\le \int_0^1 |v|^2\, |u_x|\, |u_{xxx}|\, dx \le ||v||_{L^\infty(0,\,1)} \int_0^1 |v|\, |u_x|\, |u_{xxx}|\, dx \\
&\le ||v||_{L^\infty(0,\,1)} \left(\int_\Omega |v|^4\, dx\right)^{1/4} \left(\int_\Omega \left|\frac{du}{dx}\right|^4\, dx\right)^{1/4} \left(\int_\Omega \left|\frac{d^3u}{dx^3}\right|^2\, dx\right)^{1/2} \\
&\le ||v||_{L^\infty(0,\,1)}\, ||v||_{L^4(0,\,1)} \left|\left|\frac{du}{dx}\right|\right|_{L^4(0,\,1)} \left|\left|\frac{d^3u}{dx^3}\right|\right|_{L^4(0,\,1)} \\
&\le c\, ||v||^{11/12}_{L^2(0,\,1)} \left(||v||_{L^2(0,\,1)} + \left|\left|\frac{d^3v}{dx^3}\right|\right|^2_{L^2(0,\,1)}\right)^{1/12} \times \\
&\qquad ||u||^{7/12}_{L^2(0,\,1)} \left(||u||_{L^2(0,\,1)} + \left|\left|\frac{d^3u}{dx^3}\right|\right|^2_{L^2(0,\,1)}\right)^{5/12} ||u_{xxx}||_{L^2(0,\,1)} \\
&\le c + \frac{1}{4} \left|\left|\frac{d^3u}{dx^3}\right|\right|^2_{L^2(0,\,1)} + \frac{1}{4} \left|\left|\frac{d^3v}{dx^3}\right|\right|^2_{L^2(0,\,1)}.
\end{aligned}
$$

Hence,

$$
\left|\int_0^1 v^2\, u_x\, u_{xxx}\, dx\right| \le c + \frac{1}{4}\, ||u_{xxx}||^2_{L^2(0,\,1)} + \frac{1}{4}\, ||v_{xxx}||^2_{L^2(0,\,1)} \tag{2.18}
$$

We calculate in similar form the terms

$$
\left|\int_0^1 u\, v\, v_x\, u_{xxx}\, dx\right|, \quad \left|\int_0^1 u\, v\, u_x\, v_{xxx}\, dx\right|, \quad \left|\int_0^1 u^2\, v_x\, v_{xxx}\, dx\right|, \quad \left|\int_0^1 u^2\, v\, v_x\, dx\right|.
$$

This way we have

$$
||u_x||^2_{L^2(0,\,1)} + ||v_x||^2_{L^2(0,\,1)} + \epsilon \int_0^t ||u_{xxx}||^2_{L^2(0,\,1)}\, d\sigma + \epsilon \int_0^t ||v_{xxx}||^2_{L^2(0,\,1)}\, d\sigma + \int_0^t ||v_{xx}||^2_{L^2(0,\,1)}\, d\sigma \le c
$$

or

$$
\left|\left|\frac{du_\epsilon}{dx}\right|\right|^2_{L^2(0,\,1)} + \left|\left|\frac{dv_\epsilon}{dx}\right|\right|^2_{L^2(0,\,1)} + \epsilon \int_0^t \left|\left|\frac{d^3u_\epsilon}{dx^3}\right|\right|^2_{L^2(0,\,1)} d\sigma + \epsilon \int_0^t \left|\left|\frac{d^3v_\epsilon}{dx^3}\right|\right|^2_{L^2(0,\,1)} d\sigma + \int_0^t \left|\left|\frac{d^2v_\epsilon}{dx^2}\right|\right|^2_{L^2(0,\,1)} d\sigma \le c.
$$

In particular,

$$
\left|\left|\frac{du_\epsilon}{dx}\right|\right|^2_{L^2(0,\,1)} \le c, \quad \left|\left|\frac{dv_\epsilon}{dx}\right|\right|^2_{L^2(0,\,1)} \le c, \quad \sqrt{\epsilon}\left|\left|\frac{d^3u_\epsilon}{dx^3}\right|\right|^2_{L^2(0,\,T;\,L^2(0,\,1))} \le c, \quad \sqrt{\epsilon}\left|\left|\frac{d^3v_\epsilon}{dx^3}\right|\right|^2_{L^2(0,\,T;\,L^2(0,\,1))} \le c
$$

then

$$
\left|\left|\frac{du_\epsilon}{dx}\right|\right|^2_{L^2(0,\,1)} \le c, \quad \left|\left|\frac{dv_\epsilon}{dx}\right|\right|^2_{L^2(0,\,1)} \le c, \quad \sqrt{\epsilon}\left|\left|\frac{d^3u_\epsilon}{dx^3}\right|\right|^2_{L^2(Q)} \le c, \quad \sqrt{\epsilon}\left|\left|\frac{d^3v_\epsilon}{dx^3}\right|\right|^2_{L^2(Q)} \le c
$$

or

$$
u_\epsilon,\ v_\epsilon \in L^\infty(0,\,T;\,H^1(0,\,1)) \bigcap L^2(0,\,T;\,H^3(0,\,1)) \tag{2.19}
$$

$$
v_\epsilon \in L^2(0,\,T;\,H^2(0,\,1)) \tag{2.20}
$$

Hence from (2.14)-(2.15) and (2.19)-(2.20) we have the existence of subsequences $u_{\epsilon_j} \stackrel{def}{=} u_\epsilon$ and $v_{\epsilon_j} \stackrel{def}{=} v_\epsilon$ such that

$$\begin{aligned}
u_\epsilon &\stackrel{*}{\rightharpoonup} u \quad \text{weakly in} \quad L^\infty(0,\,T;L^2(0,\,1)) \hookrightarrow L^2(0,\,T;L^2(0,\,1)) = L^2(Q)\\
v_\epsilon &\stackrel{*}{\rightharpoonup} v \quad \text{weakly in} \quad L^\infty(0,\,T;L^2(0,\,1)) \hookrightarrow L^2(0,\,T;L^2(0,\,1)) = L^2(Q)\\
\frac{\partial u_\epsilon}{\partial x} &\stackrel{*}{\rightharpoonup} \frac{\partial u}{\partial x} \quad \text{weakly in} \quad L^\infty(0,\,T;L^2(0,\,1)) \hookrightarrow L^2(0,\,T;L^2(0,\,1)) = L^2(Q)\\
\frac{\partial v_\epsilon}{\partial x} &\stackrel{*}{\rightharpoonup} \frac{\partial v}{\partial x} \quad \text{weakly in} \quad L^\infty(0,\,T;L^2(0,\,1)) \hookrightarrow L^2(0,\,T;L^2(0,\,1)) = L^2(Q)
\end{aligned}$$

from the equation (R) we deduce that

$$\begin{aligned}
\frac{\partial u_\epsilon}{\partial t} &\stackrel{*}{\rightharpoonup} \frac{\partial u}{\partial t} \quad \text{weakly in} \quad L^2(0,\,T;H^{-2}(0,\,1))\\
\frac{\partial v_\epsilon}{\partial t} &\stackrel{*}{\rightharpoonup} \frac{\partial v}{\partial t} \quad \text{weakly in} \quad L^2(0,\,T;H^{-2}(0,\,1)).
\end{aligned}$$

By other hand, we have $H^1(0,\,1) \stackrel{c}{\hookrightarrow} L^2(0,\,1) \hookrightarrow H^{-2}(0,\,1)$. Using Lions-Aubin's compactness Theorem

$$u_\epsilon \to u \quad \text{strongly in} \quad L^2(Q)$$

$$v_\epsilon \to v \quad \text{strongly in} \quad L^2(Q).$$

Then

$$\partial_x(u_\epsilon\, v_\epsilon^2) = 2\, u_\epsilon\, v_\epsilon\, \frac{\partial v_\epsilon}{\partial x} + \frac{\partial u_\epsilon}{\partial x}\, v_\epsilon\, v_\epsilon \longrightarrow 2\, u\, v\, \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\, v\, v = \partial_x(u\, v^2) \quad \text{in} \quad \mathcal{D}'(0,\,1).$$

The other terms are calculated in a similar way and therefore we can pass to the limit in the equation (R). Finally, u, v are solutions of the equation (P) and the theorem follows.

3 Carleman's estimate and unique continuation property

We consider the equation (P), then

$$(Q)\left\{\begin{array}{l} \partial_t u + \partial_x^3 u + v^2\, u_x + 2\, u\, v\, v_x = 0\\ \partial_t v + \partial_x^3 v + 2\, u\, v\, u_x + u^2\, v_x + v_x = 0 \end{array}\right.$$

We rewrite the above equations as

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} + \partial_x^3 \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} v^2 & 2\, u\, v \\ 2\, u\, v & u^2 \end{bmatrix} \partial_x \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \partial_x \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $U = U(x,\,t)$,

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \quad ; \quad B(U) = \begin{bmatrix} v^2 & 2\, u\, v \\ 2\, u\, v & u^2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \quad ; \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence in (Q) we obtain

$$U_t + U_{xxx} + (B(U) + C)\, U_x = 0, \qquad 0 \le x \le 1, \quad t \ge 0 \tag{3.1}$$

$$U(x,\,0) = U_0(x) \quad ; \quad \partial_x^k U(0,\,t) = \partial_x^k U(1,\,t), \qquad k = 0,\,1,\,2. \tag{3.2}$$

Then

$$\mathcal{L}U = \left[I\partial_t + I\,\partial_x^3 + B(U)\,\partial_x\right] U. \tag{3.3}$$

System (3.1) may be written as

$$\mathcal{L}U = 0 \tag{3.4}$$

with

$$\mathcal{L} = I\partial_t + I\,\partial_x^3 + B(U)\,\partial_x. \tag{3.5}$$

We see in (3.1) and (3.4) that the system (3.1) may be written as $\mathcal{L}U = 0$ where the operator $\mathcal{L}$ is given in (3.5). It has the form:

$$\mathcal{L} = \begin{bmatrix} L_1 & f_2\,\partial_x \\ f_3\,\partial_x & L_2 \end{bmatrix} \qquad (\alpha)$$

with

$$\begin{aligned} L_1 &= \partial_t + \partial_x^3 + f_1\,\partial_x & (3.6)\\ L_2 &= \partial_t + \partial_x^3 + f_4\,\partial_x. & (3.7) \end{aligned}$$

Proposition 3.1 (Carleman's Estimate). *Let $\delta > 0$, $\mathbb{B}_\delta = \{(x,\,t) \in \mathbb{R}^2/\ x^2 + t^2 < \delta^2\}$, $\varphi(x,\,t) = (x-\delta)^2 + \delta^2\,t^2$ and the differential operator $\mathcal{L}$ defined by (3.5). Assume that $f_k \in L^\infty(\mathbb{B}_\delta)$, $k = 1,\,2,\,3,\,4$. Then*

$$3\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\, e^{2\,\tau\,\varphi}\, dx\, dt + 12\,\tau^3 \int_{\mathbb{B}_\delta} |\Phi|^2\, e^{2\,\tau\,\varphi} dx\, dt \le 2 \int_{\mathbb{B}_\delta} |\mathcal{L}\Phi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \tag{3.8}$$

for any $\Phi \in C_0^\infty(\mathbb{B}_\delta) \times C_0^\infty(\mathbb{B}_\delta)$ and $\tau > 0$ large enough.

Proof. We consider the operator $P_1 = \partial_t + \partial_x^3$, then using the Treve inequality

$$\begin{aligned} 96\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\, e^{2\tau\,\varphi}\, dx\, dt &\le \int_{\mathbb{B}_\delta} |P_1\Phi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt & (3.9)\\ 384\,\tau^3 \int_{\mathbb{B}_\delta} |\Phi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt &\le \int_{\mathbb{B}_\delta} |P_1\Phi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt & (3.10) \end{aligned}$$

whenever $\Phi \in C_0^\infty(\mathbb{B}_\delta)$ and $\tau > 0$. Adding up the inequalities (3.9) and (3.10), we obtain

$$96\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\, e^{2\,\tau\,\varphi}\, dx\, dt + 384\,\tau^3 \int_{\mathbb{B}_\delta} |\Phi|^2 e^{2\,\tau\,\varphi}\, dx\, dt \le 2 \int_{\mathbb{B}_\delta} |P_1\Phi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt$$

for any $\Phi \in C_0^\infty(\mathbb{B}_\delta)$ and $\tau > 0$. Then

$$12\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\, e^{2\,\tau\,\varphi}\, dx\, dt + 48\,\tau^3 \int_{\mathbb{B}_\delta} |\Phi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \le \frac{1}{4} \int_{\mathbb{B}_\delta} |P_1\Phi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \tag{3.11}$$

for any $\Phi \in C_0^\infty(\mathbb{B}_\delta)$ and $\tau > 0$. Similarly, we have for the operator $P_1 = \partial_t + \partial_x^3$.

$$12\,\tau^2 \int_{\mathbb{B}_\delta} |\Psi_x|^2\, e^{2\,\tau\,\varphi}\, dx\, dt + 48\,\tau^3 \int_{\mathbb{B}_\delta} |\Psi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \le \frac{1}{4} \int_{\mathbb{B}_\delta} |P_1\Psi|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \tag{3.12}$$

for any $\Psi \in C_0^\infty(\mathbb{B}_\delta)$ and $\tau > 0$. On the other hand,

$$\int_{\mathbb{B}_\delta} |f_1\Phi_x|^2 e^{2\,\tau\,\varphi}\, dx\, dt \le ||f_1||^2_{L^\infty(\mathbb{B}_\delta)} \int_{\mathbb{B}_\delta} |\Phi_x|^2\, e^{2\,\tau\,\varphi}\, dx\, dt. \tag{3.13}$$

Let $\tau \geq \frac{\sqrt{6}}{12}\,||f_1||_{L^\infty(\mathbb{B}_\delta)}$, then $\tau^2 \geq \frac{1}{24}\,||f_1||^2_{L^\infty(\mathbb{B}_\delta)}$. This way in (3.9) we have

$$\begin{aligned}\int_{\mathbb{B}_\delta} |P_1\Phi|^2 e^{2\,\tau\,\varphi}\,dx\,dt &\geq 96\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \geq 96\,\frac{1}{24}\,||f_1||^2_{L^\infty(\mathbb{B}_\delta)} \int_{\mathbb{B}_\delta} |\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \\ &= 4\,||f_1||^2_{L^\infty(\mathbb{B}_\delta)} \int_{\mathbb{B}_\delta} |\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \geq 4 \int_{\mathbb{B}_\delta} |f_1\Phi_x|^2 e^{2\,\tau\,\varphi}\,dx\,dt \quad \text{(using (3.13))}\end{aligned}$$

hence

$$\int_{\mathbb{B}_\delta} |f_1\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \leq \frac{1}{4} \int_{\mathbb{B}_\delta} |P_1\Phi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt. \tag{3.14}$$

Then adding (3.12) and (3.14) we have

$$\begin{aligned}&\int_{\mathbb{B}_\delta} |f_1\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + 12\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + 48\,\tau^3 \int_{\mathbb{B}_\delta} |\Phi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \\ &\leq \frac{1}{2} \int_{\mathbb{B}_\delta} |P_1\Phi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt.\end{aligned} \tag{3.15}$$

But $L_1 = \partial_t + \partial_x^3 + f_1\,\partial_x = P_1 + f_1\,\partial_x$. Then $P_1\Phi = L_1\Phi - f_1\Phi_x$ and $|P_1\Phi|^2 \leq 2\,|L_1\Phi|^2 + 2\,|f_1\Phi_x|^2$. Hence, in (3.15)

$$\begin{aligned}&\int_{\mathbb{B}_\delta} |f_1\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + 12\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + 48\,\tau^3 \int_{\mathbb{B}_\delta} |\Phi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \\ &\leq \int_{\mathbb{B}_\delta} |L_1\Phi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + \int_{\mathbb{B}_\delta} |f_1\,\Phi_x|^2 e^{2\,\tau\,\varphi}\,dx\,dt\end{aligned}$$

then

$$12\,\tau^2 \int_{\mathbb{B}_\delta} |\Phi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + 48\,\tau^3 \int_{\mathbb{B}_\delta} |\Phi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \leq \int_{\mathbb{B}_\delta} |L_1\Phi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt. \tag{3.16}$$

for any $\Phi \in C_0^\infty(\mathbb{B}_\delta)$ and $\tau \geq \frac{\sqrt{6}}{12}\,||f_1||_{L^\infty(\mathbb{B}_\delta)}$. Performing similar calculations with (3.12) and the operator L_2, we obtain

$$12\,\tau^2 \int_{\mathbb{B}_\delta} |\Psi_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + 48\,\tau^3 \int_{\mathbb{B}_\delta} |\Psi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \leq \int_{\mathbb{B}_\delta} |L_2\Psi|^2\,e^{2\,\tau\,\varphi}\,dx\,dt. \tag{3.17}$$

for any $\Psi \in C_0^\infty(\mathbb{B}_\delta)$ and $\tau \geq \frac{\sqrt{6}}{12}\,||f_4||_{L^\infty(\mathbb{B}_\delta)}$. Summing up (3.16) and (3.17), we have

$$3\,\tau^2 \int_{\mathbb{B}_\delta} |\Theta_x|^2\,e^{2\,\tau\,\varphi}\,dx\,dt + 12\,\tau^3 \int_{\mathbb{B}_\delta} |\Theta|^2\,e^{2\,\tau\,\varphi}\,dx\,dt \leq \frac{1}{4} \int_{\mathbb{B}_\delta} (\,|L_1\Phi|^2 + |L_2\Psi|^2\,)\,e^{2\,\tau\,\varphi}\,dx\,dt. \tag{3.18}$$

Whenever

$$\Theta = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \in C_0^\infty(\mathbb{B}_\delta) \times C_0^\infty(\mathbb{B}_\delta) \qquad \text{and} \qquad \tau \geq \text{Max}\left\{ \frac{\sqrt{6}}{12}\,||f_1||_{L^\infty(\mathbb{B}_\delta)}\,;\, \frac{\sqrt{6}}{12}\,||f_4||_{L^\infty(\mathbb{B}_\delta)} \right\}.$$

Similarly, since $f_2,\, f_3 \in L^\infty(\mathbb{B}_\delta)$ and according to $\mathcal{L}$ and (3.6), (3.7) we have

$$|\mathcal{L}\Theta|^2 = |L_1\Phi + f_2\Psi_x|^2 + |L_2\Psi + f_3\Phi_x|^2$$

Then we can add $f_2\Psi_x$ to $L_1\Phi$ and $f_3\Phi_x$ to $L_2\Psi$ in (3.18), and obtain Carleman's estimate (3.8) when

$$\Theta = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \in C_0^\infty(\mathbb{B}_\delta) \times C_0^\infty(\mathbb{B}_\delta)$$

and τ satisfying $\tau \geq \text{Max}\left\{\frac{\sqrt{6}}{12}\,||f_1||_{L^\infty(\mathbb{B}_\delta)}\,;\,\frac{\sqrt{6}}{12}\,||f_4||_{L^\infty(\mathbb{B}_\delta)},\,\frac{\sqrt{6}}{6}\,||f_2||_{L^\infty(\mathbb{B}_\delta)}\,;\,\frac{\sqrt{6}}{6}\,||f_3||_{L^\infty(\mathbb{B}_\delta)}\right\}$.

Remark 3.2. The estimate (3.8) is invariant under changes of signs of any term in L_1 or L_2.

Corollary 3.3. *Assume that, in addition to the hypotheses of Proposition 3.1, we have for any $T > 0$ and $a > 0$ that*

$$V = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in L^2(-T,\,T;\; H^3(-a,\,a) \times H^3(-a,\,a)).$$

$$V_t = \begin{bmatrix} \xi_t \\ \eta_t \end{bmatrix} \in L^2(-T,\,T;\; H^3(-a,\,a) \times L^2(-a,\,a)).$$

and that supp ξ and supp η are compact sets in $\mathbb{B}_\delta$. Then, the estimate (3.8) holds with V instead of

$$\Theta = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}$$

Indeed,

$$3\,\tau^2 \int_{\mathbb{B}_\delta} |V_x|^2\, e^{2\,\tau\,\varphi}\, dx\, dt + 12\,\tau^3 \int_{\mathbb{B}_\delta} |V|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \leq 2 \int_{\mathbb{B}_\delta} |\mathcal{L}V|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \tag{3.19}$$

for $\tau > 0$ sufficiently large.

Proof. Choose a regularization sequence $\{\rho_\epsilon(x,\,t)\}_{\epsilon>0}$. Consider the functions

$$V^\epsilon = \rho_\epsilon * V = \begin{bmatrix} \rho_\epsilon * \xi \\ \rho_\epsilon * \eta \end{bmatrix} \qquad (* \text{ denote the usual convolution})$$

Hence, the inequality (3.8) is valid replacing

$$\Theta = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \quad \text{by} \quad V^\epsilon = \begin{bmatrix} \rho_\epsilon * \xi \\ \rho_\epsilon * \eta \end{bmatrix} \quad \text{if} \quad \epsilon > 0 \quad \text{is sufficiently small.}$$

Taking the limit as $\epsilon \to 0^+$ the result follows.

Theorem 3.4. *Let $T > 0$, $a > 0$ and $R = (-a,\,a) \times (-T,\,T)$. Let $\mathcal{L}$ be the operator defined in (α) and let*

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \in L^2(-T,\,T\,;\; H^3(-a,\,a) \times H^3(-a,\,a))$$

be a solution of the differential equation (3.4), $\mathcal{L}U = 0$. Assume that $f_k \in L^\infty(\mathbb{R})$, $k = 1, 2, 3, 4$ and $U \equiv 0$ when $x < t^2$ in a neighborhood of $(0,\,0)$. Then, there exists a neighborhood of $(0,\,0)$ for which $U \equiv 0$.

Proof. We choose a positive number $\delta < 1$ such that $\mathbb{B}_\delta$ lies in the neighborhood where $U \equiv 0$ when $x < t^2$. Let $\chi \in C_0^\infty(\mathbb{B}_\delta)$, $\chi \equiv 1$ on a neighborhood $\mathcal{N}$ of $(0,\,0)$ and set

$$V = \chi U = \begin{bmatrix} \chi\, u \\ \chi\, v \end{bmatrix}.$$

The function V satisfies the conditions of Corollary 3.3, and $\mathcal{L} = 0$ on $\mathcal{N}$ because $V = U$ on $\mathcal{N}$. Hence by (3.19)

$$6\,\tau^3 \int_{\mathbb{B}_\delta} |V|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \leq \int_{\mathbb{B}_\delta - \mathcal{N}} |\mathcal{L}V|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \tag{3.20}$$

when $\tau > 0$ is sufficiently large. On the other hand, if $(x,\,t) \in$ supp V one has $0 \leq t^2 \leq x < \delta < 1$ and

$$\begin{aligned} \varphi(x,\,t) &= (x-\delta)^2 + \delta^2\, t^2 = (\delta - x)^2 + \delta^2\, t^2 = (t^2 - \delta)^2 + \delta^2\, t^2 \\ &= t^4 - 2\, t^2\, \delta + \delta^2 + \delta^2\, t^2 = t^2\,[\,(t^2 - 2\,\delta + \delta^2) + \delta^2\,] \\ &< \delta^2\,[\,(\delta^2 - 2\,\delta + \delta^2) + \delta^2\,] < \delta^2 \end{aligned}$$

and where $\varphi(0,\,0) = \delta^2$. Therefore, if $(x,\,t) \in \text{supp}\,\mathcal{L}V$ there is an $\epsilon > 0$ such that $\varphi(x,\,t) \le \delta^2 - \epsilon$. We can choose a neighborhood $\mathcal{N}'$ of $(0,\,0)$ at which $\varphi(x,\,t) > \delta^2 - \epsilon$ and obtain from (3.20)

$$\int_{\mathcal{N}'} |V|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \le \frac{1}{6\,\tau^3} \int_{\mathbb{B}_\delta - \mathcal{N}} |\mathcal{L}V|^2\, e^{2\,\tau\,\varphi}\, dx\, dt \tag{3.21}$$

Taking limit in (3.21) as $\tau \to +\infty$ one deduces that $U = V \equiv 0$ on $\mathcal{N}'$.

Definition 3.5. *By a Holmgren's transformation we mean a transformation which is defined by* $\xi = t$, $\eta = x + t^2$ *and which maps the half-space* $x \ge 0$ *into the domain* $\Omega = \{(\eta,\,\xi) \in \mathbb{R}^2 :\ \eta - \xi^2 \ge 0\}$.

Corollary 3.6. *Under the assumptions of Theorem 3.4, consider the curve* $x = \mu_0(t)$, $\mu_0(0) = 0$, μ_0 *a continuously differentiable function in a neighborhood of* $(0,\,0)$. *Suppose that* $U \equiv 0$ *in the region* $x < \mu_0(t)$ *in a neighborhood of* $(0,\,0)$. *Then, there exists a neighborhood of* $(0,\,0)$ *where* $U \equiv 0$.

Proof. We consider the Holmgren transformation

$$(x,\,t) \longrightarrow (\eta,\,\xi) \quad , \quad \eta = x - \mu_0(t) + t^2 \quad , \quad \xi = t.$$

With this variables the function $U = U(\eta,\,\xi)$ satisfies $U \equiv 0$ when $\eta < \xi^2$ in a neighborhood of $(0,\,0)$ and $\mathcal{F}U = 0$, where

$$\mathcal{F} = \begin{bmatrix} \partial_\xi + \partial_\eta^3 + F_1\,\partial_\eta & f_2\,\partial_\eta \\ f_2\,\partial_\eta & \partial_\xi + \partial_\eta^3 + F_4\,\partial_\eta \end{bmatrix}$$

where

$$F_1(\eta,\,\xi) = f_1(x,\,t) + (2\,\xi - \mu_0'(\xi)) \qquad \text{and} \qquad F_4(\eta,\,\xi) = f_4(x,\,t) + (2\,\xi - \mu_0'(\xi))$$

Thus by using Theorem 3.4, and Holmgren's transformation we conclude that there exists a neighborhood of $(0,\,0)$ in the $x\,t-$ plane where $U \equiv 0$.

Theorem 3.7. *Let* $T > 0$ *and* $\Omega = (0,\,1) \times (0,\,T)$. *Assume that* $f_k \in L^\infty_{Loc}(\Omega)$, $k = 1,\,2,\,3,\,4$. *Let*

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \in L^2(0,\,T;\,H^3(0,\,1) \times H^3(0,\,1))$$

be a solution of the equation (3.1). If $U \equiv 0$ *in an open subset* Ω_1 *of* Ω, *then* $U \equiv 0$ *in the horizontal component* Ω_h *of* Ω_1 *in* Ω.

Proof. We prove the theorem for the equation (3.1) or the equivalent equation $\mathcal{L}U = 0$, where $\mathcal{L}$ is the operator defined in (2.4). The proof follows as in [21] applying Corollary 3.6 and considering Remark 3.2.

Corollary 3.8. *Let* $T > 0$, $\Omega = (0,\,1) \times (0,\,T)$ *and let*

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \in C(0,\,T;\,H^3_p(0,\,1) \times H^3_p(0,\,1))$$

be a solution of equation (3.1). Suppose that U *vanishes in an open subset* Ω_1 *of* Ω. *Then* $U \equiv 0$ *in* Ω.

4 The Main Theorem

The first result is concerned with the decay properties of solutions to the coupled system. The idea goes back to T. Kato [11].

Lemma 4.1. *Let* $(u,\, v) \;=\; (u(x,\, t),\, v(x,\, t))$ *be a solution of the coupled system equations* (P) *such that*

$$\sup_{t\in[0,\,1]} \|u(\,.\,,\, t\,)\|_{H^1(\mathbb{R})} < +\infty \quad ; \quad \sup_{t\in[0,\,1]} \|v(\,.\,,\, t\,)\|_{H^1(\mathbb{R})} < +\infty$$

and $e^{\beta x}\, u_0 \in L^2(\mathbb{R})$; $\;e^{\beta x}\, v_0 \in L^2(\mathbb{R}),\ \forall\, \beta > 0$. *Then* $e^{\beta x} u \in C([0,\,1];\, L^2(\mathbb{R}))$; $\;e^{\beta x} v \in C([0,\,1];\, L^2(\mathbb{R}))$.

Proof. Let $\varphi_n \in C^{\infty}(\mathbb{R})$ be defined by

$$\varphi_n(x) = \begin{cases} e^{\beta\, x} & , \quad \text{for } \ x \le n \\ e^{2\,\beta\, x} & , \quad \text{for } \ x > 10\, n \end{cases}$$

with

$$\varphi_n(x) \le e^{2\,\beta\, x} \quad ; \quad 0 \le \varphi_n'(x) \le \beta\, \varphi_n(x) \quad ; \quad |\varphi_n^{(j)}(x)| \le \beta^j\, \varphi_n(x) \qquad j = 2,\, 3. \tag{4.1}$$

Examples.

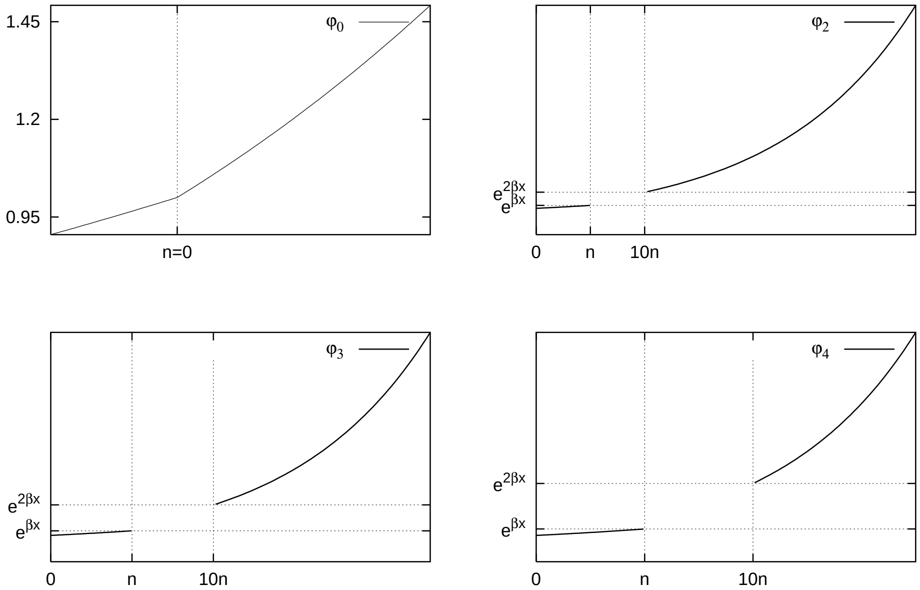


Figure 1: These are sample figures for different values of n.

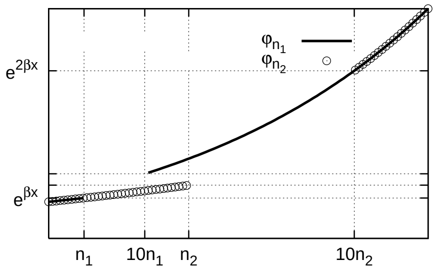


Figure 2: This is a figure comparing two functions with different values of n.

Let

$$u_t + u_{xxx} + u_x\, v^2 + u\,(v^2)_x = 0. \tag{4.2}$$

Multiplying the equation (4.2) by $u\,\varphi_n$, and integrating by parts we get

$$\int_{\mathbb{R}} u\, u_t\, \varphi_n\, dx + \int_{\mathbb{R}} u\, u_{xxx}\, \varphi_n\, dx + \int_{\mathbb{R}} u\, u_x\, v^2\, \varphi_n\, dx + \int_{\mathbb{R}} u^2\,(v^2)_x\, \varphi_n\, dx = 0$$

hence

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}} u^2\, \varphi_n\, dx + \int_{\mathbb{R}} u\, u_{xxx}\, \varphi_n\, dx + \frac{1}{2}\int_{\mathbb{R}} (u^2)_x\, v^2\, \varphi_n\, dx + \int_{\mathbb{R}} u^2\,(v^2)_x\, \varphi_n\, dx = 0.$$

Integrating by parts we obtain

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}} u^2\, \varphi_n\, dx + \frac{3}{2}\int_{\mathbb{R}} u_x^2\, \varphi_n'\, dx - \frac{1}{2}\int_{\mathbb{R}} u^2\, \varphi_n'''\, dx + \frac{1}{2}\int_{\mathbb{R}} (u^2)_x\, v^2\, \varphi_n\, dx\\ &-\int_{\mathbb{R}} (u^2)_x\, v^2\, \varphi_n\, dx - \int_{\mathbb{R}} u^2 v^2\, \varphi_n'\, dx = 0\end{aligned}$$

then

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} u^2\, \varphi_n\, dx + 3\int_{\mathbb{R}} u_x^2\, \varphi_n'\, dx - \int_{\mathbb{R}} u^2\, \varphi_n'''\, dx - \int_{\mathbb{R}} (u^2)_x\, v^2\, \varphi_n\, dx\\ &-2\int_{\mathbb{R}} u^2\, v^2\, \varphi_n'\, dx = 0.\end{aligned} \tag{4.3}$$

Similarly, multiplying the equation

$$v_t + v_{xxx} + (u^2)_x\, v + v_x\, u^2 = 0 \tag{4.4}$$

by $v\,\varphi_n$ and integrating by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} v^2\, \varphi_n\, dx + 3\int_{\mathbb{R}} v_x^2\, \varphi_n'\, dx - \int_{\mathbb{R}} v^2\, \varphi_n'''\, dx - \int_{\mathbb{R}} u^2\,(v^2)_x\, \varphi_n\, dx\\ &-2\int_{\mathbb{R}} u^2\, v^2\, \varphi_n'\, dx = 0.\end{aligned} \tag{4.5}$$

Hence, from (4.3) and (4.5)

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} [u^2+v^2]\, \varphi_n\, dx + 3\int_{\mathbb{R}} [u_x^2+v_x^2]\, \varphi_n'\, dx - \int_{\mathbb{R}} [u^2+v^2]\, \varphi_n'''\, dx\\ &-\int_{\mathbb{R}} [(u^2)_x\, v^2 + u^2\,(v^2)_x]\, \varphi_n dx - 4\int_{\mathbb{R}} u^2\, v^2\, \varphi_n'\, dx = 0\end{aligned}$$

then

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} [u^2+v^2]\, \varphi_n dx + 3\int_{\mathbb{R}} [u_x^2+v_x^2]\, \varphi_n'\, dx - \int_{\mathbb{R}} [u^2+v^2]\, \varphi_n'''\, dx\\ &-\int_{\mathbb{R}} (u^2\, v^2)_x\, \varphi_n\, dx - 4\int_{\mathbb{R}} u^2\, v^2\, \varphi_n'\, dx = 0\end{aligned}$$

hence,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} [u^2+v^2]\, \varphi_n\, dx + 3\int_{\mathbb{R}} [u_x^2+v_x^2\,]\, \varphi_n'\, dx - \int_{\mathbb{R}} [u^2+v^2]\, \varphi_n'''\, dx\\ &+\int_{\mathbb{R}} u^2\, v^2\, \varphi_n'\, dx - 4\int_{\mathbb{R}} u^2\, v^2\, \varphi_n'\, dx = 0\end{aligned}$$

then

$$\frac{d}{dt}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx+3\int_{\mathbb{R}}[u_x^2+v_x^2]\,\varphi_n'\,dx-\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n'''\,dx$$
$$-3\int_{\mathbb{R}}u^2\,v^2\,\varphi_n'\,dx=0$$

hence,

$$\frac{d}{dt}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx+3\int_{\mathbb{R}}[u_x^2+v_x^2]\,\varphi_n'\,dx=\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n'\,dx+3\int_{\mathbb{R}}u\,u\,v\,v\,\varphi_n'\,dx$$
$$\le\beta^3\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx+3\,\beta\int_{\mathbb{R}}u\,u\,v\,v\,\varphi_n\,dx.\qquad\text{(using (4.1))}$$

Using that $\varphi_n\ge 0$ and the Holder inequality

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx &\le \beta^3\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx+3\,\beta\,||u||_{L^\infty(\mathbb{R})}\,||v||_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}u\,v\,\varphi_n\,dx\\
&\le \beta^3\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx\\
&\quad+3\,\beta\,||u||_{L^\infty(\mathbb{R})}\,||v||_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}}u^2\,\varphi_n\,dx\right)^{1/2}\left(\int_{\mathbb{R}}v^2\,\varphi_n\,dx\right)^{1/2}\\
&\le \beta^3\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx+\frac{3}{2}\,\beta\,||u||_{L^\infty(\mathbb{R})}\,||v||_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx\\
&\le \left[\beta^3+\frac{3}{2}\,\beta\,||u||_{L^\infty(\mathbb{R})}\,||v||_{L^\infty(\mathbb{R})}\right]\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx
\end{aligned}$$

then

$$\frac{d}{dt}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx\le c_1\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx$$

where $c_1=\beta^3+\frac{3}{2}\,\beta\,||u||_{L^\infty(\mathbb{R})}\,||v||_{L^\infty(\mathbb{R})}$. Thus,

$$\frac{\frac{d}{dt}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx}{\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx}\le c_1.$$

Integrating over $t\in[0,\,1]$ we have

$$\int_0^t\frac{\frac{d}{dt}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx}{\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx}\,ds\le c_1\,t$$

then

$$Ln\,\left[\frac{\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx}{\int_{\mathbb{R}}[u_0^2+v_0^2]\,\varphi_n\,dx}\right]\le c_1\,t$$

hence

$$\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx\le\left[\int_{\mathbb{R}}[u_0^2+v_0^2]\,\varphi_n\,dx\right]\,e^{c_1\,t}$$

thus

$$\sup_{t\in[0,\,1]}\int_{\mathbb{R}}[u^2+v^2]\,\varphi_n\,dx\le\left[\int_{\mathbb{R}}[u_0^2+v_0^2]\,\varphi_n\,dx\right]e^{c_1}\le\left[\int_{\mathbb{R}}[u_0^2+v_0^2]\,e^{2\,\beta\,x}\,dx\right]e^{c_1}$$

Now, taking $n \to +\infty$ we obtain

$$\sup_{t\in[0,\,1]} \int_{\mathbb{R}} [u^2 + v^2]\, e^{2\,\beta\,x}\, dx \leq \left[\int_{\mathbb{R}} [u_0^2 + v_0^2]\, e^{2\,\beta\,x}\, dx \right] e^{c_1}$$

This way if

$$\sup_{t\in[0,\,1]} ||u(\,.\,,\,t)||_{H^1(\mathbb{R})} < \infty \qquad ; \qquad \sup_{t\in[0,\,1]} ||v(\,.\,,\,t)||_{H^1(\mathbb{R})} < +\infty$$

and

$$e^{\beta\,x}\, u_0 \in L^2(\mathbb{R}) \quad ; \quad e^{\beta\,x}\, v_0 \in L^2(\mathbb{R}), \qquad \forall\, \beta > 0$$

then

$$e^{\beta\,x}\, u \in C([0,\,1];\, L^2(\mathbb{R})) \quad ; \quad e^{\beta\,x}\, v \in C([0,\,1];\, L^2(\mathbb{R}))$$

with

$$c_1 = \beta^3 + \frac{3}{2}\,\beta\, ||u||_{L^\infty(\mathbb{R}\times[0,\,1])}\; ||v||_{L^\infty(\mathbb{R}\times[0,\,1])}.$$

We have the following extension to higher derivatives.

Lemma 4.2. *Let $j \in \mathbb{N}$. Let $(u,\,v) = (u(x,\,t),\,v(x,\,t))$ be a solution of the coupled system equations (P) such that*

$$\sup_{t\in[0,\,1]} ||u(\,.\,,\,t)||_{H^{j+1}(\mathbb{R})} < +\infty \qquad ; \qquad \sup_{t\in[0,\,1]} ||v(\,.\,,\,t)||_{H^{j+1}(\mathbb{R})} < +\infty$$

and

$$e^{\beta\,x}\, u_0,\, \dots\,,\, e^{\beta\,x}\, \partial_x^j u_0 \in L^2(\mathbb{R}) \quad ; \quad e^{\beta\,x}\, v_0,\, \dots\,,\, e^{\beta\,x}\, \partial_x^j v_0 \in L^2(\mathbb{R}), \qquad \forall\, \beta > 0$$

then

$$\sup_{t\in[0,\,1]} ||u(t)||_{C^{j-1}(\mathbb{R})} \leq c_j = c_j(u_0,\,c_1) \quad ; \quad \sup_{t\in[0,\,1]} ||v(t)||_{C^{j-1}(\mathbb{R})} \leq c_j = c_j(v_0,\,c_1)$$

with $c_1 = \beta^3 + \frac{3}{2}\,\beta\, ||u||_{L^\infty(\mathbb{R}\times[0,\,1])}\; ||v||_{L^\infty(\mathbb{R}\times[0,\,1])}$.

Lemma 4.3. *If $(u,\,v) \in C_0^{3,\,1}(\mathbb{R}^2) \times C_0^{3,\,1}(\mathbb{R}^2)$, then*

$$||e^{\lambda\,x}\, u||_{L^8(\mathbb{R}^2)} \leq ||e^{\lambda\,x} \{\partial_t + \partial_x^3\}\, u||_{L^{8/7}(\mathbb{R}^2)} \quad ; \quad ||e^{\lambda\,x}\, v||_{L^8(\mathbb{R}^2)} \leq ||e^{\lambda\,x} \{\partial_t + \partial_x^3\}\, v||_{L^{8/7}(\mathbb{R}^2)}$$

for all $\lambda \in \mathbb{R}$, with c independent of λ.

Proof. Similar to those in [15].

Lemma 4.4. *If $(u,\,v) \in C^{3,\,1}(\mathbb{R}^2) \times C^{3,\,1}(\mathbb{R}^2)$, such that*

$$\operatorname{supp} u \subseteq [-M,\,M] \times [0,\,1] \quad ; \quad \operatorname{supp} v \subseteq [-M,\,M] \times [0,\,1] \tag{4.6}$$

and

$$u(x,\,0) = u(x,\,1) = 0 \quad \text{and} \quad v(x,\,0) = v(x,\,1) = 0, \qquad \forall\, x \in \mathbb{R}, \tag{4.7}$$

then

$$||e^{\lambda\,x}\, u||_{L^8(\mathbb{R}\times[0,\,1])} \;\leq\; c\, ||e^{\lambda\,x} \{\partial_t + \partial_x^3\}\, u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \tag{4.8}$$

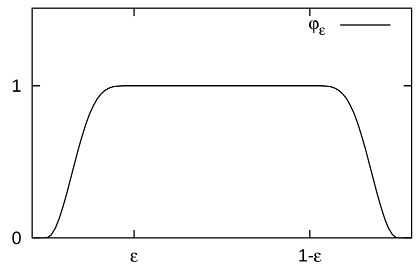
$$||e^{\lambda\,x}\, v||_{L^8(\mathbb{R}\times[0,\,1])} \;\leq\; c\, ||e^{\lambda\,x} \{\partial_t + \partial_x^3\}\, v||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \tag{4.9}$$

for all $\lambda \in \mathbb{R}$, *with* c *independent of* λ.

Proof. Let $\varphi_\epsilon \in C_0^\infty(\mathbb{R})$ with

$$\begin{aligned}&\text{supp } \varphi_\epsilon \subseteq [0,\, 1]\\ &\varphi_\epsilon(t) = 1 \quad \text{for} \quad t \in (\epsilon,\, 1-\epsilon),\\ &0 \leq \varphi_\epsilon(t) \leq 1 \quad \text{and} \quad |\varphi_\epsilon'(t)| \leq c/\epsilon, \quad \text{c constant positive.}\end{aligned}$$

Example.



Let $u_\epsilon(x,\, t) = \varphi_\epsilon(t)\, u(x,\, t)$, hence

$$\text{supp } u_\epsilon(x,\, t) \quad \subseteq \quad \text{supp } \varphi_\epsilon(t) \cap \text{supp } u(x,\, t) \subseteq \text{supp } u(x,\, t) \subseteq [-M,\, M] \times [0,\, 1].$$

Then, on one hand we have that

$$||e^{\lambda\, x}\, u_\epsilon||_{L^8(\mathbb{R}^2)} \longrightarrow ||e^{\lambda\, x}\, u||_{L^8(\mathbb{R}\times[0,\,1])} \quad \text{as} \quad \epsilon \downarrow 0 \tag{4.10}$$

and on the other hand,

$$\{\partial_t + \partial_x^3\}\, u_\epsilon(x,\, t) = \{\partial_t + \partial_x^3\}\, [\, \varphi_\epsilon(t)\, u(x,\, t)\,] = \varphi_\epsilon(t)\, \{\partial_t + \partial_x^3\}\, u + \varphi_\epsilon'(t)\, u. \tag{4.11}$$

Hence,

$$||\varphi_\epsilon(t)\, \{\partial_t + \partial_x^3\}\, u||_{L^{8/7}(\mathbb{R}^2)} \longrightarrow ||\{\partial_t + \partial_x^3\}\, u||_{L^{8/7}(\mathbb{R}^2)}$$

and

$$\begin{aligned}||\varphi_\epsilon'(t)\, u||_{L^{8/7}(\mathbb{R}^2)} &= \left[\int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi_\epsilon'(t)\, u(x,\, t)|^{8/7}\, dx\, dt\right]^{7/8}\\ &= \left[\int_0^1 \int_{-M}^{M} |\varphi_\epsilon'(t)|^{8/7}\, |u(x,\, t)|^{8/7}\, dx\, dt\right]^{7/8} \qquad \text{(using (4.6))}\\ &= \left[\int_0^\epsilon \int_{-M}^{M} |\varphi_\epsilon'(t)|^{8/7}\, |u(x,\, t)|^{8/7}\, dx\, dt + \int_{1-\epsilon}^1 \int_{-M}^{M} |\varphi_\epsilon'(t)|^{8/7}\, |u(x,\, t)|^{8/7}\, dx\, dt\right]^{7/8}\\ &\leq \frac{c}{\epsilon}\left[\int_0^\epsilon \int_{-M}^{M} |u(x,\, t)|^{8/7}\, dx\, dt + \int_{1-\epsilon}^1 \int_{-M}^{M} |u(x,\, t)|^{8/7}\, dx\, dt\right]^{7/8}. \end{aligned} \tag{4.12}$$

Defining

$$G(t) = \int_{-M}^{M} |u(x,\, t)|^{8/7} dx$$

then, by (4.7) we have that $G(0) = G(1) = 0$. G is continuous and differentiable with

$$G'(t) = \frac{8}{7} \int_{-M}^{M} |u(x,\, t)|^{1/7}\, \partial_t u(x,\, t)\, \mathrm{sgn}(u(x,\, t))\, dx$$

hence, $G'(t)$ is continuous,

$$|G'(t)| \le c\,|t| \quad , \quad |G'(t)| \le c\,|1-t|$$

and

$$\int_0^{\epsilon} G(t)\, dt + \int_{1-\epsilon}^{1} G(t)\, dt \le c\, \epsilon^2. \tag{4.13}$$

Inserting (4.13) in (4.12) we have

$$||\varphi_\epsilon'(t)\, u||_{L^{8/7}(\mathbb{R}^2)} \le c\, \frac{1}{\epsilon}\, \epsilon^{7/4} \to 0 \quad \text{as} \quad \epsilon \downarrow 0$$

and (4.8) follows. Similarly, we obtain (4.9).

Lemma 4.5. *Let* $(u,\, v) \in C^{3,\,1}(\mathbb{R}\times[0,\, 1]) \times C^{3,\,1}(\mathbb{R}\times[0,\, 1])$. *Suppose that*

$$\sum_{j\le 2} |\partial_x^j u(x,\, t)| \le C_\beta\, e^{-\,\beta\,|x|}\ , \quad \sum_{j\le 2} |\partial_x^j v(x,\, t)| \le C_\beta\, e^{-\,\beta\,|x|},\ t\in[0,\, 1],\ \forall\,\beta > 0, \tag{4.14}$$

and

$$u(x,\, 0) = u(x,\, 1) = 0 \quad ; \quad v(x,\, 0) = v(x,\, 1) = 0, \quad \forall\, x \in \mathbb{R}.$$

Then

$$||e^{\lambda\, x}\, u||_{L^8(\mathbb{R}\times[0,\,1])} \le c_0\, ||e^{\lambda\, x}\{\partial_t + \partial_x^3\}\, u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \tag{4.15}$$

$$||e^{\lambda\, x}\, v||_{L^8(\mathbb{R}\times[0,\,1])} \le c_0\, ||e^{\lambda\, x}\{\partial_t + \partial_x^3\}\, v||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \tag{4.16}$$

for all $\lambda \in \mathbb{R}$, *with* c_0 *independent of* λ.

Proof. Let $\phi \in C_0^\infty(\mathbb{R})$ be an even, nonincreasing function for $x > 0$ with

$$\begin{aligned} &\phi(x) = 1, \quad |x| \le 1, \\ &\mathrm{supp}\ \phi \subseteq [-2,\, 2]. \end{aligned}$$

For each M we consider the sequence $\{\phi_M\}$ in $C_0^\infty(\mathbb{R})$ defined by $\phi_M(x) = \phi(\frac{x}{M})$, then $\phi_M \equiv 1$ in a neighborhood of 0 and supp $\phi_M \subseteq [-M,\, M]$. Let $u_M(x,\, t) = \phi_M(x)\, u(x,\, t)$, then supp $u_M \subseteq [-M,\, M]\times[0,\, 1]$. Hence,

$$\begin{aligned} \{\partial_t + \partial_x^3\}\, u_M(x,\, t) &= \{\partial_t + \partial_x^3\}\, [\, \phi_M(x)\, u(x,\, t)\,] \\ &= \phi_M\, \{\partial_t + \partial_x^3\}\, u(x,\, t) + 3\, \partial_x\phi_M\, \partial_x^2 u + 3\, \partial_x^2\phi_M\, \partial_x u + \partial_x^3\phi_M\, u \\ &= \phi_M\, \{\partial_t + \partial_x^3\}\, u(x,\, t) + E_1 + E_2 + E_3 \end{aligned} \tag{4.17}$$

using Lemma 4.4 to $u_M(x,\, t)$ we get

$$\begin{aligned} ||e^{\lambda\, x}\, u_M||_{L^8(\mathbb{R}\times[0,\,1])} &\le c\, ||e^{\lambda\, x}\{\partial_t + \partial_x^3\} u_M||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \\ &\le c\, ||e^{\lambda\, x}\, \phi_M\{\partial_t + \partial_x^3\}\, u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} + c \sum_{j=1}^{3} ||e^{\lambda\, x}\, E_j||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \end{aligned} \tag{4.18}$$

We show that the terms involving the $L^{8/7}$-norm of the error E_1, E_2 and E_3 in (4.18) tend to zero as $M \to \infty$.
We consider the case $x > 0$ and $\lambda > 0$. From (4.14) with $\beta > \lambda$ it follows that

$$
\begin{aligned}
\|e^{\lambda x} E_1\|^{8/7}_{L^{8/7}(\mathbb{R}\times[0,\,1])} &= 3^{8/7}\int_0^1\int_M^{2M} |e^{\lambda x}\partial_x\phi_M\,\partial_x^2 u|^{8/7}\,dx\,dt \\
&\leq c\int_0^1\int_M^{2M} \left|\frac{e^{\lambda x}}{M}\,\partial_x^2 u\right|^{8/7} dx\,dt \\
&\leq c\int_0^1\int_M^{2M} e^{8\,\lambda\,x/7}e^{-\,8\,\beta\,x/7}\,dx\,dt \qquad (4.19)\\
&= c\int_0^1\int_M^{2M} e^{-\frac{8}{7}(\beta-\lambda)\,x}\,dx\,dt \to 0 \quad \text{as} \quad M\to\infty. \qquad (4.20)
\end{aligned}
$$

Thus taking the limits as $M \to \infty$ in (4.17) and using (4.19) we obtain (4.15). In a similar way we have (4.17) and the Lemma 4.5 follows.

Lemma 4.6. *Suppose that*

$$(u,\,v) \in C([0,\,1];\; H^4(\mathbb{R})) \cap C^1([0,\,1];\; H^1(\mathbb{R})) \times C([0,\,1];\; H^4(\mathbb{R})) \cap C^1([0,\,1];\; H^1(\mathbb{R}))$$

satisfies the system

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u\,v^2) = 0 \qquad (x,\,t) \in \mathbb{R}\times[0,\,1] \\ \partial_t v + \partial_x^3 v + \partial_x(u^2\,v) = 0 \end{cases}$$

with

$$\text{supp}\; u(x,\,0) \subseteq (-\infty,\,b] \quad ; \quad \text{supp}\; v(x,\,0) \subseteq (-\infty,\,b].$$

Then for any $\beta > 0$,

$$\sum_{j\leq 2} |\partial_x^j u(x,\,t)| \leq c_b\,e^{-\beta\,x} \quad ; \qquad \sum_{j\leq 2} |\partial_x^j v(x,\,t)| \leq c_b\,e^{-\beta\,x}, \quad \text{for} \quad x>0, \quad t\in[0,\,1].$$

Proof. From Lemma 4.2.

Theorem 4.7. *Suppose that* $(u(x,\,t),\,v(x,\,t))$ *is a sufficiently smooth solution of the dispersive coupled system* (P). *If*

$$\text{supp}\; u(\,.\,,\,t_j) \subseteq (-\infty,\,b) \quad \textit{and} \quad \text{supp}\; v(\,.\,,\,t_j) \subseteq (-\infty,\,b), \qquad j=1,\,2.$$

or

$$\text{supp}\; u(\,.\,,\,t_j) \subseteq (a,\,\infty) \quad \textit{and} \quad \text{supp}\; v(\,.\,,\,t_j) \subseteq (a,\,\infty), \qquad j=1,\,2.$$

then

$$u(x,\,t) \equiv 0 \qquad \textit{and} \qquad v(x,\,t)\equiv 0.$$

Proof. Without loss of generality we assume that $t_1 = 0$ and $t_2 = 1$. Thus,

$$\text{supp}\; u(\,.\,,\,0),\; \text{supp}\; u(\,.\,,\,1) \subseteq (-\infty,\,b) \quad ; \quad \text{supp}\; v(\,.\,,\,0),\; \text{supp}\; v(\,.\,,\,1) \subseteq (-\infty,\,b).$$

We will show that there exists a large number $R > 0$ such that

$$\text{supp}\; u(\,.\,,\,t) \subseteq (-\infty,\,2R] \quad ; \quad \text{supp}\; v(\,.\,,\,t) \subseteq (-\infty,\,2R], \quad \forall\, t\in[0,\,1].$$

Then the result will follow from Theorem 3.7.

Let $\mu \in C_0^\infty(\mathbb{R})$ be a nondecreasing function such that

$$\mu(x) = \begin{cases} 0 & , \quad x \le 1 \\ 1 & , \quad x \ge 2 \end{cases}$$

and $0 \le \mu(x) \le 1,\ \forall\, x \in \mathbb{R}$. For each $R \neq 0$ we define $\mu_R(x) = \mu\left(\frac{x}{R}\right)$, i. e.,

$$\mu_R(x) = \begin{cases} 0 & , \quad x \le R \\ 1 & , \quad x \ge 2\,R. \end{cases}$$

Let $u_R(x,\,t) = \mu_R(x)\, u(x,\,t)$, then $u_R(x,\,t) \in C_0^\infty(\mathbb{R})$, since $(\mu_R\, u) \in C^\infty(\mathbb{R})$ and moreover

$$\text{supp}\ u_R = \text{supp}\ (\mu_R\, u) \subseteq \text{supp}\ \mu_R \cap \text{supp}\ u \subseteq \text{supp}\ \mu_R.$$

From the above inequality we have that supp $u_R \subseteq (-\infty,\, 2\,R]$. Using that u(v respectively) is a sufficiently smooth function (see [19]) and Lemma 4.6 we can apply Lemma 4.5 to $u_R(x,\,t)$ for R sufficiently larger. Thus

$$\begin{aligned} \{\partial_t + \partial_x^3\}[\,\mu_R \cdot u\,] &= \mu_R\,\{\partial_t u + \partial_x^3 u\} + 3\,\partial_x\mu_R \cdot \partial_x^2 u + 3\,\partial_x^2\mu_R \cdot \partial_x u + \partial_x^3\mu_R \cdot u \\ &= -\mu_R\,\{\partial_x u \cdot v^2 + u\,\partial_x(v^2)\} + 3\partial_x\mu_R \cdot \partial_x^2 u + 3\,\partial_x^2\mu_R \cdot \partial_x u + \partial_x^3\mu_R \cdot u \\ &= -\mu_R \cdot V_1 \cdot v + 3\,\partial_x\mu_R \cdot \partial_x^2 u + 3\,\partial_x^2\mu_R \cdot \partial_x u + \partial_x^3\mu_R \cdot u \\ &= -\mu_R \cdot V_1 \cdot v + F_1 + F_2 + F_3 \\ &= -\mu_R \cdot V_1 \cdot v + F_R \end{aligned} \tag{4.21}$$

where $V_1(x,\,t) = \partial_x u \cdot v + 2\,u \cdot \partial_x v$. The similar form

$$\begin{aligned} \{\partial_t + \partial_x^3\}\mu_R v &= -\mu_R \cdot V_2 \cdot u + 3\,\partial_x\mu_R \cdot \partial_x^2 v + 3\,\partial_x^2\mu_R \cdot \partial_x v + \partial_x^3\mu_R \cdot v \\ &= -\mu_R \cdot V_2 \cdot u + H_1 + H_2 + H_3 \\ &= -\mu_R \cdot V_2 \cdot u + H_R \end{aligned} \tag{4.22}$$

where $V_2(x,\,t) = \partial_x v \cdot u + 2\,v \cdot \partial_x u$. Then, by using Lemma 4.5

$$\begin{aligned} ||e^{\lambda\,x}\,\mu_R \cdot u||_{L^8(\mathbb{R}\times[0,\,1])} &\le c_0\,||e^{\lambda\,x}\{\partial_t + \partial_x^3\}\,\mu_R \cdot u||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \\ &\le c_0\,||e^{\lambda\,x}\,\mu_R \cdot V_1 \cdot v + e^{\lambda x}F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \qquad \text{(using (4.21))} \\ &\le c_0\,||e^{\lambda\,x}\mu_R \cdot V_1 \cdot v||_{L^{8/7}(\mathbb{R}\times[0,\,1])} + c_0\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \end{aligned} \tag{4.23}$$

We estimate the term $||e^{\lambda\,x}\,\mu_R \cdot V_1 \cdot v||_{L^{8/7}(\mathbb{R}\times[0,\,1])}$.

$$\begin{aligned} ||e^{\lambda\,x}\,\mu_R \cdot V_1 \cdot v||_{L^{8/7}(\mathbb{R}\times[0,\,1])} &= \left(\int_0^1 \int_{\mathbb{R}} |e^{\lambda\,x}\,\mu_R \cdot V_1 \cdot v|^{8/7}\,dx\,dt\right)^{8/7} \\ &= \left(\int_0^1 \int_{x>R} |e^{\lambda\,x}\,\mu_R \cdot v \cdot V_1|^{8/7}\,dx\,dt\right)^{8/7} \\ &\le \left(\int_0^1 \left[\int_{x\ge R} |e^{\lambda\,x}\,\mu_R \cdot v|^8\,dx\right]^{1/7} \left[\int_{x\ge R} |V_1|^{8/6}\,dx\right]^{6/7} dt\right)^{7/8} \\ &= \left(\int_0^1 \left[\int_{x\ge R} |e^{\lambda\,x}\,\mu_R \cdot v|^8 dx\right]^{1/7} \left[\int_{x\ge R} |V_1|^{4/3}\,dx\right]^{6/7} dt\right)^{7/8} \\ &= \left(\int_0^1 \left[\int_{\mathbb{R}} |e^{\lambda\,x}\mu_R \cdot v|^8\,dx\right]^{1/7} \left[\int_{x\ge R} |V_1|^{4/3}\,dx\right]^{6/7} dt\right)^{7/8} \\ &\le \left(\int_0^1 \int_{\mathbb{R}} |e^{\lambda\,x}\mu_R \cdot v|^8\,dx\,dt\right)^{1/8} \left(\int_0^1 \int_{x\ge R} |V_1|^{4/3}\,dx\,dt\right)^{3/4} \\ &= ||e^{\lambda\,x}\mu_R \cdot v||_{L^8(\mathbb{R}\times[0,\,1])}\,||V_1||_{L^{4/3}(\{x\ge R\}\times[0,\,1])} \end{aligned}$$

then

$$c_0\,||e^{\lambda\,x}\,\mu_R\cdot V_1\cdot v||_{L^{8/7}(\mathbb{R}\times[0,\,1])}\leq c_0\,||e^{\lambda\,x}\,\mu_R\cdot v||_{L^{8}(\mathbb{R}\times[0,\,1])}\,||V_1||_{L^{4/3}(\{x\geq R\}\times[0,\,1])}. \tag{4.24}$$

We define $V_1(x,\,t)=\partial_x u\cdot v+2\,u\cdot\partial_x v\in L^q(\mathbb{R}\times[0,\,1])$ with $q\in[0,\,\infty)$. Now, we fix R so large such that

$$c\,||V_1||_{L^{4/3}(\{x\geq R\}\times[0,\,1])}\leq\frac{1}{2}$$

then

$$c\,||e^{\lambda\,x}\,\mu_R\cdot V_1\cdot v||_{L^{8/7}(\mathbb{R}\times[0,\,1])}\leq\frac{1}{2}\,||e^{\lambda\,x}\,\mu_R\cdot v||_{L^{8}(\mathbb{R}\times[0,\,1])}$$

this way

$$||e^{\lambda\,x}\,\mu_R\cdot v||_{L^{8}(\mathbb{R}\times[0,\,1])}\leq\frac{1}{2}\,||e^{\lambda\,x}\,\mu_R\cdot v||_{L^{8}(\mathbb{R}\times[0,\,1])}+c\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])}$$

hence

$$\frac{1}{2}\,||e^{\lambda\,x}\,\mu_R\cdot v||_{L^{8}(\mathbb{R}\times[0,\,1])}\leq c\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])}$$

thus

$$||e^{\lambda\,x}\mu_R\cdot v||_{L^{8}(\mathbb{R}\times[0,\,1])}\leq 2\,c\,||e^{\lambda\,x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \tag{4.25}$$

To estimate the F_R term it suffices to consider one of the terms in F_R, say F_2, since the proofs for F_1, and F_3, are similar. We have that

$$\begin{aligned}F_R &= F_1+F_2+F_3\\ &= 3\,\partial_x\mu_R\cdot\partial_x^2 u+3\,\partial_x^2\mu_R\cdot\partial_x u+\partial_x^3\mu_R\cdot u\end{aligned}$$

and $\operatorname{supp}F_i,\subseteq[R,\,2R]$, $i=1,\,2,\,3$. We estimate F_2.

$$\begin{aligned}2\,c\,||e^{\lambda\,x}\,F_2||_{L^{8/7}(\mathbb{R}\times[0,\,1])} &= 2\,c\left(\int_0^1\int_{\mathbb{R}}|e^{\lambda\,x}\,F_2|^{8/7}\,dx\,dt\right)^{7/8}\\ &= 2\,c\left(\int_0^1\int_R^{2\,R}|3\,e^{\lambda\,x}\,\partial_x^2\mu_R\cdot\partial_x u|^{8/7}\,dx\,dt\right)^{7/8}\\ &= 2\,\frac{c}{R^2}\left(\int_0^1\int_R^{2R}e^{\frac{8}{7}\,\lambda\,x}\,|\partial_x^2\mu\cdot\partial_x u|^{8/7}\,dx\,dt\right)^{7/8}\\ &\leq 2\,\frac{c}{R^2}\left(\int_0^1\int_R^{2R}e^{\frac{8}{7}\,\lambda\,x}\,|\partial_x u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}.\end{aligned}$$

Then

$$2\,c\,||e^{\lambda\,x}\,F_2||_{L^{8/7}(\mathbb{R}\times[0,\,1])}\leq 2\,\frac{c}{R^2}\,e^{2\,\lambda\,R}\left(\int_0^1\int_R^{2R}|\partial_x u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}. \tag{4.26}$$

On the other hand,

$$||e^{\lambda\,x}\mu_R\cdot u||_{L^{8}(\mathbb{R}\times[0,\,1])}\geq\left(\int_0^1\int_{x>2R}e^{8\,\lambda\,x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8}$$

then

$$\left(\int_0^1 \int_{x>2R} e^{8\lambda x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8} \begin{aligned}[t] &\leq ||e^{\lambda x}\mu_R \cdot u||_{L^8(\mathbb{R}\times[0,\,1])} \\ &\leq 2\,c\,||e^{\lambda x}\,F_R||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \\ &\leq 2\,\frac{c}{R^2}\,e^{2\lambda R}\left(\int_0^1\int_R^{2R}|\partial_x u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}\end{aligned}$$

hence,

$$\left(\int_0^1 \int_{x>2R} e^{8\lambda x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8} \leq c_0\left(\int_0^1\int_R^{2R}|\partial_x u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}.$$

This way we have

$$\left(\int_0^1 \int_{x>2R} e^{8\lambda x}\,|u(x,\,t)|^8\,dx\,dt\right)^{1/8} \begin{aligned}[t] &\leq ||e^{\lambda x}\,\mu_R \cdot u||_{L^8(\mathbb{R}\times[0,\,1])} \\ &\leq 2\,c\,||e^{\lambda x}\,F_2||_{L^{8/7}(\mathbb{R}\times[0,\,1])} \qquad \text{(using (4.25))} \\ &\leq 2\,\frac{c}{R^2}\,e^{2\lambda R}\left(\int_0^1\int_R^{2R}|\partial_x u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8} \qquad \text{(using (4.26))}\end{aligned}$$

then

$$\left(\int_0^1 \int_{x>2R} e^{8\lambda(x-2R)}|u(x,\,t)|^8\,dx\,dt\right)^{1/8} \leq 2\,\frac{c}{R^2}\left(\int_0^1\int_R^{2R}|\partial_x u(x,\,t)|^{8/7}\,dx\,dt\right)^{7/8}$$

letting $\lambda \to \infty$ it follows that

$$u(x,\,t) \equiv 0 \qquad \text{for} \qquad x > 2R, \qquad t \in [0,\,1]$$

and in a similar way we obtain that

$$v(x,\,t) \equiv 0 \qquad \text{for} \qquad x > 2R, \qquad t \in [0,\,1]$$

which yields the proof.

Acknowledgement. The authors want to thank Prof. Carlos Picarte (Universidad del Bío-Bío) for his valuable help in the typesetting of this paper.

References

[1] M Ablowitz, D. Kaup, A. Newell, H. Segur, *Nonlinear evolution equations of physical significance*, Phys. Rev. Lett. 31 (2)(1973) 125-127.

[2] E. Alarcón, J. Angulo, J. F. Montenegro, *Stability and instability of solitary waves for a nonlinear dispersive system*, Nonlinear Analysis, 36(1999) 1015-1035.

[3] J. Angulo, F. Linares, *Global existence of solutions of a nonlinear dispersive model*, J. Math. Anal. Appl. 195(1995) 797-808.

[4] J. Bona, G. Ponce, J-C. Saut,M. M. Tom, *A model system for strong interaction between internal solitary waves*, Commun,. Math. Phys. 142(1992) 287-313.

[5] J. Bona, R. Smith, *The initial value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London, A278(1975) 555-604.

[6] J. Bourgain, *On the compactness of the support of solutions of dispersive equations*, Internat. Math. Res. Notices 9(1997) 437-447.
[7] T. Carleman, *Sur les systemes linéaires aux dérivées partielles du premier ordre a deux variables*, C. R. Acad. Sci. Paris, 197(1933) 471-474.
[8] J. Gear, R. Grimshaw, *Weak and strong interactions between internal solitary waves equation*, Stud. Appl. Math. 70(1984) 235-258.
[9] J Ginibre, Y. Tsutsumi, *Uniqueness of solutions for the generalized Korteweg-de Vries equation*, SIAM J. Math. Anal, 20(1989) 1388-1425.
[10] L. Hormander, *Linear Partial Differential Operators*, Springer.Verlag, Berlin/Heidelberg/New York, 1969.
[11] T. Kato, *On the cauchy problem for the (generalized) Korteweg-de Vries equation*, Advances in Mathematics Supplementary Studies in Applied Math. 8(1983) 93-128.
[12] C. E. Kenig, G. Ponce, L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana University Math. J. 40(1991) 33-69.
[13] C. E. Kenig, G. Ponce, L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. 46(1993) 527-620.
[14] C. E. Kenig, G. Ponce, L. Vega, *Higher-order nonlinear dispersive equations*, Proc. Amer. Math. Soc. 122(1994) 157-166.
[15] C. E. Kenig, G. Ponce, L. Vega, *On the support of solutions to the generalized KdV equation*, Analise non linéare 19(1992) 191-208.
[16] C. E. Kenig, A. Ruiz, C. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J. 55(1987) 329-347.
[17] C. E. Kenig, C. Sogge, *A note on unique continuation for Schrodinger's operator*, Proc. Amer. Math. Soc. 103(1988) 543-546.
[18] V. Komornik, V. D. L. Russell, B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation on a periodic domain*, J. Differential Equations, to appear.
[19] F. Linares, M. Panthee, *On the Cauchy problem for a coupled system of KdV equation*, Communications on Pure and Applied Analysis (2003) 417-431.
[20] S. Mizohata, *Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques*, Mem. Coll. Sci. Univ. Kyoto A31(1958) 219-239.
[21] L. Nirenberg, *Uniqueness of Cauchy problems for differential equations with constant leading coefficient*, Comm. Pure Appl. Math. 10(1957) 89-105.
[22] J. Peetre, *Espaces d'interpolation et théoréme de Sobolev*, Ann. Inst. Fourier, 16(1966) 279-317.
[23] L. Robbiano, *Théoréme d'unicité adapté au contróle des solutions des problemes hyperboliques*, Comm. PDE 16(1991) 789-800.
[24] J-C. Saut, B. Scheurer, *Unique continuation for some evolution equations*, J. Diff. Eqs. 66(1987) 118-139.
[25] J-C. Saut, R. Temam, *Remark on the Korteweg-de Vries equation*, Israel J. Math. 24(1976) 78-87.
[26] M. Schechter, B. Simon, *Unique continuation for Schrodinger o perators with unbounded potentials*, J. Math. Anal. Appl. 77(1980) 482-492.
[27] T. Schombek, Uniqueness for a non-linear heat equation, Math. Comput. Modelling 18(1993) 65-77.
[28] R. Temam, *Sur un probleme non linéaire*, J. Math. Pures Appl. 48(1969) 157-172.
[29] O. Vera, *Gain of regularity for a coupled system of K-dV equations*, Ph. D. Thesis, U.F.R.J. Brasil(2001).
[30] O. Vera, *Gain of regularity for a generalized coupled system of K-dV equations*, Ph. D. Thesis, U.F.R.J. Brasil(2001).
[31] B.-Y. Zhang, *Hardy function and unique continuation for evolution equations*, J. Math. Anal. Appl. 178 (1993) 381-403.
[32] B.-Y. Zhang, *Unique continuation for the Korteweg-de Vries equation*, SIAM J. Math. Anal. 23, 55-71.

(Received September 28, 2004)