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Existence Results for Impulsive Dynamic Inclusions on Time Scales

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Dedicated to the memory of Professor Bernd Aulbach

Abstract

In this paper, we investigate the existence of solutions and extremal solutions for a first order impulsive dynamic inclusion on time scales. By using suitable fixed point theorems, we study the case when the right hand side has convex as well as nonconvex values.

Key words and phrases: Impulsive dynamic inclusions, delta derivative, contraction, extremal solutions, fixed point, time scales.
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1 Introduction

This paper is concerned with the existence of solutions and extremal solutions for a class of initial value problem for impulsive dynamic inclusions on time scales. More precisely, in Section 3, we consider the following problem:

$$y^{\Delta}(t) + p(t)y^{\sigma}(t) \in F(t, y(t)),\ t \in J := [0, b] \cap \mathbb{T},\ t \neq t_k, \quad k = 1, \ldots, m, \tag{1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{2}$$

$$y(0) = \eta, \tag{3}$$

where $\mathbb{T}$ is a time scale, $F : [0, b] \times \mathrm{I\!R} \to \mathcal{P}(\mathrm{I\!R})$ is a compact valued multi-valued map, $\mathcal{P}(\mathrm{I\!R})$ is the family of all nonempty subsets of $\mathrm{I\!R}$, $I_k \in C(\mathrm{I\!R}, \mathrm{I\!R}), \eta \in \mathrm{I\!R}$, $0 = t_0 < t_1 < ... < t_m < t_{m+1} = b$, and for each $k = 1, \ldots, m$,

$y(t_k^+) = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$ in the sense of time scales, that is, $t_k + h \in [0, b] \cap \mathbb{T}$ for each h in a neighborhood of 0 and in addition, if t_k is right scattered, then $y(t_k^+) = y(t_k)$, whereas,

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if t_k is left scattered, then $y(t_k^-) = y(t_k)$, σ is a function that will be defined later and $y^\sigma(t) = y(\sigma(t))$.

Impulsive differential equations have become important in recent years in mathematical models of real processes and they arise in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Lakshmikantham *et al* [22], Samoilenko and Perestyuk [25] and the references therein. In recent years dynamic equations on time scales have received much attention. We refer to the books by Bohner and Peterson [9, 10], Lakshmikantham *et al* [23] and to the references cited therein. The time scales calculus has tremendous potential for applications in mathematical models of real processes and phenomena, for example in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, see the monographs of Aulbach and Hilger [2], Bohner and Peterson [9, 10], Lakshmikantham *et al* [23] and to the references therein. Recently Henderson [18] and Benchohra *et al* [1, 7, 8] have initiated the study of impulsive dynamic equations on time scales. The first paper for impulsive dynamic inclusions was proposed by Belarbi, Benchohra and Ouahab [4]. In this paper, we continue this study by considering more general classes of impulsive dynamic inclusions on time scales. We shall provide existence results for the problem (1)-(3). The first one relies on the nonlinear alternative of Leray-Schauder type [16] when the right hand side is convex valued, the second and the third rely also on the nonlinear alternative of Leray-Schauder type but under weaker conditions on the functions $I_k \, (k = 1, ..., m)$ and the mixed generalized Lipschitz and Carathéodory's conditions and the last one on the fixed point theorem for contraction multi-valued maps due to Covitz and Nadler [13] when the right hand side is not necessarily convex valued. The last section is concerned with the existence of extremal solutions of the above mentioned problem by using a recent fixed point theorem due to Dhage [14] for the sum of a contraction multivalued map and a completely continuous one defined on ordered Banach spaces. These results complement the few existence results devoted to dynamic inclusions on time scales.

2 Preliminaries

We will briefly recall some basic definitions and facts from times scales calculus that we will use in the sequel.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) =$

$t,\ \sigma(t) > t$ respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The notations $[0,b], [0,b)$, and so on, will denote time scales intervals

$$[0,b] = \{t \in \mathbb{T} : a \le t \le b\},$$

where $0, b \in \mathbb{T}$ with $0 < \rho(b)$.

Definition 2.1 *Let X be a Banach space. The function $f : \mathbb{T} \to X$ will be called rd−continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write $f \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, X)$.*

Definition 2.2 *Let $t \in \mathbb{T}^k$, the Δ derivative of f at t, denoted $f^{\Delta}(t)$, be the number(provided it exists) if for all $\varepsilon > 0$ there exists a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$, at fix t.

A function F is called antiderivative of $f : \mathbb{T} \to X$ provided

$$F^{\Delta}(t) = f(t) \ \text{ for each } t \in \mathbb{T}^k.$$

Remark 2.3 *(i) If f is continuous, then f rd−continuous.*

(ii) If f is delta differentiable at t then f is continuous at t.

A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* if

$$1 + \mu(t)p(t) \ne 0 \ \text{ for all } \ t \in \mathbb{T},$$

where $\mu(t) = \sigma(t) - t$, which is called the *graininess function*. We denote by $\mathcal{R}^+$ the set of the regressive functions. The generalized exponential function e_p is defined as the unique solution of the initial value problem $y^{\Delta} = p(t)y,\ y(0) = 1$, where p is a regressive function. An explicit formula for $e_p(t,0)$ is given by

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \text{ with } \xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1+\mathrm{hz})}{h} & \text{if } h \ne 0, \\ z & \text{if } h = 0. \end{cases}$$

For more details, see [9]. Clearly, $e_p(t,s)$ never vanishes. We now give some fundamental properties of the exponential function. Let $p, q : \mathbb{T} \to \mathbb{R}$ two regressive functions. We define

$$p \oplus q = p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Theorem 2.4 *[9] Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are regressive functions, then the following hold:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s)\dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

$C([0,b],\mathbb{R})$ is the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in [0,b]\}.$$

$L^1([0,b],\mathbb{R})$ denote the space of functions from $[0,b]$ into $\mathbb{R}$ which are Lebesgue integrable in the time scale sense normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\Delta t \text{ for each } y \in L^1([0,b],\mathbb{R})$$

$AC((0,b),\mathbb{R})$ is the space of differentiable functions $y : (0,b) \to \mathbb{R}$ whose first delta derivative, y^Δ, is absolutely continuous.

Let $(X, |\cdot|)$ be a normed space, $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_c(X) = \{Y \in P(X) : Y \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$. A multivalued map $N : [0,b] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be *measurable*, if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, N(t)) = \inf\{|y-z| : z \in N(t)\}$ is measurable where d is the metric induced by the Banach space $\mathbb{R}$. In what follows, we will assume that the function $F : [0,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory, i.e.

(i) $t \to F(t,x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \to F(t,x)$ is upper semicontinuous for almost all $t \in [0,b]$,

For each $y \in C([0,b],\mathbb{R})$, let $S_{F,y}$ the set of selections of F defined by

$$S_{F,y} = \{v \in L^1([0,b],\mathbb{R}) : v(t) \in F(t,y(t)),\ a.e.\ t \in [0,b]\}.$$

The following Lemma is crucial in the proof of our main results when the multivalued map has convex values:

Lemma 2.5 *[24]. Let X be a Banach space. Let $F : J \times X \longrightarrow P_{cp,c}(X)$ be a Carathéodory multivalued map and let Γ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$, then the operator*

$$\begin{array}{rl}\Gamma \circ S_F : C(J,X) & \longrightarrow P_{cp,c}(C(J,X)), \\ y & \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F(y)})\end{array}$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

3 Existence Results

We will assume for the remainder of this paper that, for each $k = 1, \ldots, m$, the points of impulse t_k are right dense. In order to define the solution of (1)–(3), we shall consider the following space:

$$\begin{aligned} PC \ &= \{y : [0,b] \longrightarrow \mathbb{R} : y_k \in C(J_k, \mathbb{R}), k = 0, \ldots, m, \text{ and there exist } y(t_k^-) \\ &\quad \text{and } y(t_k^+) \text{ with } y(t_k^-) = y(t_k),\ k = 1, \ldots, m\}, \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_{J_k},\ k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}] \subset [0,b],\ k = 1, \ldots, m$, and $J_0 = [t_0, t_1]$.

Let us start by defining what we mean by a solution of problem (1)-(3).

Definition 3.1 *A function $y \in PC \cap AC(J \backslash \{t_1, \ldots t_m\}, \mathbb{R})$ is said to be a solution of (1)–(3) if there exists a function $v \in L^1([0,b], \mathbb{R})$ such that*

$$y^{\Delta}(t) + p(t) y^{\sigma}(t) = v(t) \qquad a.e. \ on \ J \backslash \{t_k\},\ k = 1, \ldots, m,$$

and for each $k = 1, \ldots, m$, the function y satisfies the condition $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-))$, and the intial condition $y(0) = \eta$.

We need the following auxiliary result (see [7]).

Lemma 3.2 *Let $p : \mathbb{T} \to \mathbb{R}$ be $rd-$continuous and regressive. Suppose $f : \mathbb{T} \to \mathbb{R}$ $rd-$continuous. Let $t_0 \in \mathbb{T}$, and $y_0 \in \mathbb{R}$. Then, y is the unique solution of the initial value problem*

$$y^{\Delta}(t) + p(t) y^{\sigma}(t) = f(t),\ t \in [0,b] \cap \mathbb{T},\ t \neq t_k, k = 1, \ldots, m \tag{4}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{5}$$

$$y(0) = y_0, \tag{6}$$

if and only if

$$y(t) = e_{\ominus p}(t, 0) y_0 + \int_0^t e_{\ominus p}(t,s) f(s) \Delta s + \sum_{0 < t_k < t} e_{\ominus p}(t, t_k) I_k(y(t_k^-)). \tag{7}$$

Let us introduce the following hypotheses which are assumed hereafter:

(H1) The function $F : [0,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory.

(H2) There exist constants $c_k > 0$ such that

$$|I_k(x)| \leq c_k \quad \text{for each } k = 1, \ldots, m \text{ and for all } x \in \mathbb{R}.$$

(H3) There exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$, a function $p \in L^1([0,b], \mathbb{R}_+)$ and a constant $M > 0$ such that

$$\|F(t,x)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,x)\} \leq p(t)\psi(|x|) \text{ for each } (t,x) \in [0,b] \times \mathbb{R},$$

and

$$\frac{M}{|\eta| \sup\limits_{t\in[0,b]} e_{\ominus p}(t,0) + \sum\limits_{k=1}^{m} c_k \sup\limits_{t\in[0,b]} e_{\ominus p}(t,t_k) + \sup\limits_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\psi(M) \int_0^b p(s)\Delta s} > 1.$$

Theorem 3.3 *Suppose that hypotheses (H1)–(H3) hold. Then the impulsive dynamic inclusions (1)–(3) has at least one solution on* $[0,b]$.

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operator $N : PC \longrightarrow \mathcal{P}(PC)$ defined by

$$\begin{aligned} N(y) &= \{h \in PC : \ h(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v(s)\Delta s \\ &\quad + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)), \ v \in S_{F,y}\}. \end{aligned}$$

Remark 3.4 *Clearly, from Lemma 3.2, the fixed points of* N *are solutions to (1)–(3).*

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in PC$.

Indeed, if $h_1, \ h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in [0,b]$ we have

$$h_i(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v_i(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)) \, (i = 1,2).$$

Let $0 \leq d \leq 1$. Then, for each $t \in [0,b]$ we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) &= e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)[dv_1(s) + (1-d)v_2(s)]\Delta s \\ &\quad + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)). \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *N maps bounded sets into bounded sets in PC.*

Let $B_q = \{y \in PC : \|y\|_{PC} \leq q\}$ be a bounded set in PC and $y \in B_q$, then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that for each $t \in [0,b]$,

$$h(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)).$$

From (H2) and (H3) we have

$$\begin{aligned}
|h(t)| &\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s) \int_0^b |v(s)|\Delta s \\
&+ \sum_{k=0}^{m} e_{\ominus p}(t,t_k)c_k \\
&\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s) \int_0^b \psi(q)p(s)\Delta s \\
&+ \sum_{k=0}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)c_k \\
&\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\psi(q)\|p\|_{L^1} \\
&+ \sum_{k=0}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)c_k.
\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of PC.*

Let $u_1, u_2 \in J,\ u_1 < u_2$ and B_q be a bounded set of PC as in Step 2 and $y \in B_q$. For each $h \in N(y)$, there exists $v \in S_{F,y}$ such that for each $t \in [0,b]$,

$$h(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)).$$

Then, we have

$$\begin{aligned}|h(u_2)-h(u_1)| &\le |e_{\ominus p}(u_2,0)-e_{\ominus p}(u_1,0)||\eta| \\ &+\psi(q)\|p\|_{L^1}\int_0^{u_1}|e_{\ominus p}(u_2,s)-e_{\ominus p}(u_1,s)|\Delta s \\ &+\psi(q)\|p\|_{L^1}\int_{u_1}^{u_2}e_{\ominus p}(u_2,s)\Delta s \\ &+\sum_{0\le t_k<u_1}|e_{\ominus p}(u_2,t_k)-e_{\ominus p}(u_1,t_k)|c_k \\ &+\sum_{u_1\le t_k<u_2}e_{\ominus p}(u_2,t_k)c_k.\end{aligned}$$

The right hand side tends to zero as $u_2-u_1\to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli Theorem, we can conclude that $N: PC \longrightarrow \mathcal{P}(PC)$ is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n\to y_*$, $h_n\in N(y_n)$ and $h_n\to h_*$. We need to show that $h_*\in N(y_*)$. $h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that for each $t\in[0,b]$,

$$h_n(t)=e_{\ominus p}(t,0)\eta+\int_0^t e_{\ominus p}(t,s)v_n(s)\Delta s+\sum_{0<t_k<t}e_{\ominus p}(t,t_k)I_k(y_n(t_k^-)).$$

We must show that there exists $h_*\in S_{F,y_*}$ such that for each $t\in[0,b]$,

$$h_*(t)=e_{\ominus p}(t,0)\eta+\int_0^t e_{\ominus p}(t,s)v_*(s)\Delta s+\sum_{0<t_k<t}e_{\ominus p}(t,t_k)I_k(y_*(t_k^-)).$$

Clearly, since $I_k,\ k=1,\dots,m$, are continuous, we have

$$\Big\|\Big(h_n-\sum_{0<t_k<t}e_{\ominus p}(t,t_k)I_k(y_n(t_k^-))\Big)-\Big(h_*-\sum_{0<t_k<t}e_{\ominus p}(t,t_k)I_k(y_*(t_k^-))\Big)\Big\|_{PC}\longrightarrow 0,\ \text{as } n\to\infty.$$

Consider the continuous linear operator

$$\Gamma: L^1([0,b],\mathrm{I\!R})\to C([0,b],\mathrm{I\!R})$$

given by

$$v\longmapsto(\Gamma v)(t)=\int_0^t e_{\ominus p}(t,s)v(s)ds.$$

>From Lemma 2.5, it follows that $\Gamma\circ S_F$ is a closed graph operator. Moreover, we have

$$\Big(h_n(t)-\sum_{0<t_k<t}e_{\ominus p}(t,t_k)I_k(y_n(t_k^-))\Big)\in\Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 2.5 that for each $t \in [0,b]$,

$$h_*(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v_*(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y_*(t_k^-)),$$

for some $v_* \in S_{F,v_*}$.

Step 5: *A priori bounds on solutions.*

Let y be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then, there exists $v \in S_{F,y}$ such that for each $t \in [0,b]$,

$$y(t) = \lambda e_{\ominus p}(t,0)\eta + \lambda \int_0^t e_{\ominus p}(t,s)v(s)\Delta s + \lambda \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)).$$

This implies by (H2) and (H3) that, for each $t \in [0,b]$,

$$\begin{aligned} |y(t)| &\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sum_{k=1}^m c_k \sup_{t\in[0,b]} e_{\ominus p}(t,t_k) \\ &\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s) \int_0^b p(s)\psi(|y(s)|)\Delta s \\ &\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sum_{k=1}^m c_k \sup_{t\in[0,b]} e_{\ominus p}(t,t_k) \\ &\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\psi(\|y\|_{PC}) \int_0^b p(s)\Delta s. \end{aligned}$$

Consequently

$$\frac{\|y\|_{PC}}{|\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sum_{k=1}^m c_k \sup_{t\in[0,b]} e_{\ominus p}(t,t_k) + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\psi(\|y\|_{PC}) \int_0^b p(s)\Delta s} \leq 1.$$

Then by (H3), there exists M such that $\|y\|_{PC} \neq M$.
Let

$$U = \{y \in PC : \|y\|_{PC} < M\}.$$

The operator $N : \overline{U} \to \mathcal{P}(PC)$ is upper semicontinuous and completely continuous. >From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [16], we deduce that N has a fixed point y in U which is a solution of the problem (1)–(3).

We now present two other existence results for the problem (1)–(3) when the right hand side has convex values under weaker conditions on the functions I_k $(k = 1, ..., m)$ (as used in [11] for impulsive differential inclusions).

Theorem 3.5 *In addition to (H1), assume that the following conditions hold:*

(H4) There exist constants $c_k > 0$ such that

$$|I_k(x)| \leq c_k|x| \text{ for each } k = 1, ..., m \text{ and all } x \in \mathbb{R}.$$

(H5) $H_d(F(t,y), F(t,\overline{y})) \leq l(t)|y-\overline{y}|$ for each $t \in [0,b]$ and all $y, \overline{y} \in \mathbb{R}$ where $l \in L^1([0,b], \mathbb{R}_+) \cap \mathcal{R}^+$ and $d(0, F(t,0)) \leq l(t)$ a.e $t \in [0,b]$.

If

$$\sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1} + \sum_{k=1}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)c_k < 1,$$

then the problem (1)-(3) has at least one solution on $[0,b]$.

Proof. Let y be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then, there exist $v \in S_{F,y}$ such that for each $t \in [0,b]$,

$$y(t) = \lambda e_{\ominus p}(t,0)\eta + \lambda \int_0^t e_{\ominus p}(t,s)v(s)\Delta s + \lambda \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)).$$

This implies by (H4) and (H5) that for each $t \in [0,b]$,

$$\begin{aligned}
|y(t)| &\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sum_{k=1}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)c_k|y(t_k^-)| \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s) \int_0^b |v(s)|\Delta s. \\
&\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sum_{k=1}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)c_k|y(t_k^-)| \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s) \int_0^b |l(s)y(s) + l(s)|\Delta s \\
&\leq |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sum_{k=1}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)c_k\|y\|_{PC} \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|y\|_{PC}\|l\|_{L^1} \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1}.
\end{aligned}$$

Consequently

$$\|y\|_{PC} \leq \frac{|\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1}}{1 - \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1} - \sum_{k=1}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)c_k} := M.$$

Let

$$U = \{y \in PC : \|y\|_{PC} < M+1\}.$$

The operator $N : \overline{U} \to \mathcal{P}(PC)$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [16], we deduce that N has a fixed point y in U which is a solution of the problem (1)–(3).

Theorem 3.6 *In addition to (H1) and (H5), assume that the following conditions hold*

(H6) $\lim_{|x|\to+\infty} \frac{I_k(x)}{x} = 0$ *for each* $k = 1, ..., m.$

If

$$\sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1} + \sum_{k=1}^{m} \sup_{t\in[0,b]} e_{\ominus p}(t,t_k)\varepsilon_k < 1,$$

where $\varepsilon_k,\ k = 1,\ldots,m$ *are positive constants that will specified later, then the problem (1)–(3) has at least one solution on* $[0,b]$.

Proof. Let y be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then, there exist $v \in S_{F,y}$ such that for each $t \in [0,b]$,

$$y(t) = \lambda e_{\ominus p}(t,0)\eta + \lambda \int_0^t e_{\ominus p}(t,s)v(s)\Delta s + \lambda \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)).$$

(H6) implies that for each $\varepsilon_k > 0$, there exists a constant $A > 0$ such that

$$|x| \geq A \Rightarrow |I_k(x)| \leq \varepsilon_k |x|.$$

Let

$$E_1 = \{t; t \in [0,b] : \ |x(t)| < A\},$$

$$E_2 = \{t; t \in [0,b] : \ |x(t)| \geq A\}$$

and

$$C_1 = \max\{|I_k(x(t))|, \ t \in E_1\}.$$

By (H5) and (H6), for each $t \in [0, b]$,

$$\begin{aligned}
|y(t)| &\le |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + \sum_{t_k \in E_1} e_{\ominus p}(t,t_k)|I_k(y(t_k^-))| \\
&\quad + \sum_{t_k \in E_2} e_{\ominus p}(t,t_k)|I_k(y(t_k^-))| \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|y\|_{PC}\|l\|_{L^1} \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1} \\
&\le |\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + C_1 \sum_{k=1}^{m} \sup_{t\in E_1} e_{\ominus p}(t,t_k) \\
&\quad + \sum_{k=1}^{m} \sup_{t\in E_2} e_{\ominus p}(t,t_k)\varepsilon_k \|y\|_{PC} \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|y\|_{PC}\|l\|_{L^1} \\
&\quad + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1}.
\end{aligned}$$

Consequently

$$\|y\|_{PC} \le \frac{|\eta| \sup_{t\in[0,b]} e_{\ominus p}(t,0) + C_1 \sum_{k=1}^{m} \sup_{t\in E_1} e_{\ominus p}(t,t_k) + \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1}}{1 - \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t,s)\|l\|_{L^1} - \sum_{k=1}^{m} \sup_{t\in E_2} e_{\ominus p}(t,t_k)\varepsilon_k} := \bar{M}.$$

Let

$$U = \{y \in PC : \|y\|_{PC} < \bar{M} + 1\}.$$

The operator $N : \overline{U} \to \mathcal{P}(PC)$ is upper semicontinuous and completely continuous. >From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [16], we deduce that N has a fixed point y in U which is a solution of the problem (1)–(3).

We present now a result for the problem (1)-(3) with a nonconvex valued right hand side. Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{ \sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b) \right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [21]).

Definition 3.7 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multivalued operator N will be denoted by $FixN$. For more details on multivalued maps we refer to the books of Deimling [15], Gorniewicz [17], Hu and Papageorgiou [20] and Tolstonogov [26].

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [13] (see also Deimling, [15] Theorem 11.1).

Lemma 3.8 *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Let us introduce hypotheses which are assumed hereafter

(H8) $F : [0, b] \times \mathbb{R} \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ has the property that $F(\cdot, y) : [0, b] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$;

(H9) There exist constants $d_k \geq 0$ such that

$$|I_k(y) - I_k(\overline{y})| \leq d_k|y - \overline{y}| \text{ for each } \quad y, \overline{y} \in \mathbb{R}.$$

Theorem 3.9 *Assume that (H5), (H8)-(H9) are satisfied. Let $\tau > 1$. If*

$$\frac{1}{\tau} + \sum_{k=1}^{m} G_* d_k < 1, \text{ where } G_* = \sup_{(t,s)\in[0,b]\times[0,b]} e_{\ominus p}(t, s),$$

then the IVP (1)-(3) has at least one solution on $[0, b]$.

Remark 3.10 *For each $y \in PC$, the set $S_{F,y}$ is nonempty since by (H8), F has a measurable selection (see [12], Theorem III.6).*

Proof. We shall show that N satisfies the assumptions of Lemma 3.8. The proof will be given in two steps.

Step 1: *$N(y) \in \mathcal{P}_{cl}(PC)$ for each $y \in PC$.*

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in PC. Then $\tilde{y} \in PC$ and there exists $v_n \in S_{F,y}$ such that for each $t \in [0, b]$

$$y_n(t) = e_{\ominus p}(t, 0)\eta + \int_0^t e_{\ominus p}(t, s)v_n(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t, t_k)I_k(y(t_k^-)).$$

Using the fact that F has compact values and from (H5), we may pass to a subsequence if necessary to get that v_n converges to v in $L^1([0,b],\mathbb{R})$ and hence $v \in S_{F,y}$. Then, for each $t \in [0,b]$,

$$y_n(t) \longrightarrow \tilde{y}(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)).$$

So, $\tilde{y} \in N(y)$.

Step 2: *There exists $\gamma < 1$ such that*

$$H_d(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\| \text{ for each } y, \overline{y} \in PC.$$

Let $y, \overline{y} \in PC$ and $h_1 \in N(y)$. Then, there exists $v_1(t) \in F(t, y(t))$ such that for each $t \in [0,b]$,

$$h_1(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v_1(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(y(t_k^-)).$$

From (H5) it follows that for each $t \in [0,b]$,

$$H_d(F(t,y(t)), F(t,\overline{y}(t))) \leq l(t)|y(t) - \overline{y}(t)|.$$

Hence, there exists $w \in F(t, \overline{y}(t))$ such that for each $t \in [0,b]$,

$$\|v_1(t) - w\| \leq l(t)|y(t) - \overline{y}(t)|.$$

Consider $U : [0,b] \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : \|v_1(t) - w\| \leq l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition III.4 in [12]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}(t))$ and for each $t \in [0,b]$,

$$\|v_1(t) - v_2(t)\| \leq l(t)|y(t) - \overline{y}(t)|.$$

Let us define for each $t \in [0,b]$,

$$h_2(t) = e_{\ominus p}(t,0)\eta + \int_0^t e_{\ominus p}(t,s)v_2(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t,t_k)I_k(\overline{y}(t_k^-))$$

We define on PC an equivalent norm by

$$\|y\|_* = \sup_{t \in J} e_{\ominus(\tau G_* l)}(t,0)|y(t)| \quad \text{for all } y \in PC,$$

where $e_{\ominus(\tau G_* l)}(t,0)$ is the unique solution of the problem

$$y^{\Delta}(t) = \tau G_* l(t) y(t),\ \ y(0) = 1,$$

where $\tau G_* l$ is a regressive function. >From (H5) and (H9), for each $t \in [0,b]$,

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq \int_0^1 e_{\ominus p}(t,s)\|v_1(s) - v_2(s)\|\Delta s \\
&+ \sum_{k=1}^{m} e_{\ominus p}(t,s)|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\
&\leq \int_0^t G_* l(s)|y(s) - \overline{y}(s)|\Delta s \\
&+ \sum_{k=1}^{m} G_* d_k |y(t_k^-) - \overline{y}(t_k^-)| \\
&\leq \tfrac{1}{\tau}\int_0^t \tau G_* l(s)|y(s) - \overline{y}(s)|\Delta s \\
&+ \sum_{k=1}^{m} d_k G_* |y(t_k^-) - \overline{y}(t_k^-)| \\
&\leq \tfrac{1}{\tau}\int_0^t \tau G_* l(s) e_{\tau G_* l}(s,0) e_{\tau G_* l}(0,s)|y(s) - \overline{y}(s)|\Delta s \\
&+ \frac{\tau}{\tau}\sum_{k=1}^{m} d_k G_* |y(t_k^-) - \overline{y}(t_k^-)| \\
&\leq \frac{1}{\tau}\int_0^t [e_{\tau G_* l}(s,0)]^{\Delta} e_{\ominus(\tau G_* l)}(s,0)|y(s) - \overline{y}(s)|\Delta s \\
&+ \frac{\tau}{\tau}\sum_{k=1}^{m} d_k G_* |y(t_k^-) - \overline{y}(t_k^-)| \\
&\leq \frac{1}{\tau} e_{\tau G_* l}(t,0)\|y - \overline{y}\|_* + \sum_{k=1}^{m} d_k G_* e_{\tau G_* l}(t,0)\|y - \overline{y}\|_*.
\end{aligned}$$

Thus

$$\|h_1 - h_2\|_* \leq \left(\frac{1}{\tau} + \sum_{k=1}^{m} G_* d_k\right)\|y - \overline{y}\|_*.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \left(\frac{1}{\tau} + \sum_{k=1}^{m} G_* d_k\right)\|y - \overline{y}\|_*.$$

So, N is a contraction and thus, by Lemma 3.8, N has a fixed point y which is solution to (1)-(3).

4 Extremal Solutions

We equip the space PC with the order relation " $\leq$ " defined by the cone K in PC, that is

$$K = \{y \in PC : y(t) \geq 0, \ t \in [0, b]\}.$$

It is known that the cone K is normal in PC. The details of cones and their properties may be found in Heikkila and Lakshmikantham [19]. Let $\alpha, \beta \in PC$ such that $\alpha \leq \beta$. then by an order interval $[\alpha, \beta]$ we mean a set of points in PC given by

$$[\alpha, \beta] = \{y \in PC : \alpha \leq y \leq \beta\}.$$

Let $D, Q \in \mathcal{P}_{cl}(PC)$. Then by $D \leq Q$ we mean $\alpha \leq \beta$ for all $\alpha \in D$ and $\beta \in Q$. Thus $\alpha \leq D$ implies that $\alpha \leq y$ for all $y \in Q$ in particular, if $D \leq D$, then it follows that D is a singleton set.

Definition 4.1 *Let X be an ordered Banach space. A mapping $N : X \to \mathcal{P}_{cl}(X)$ is called isotone increasing if $x, y \in X$ with $x < y$, then we have that $N(x) \leq N(y)$.*

Theorem 4.2 *[14] Let $[a, b]$ be an order interval in a Banach space and let $A, B : [a, b] \to \mathcal{P}_{cl}(X)$ be two multivalued operators satisfying*

(i) A is multivalued contraction,

(ii) B is completely continuous,

(iii) A and B are isotone increasing, and

(iv) $A(x) + B(x) \subset [a, b]$ for all $x \in [a, b]$.

Further if the cone K in X is normal, then the operator inclusion $x \in A(x) + B(x)$ has a least fixed point x_* and a greatest fixed point x^* in $[a, b]$. Moreover $x_* = \lim_{n\to\infty} x_n$ and $x^* = \lim_{n\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by

$$x_{n+1} \in A(x_n) + B(x_n), x_0 = a \ \textit{and} \ y_{n+1} \in A(y_n) + B(y_n), \ y_0 = b.$$

The following concept of lower and upper solutions for (1)-(3) has been introduced by Benchohra, Henderson, and Ntouyas [6] for periodic boundary value problems for impulsive differential inclusions at fixed moments (see also [5]). It will be the basic tool in the approach that follows.

Definition 4.3 *A function $\alpha \in PC$ is said to be a lower solution of (1)-(3) if there exists $v_1 \in L^1(J, \mathbb{R})$ such that $v_1(t) \in F(t, \alpha(t))$ a.e. on J, $\alpha^\Delta(t) + p(t)\alpha^\sigma(t) \leq v_1(t)$ a.e. on J, $t \neq t_k$, $\alpha(t_k^+) - \alpha(t_k^-) \leq I_k(\alpha(t_k^-))$, $t = t_k$, $k = 1, \ldots, m$, and $\alpha(0) \leq \eta$. Similarly, a function $\beta \in PC$ is said to be an upper solution of (1)-(3) if there exists $v_2 \in L^1(J, \mathbb{R})$ such that $v_2(t) \in F(t, \beta(t))$ a.e. on J, $\beta^\Delta(t) + p(t)\beta^\sigma(t) \geq v_2(t)$ a.e. on J, $t_k \neq t_k$, $\beta(t_k^+) - \beta(t_k^-) \geq I_k(\beta(t_k^-))$, $t = t_k$, $k = 1, \ldots, m$, and $\beta(0) \geq \eta$.*

The following hypotheses will be assumed hereafter:

(A1) The multifunction $F(t, y)$ is nondecreasing in y almost everywhere for $t \in [0, b]$;

(A2) The functions $I_k,\ k = 1, \ldots, m$ are continuous and nondecreasing.

(A3) There exist α and $\beta \in PC$, respectively lower and upper solutions for the problem (1)–(3) such that $\alpha \leq \beta$.

Theorem 4.4 *Suppose that hypotheses (h5), (H9), (A1)–(A3) are satisfied. Then the impulsive IVP (1)–(3) has minimal and maximal solutions on* $[0, b]$.

Proof. Define two multivalued maps $A, B : PC \to \mathcal{P}(PC)$ by

$$A(y) = \{h \in PC : \ h(t) = e_{\ominus p}(t, 0)\eta + \int_0^t e_{\ominus p}(t, s)v(s)\Delta s, \ v \in S_{F,y}\},$$

and

$$B(y) = \{h \in PC : \ h(t) = \sum_{0<t_k<t} e_{\ominus p}(t, t_k)I_k(y(t_k^-))\}.$$

It can be shown, as in the proofs of Theorems 3.3 and 3.9, that A and B define the multi-valued operators $A : [\alpha, \beta] \to \mathcal{P}_{cl,cv,bd}(PC)$ and $B : [\alpha, \beta] \to \mathcal{P}_{cp,cv}(PC)$. It can be similarly shown that A and B are respectively multi-valued contraction and completely continuous on $[\alpha, \beta]$. We shall show that A and B are isotone increasing on $[\alpha, \beta]$. Let $x, y \in [\alpha, \beta]$ be such that $x < y$. Then by (A1), for each $t \in [0, b]$,

$$\begin{aligned} A(x) &= \{h \in PC : \ h(t) = e_{\ominus p}(t, 0)\eta + \int_0^t e_{\ominus p}(t, s)v(s)\Delta s, \ v \in S_{F,x}\} \\ &\leq \{h \in PC : \ h(t) = e_{\ominus p}(t, 0)\eta + \int_0^t e_{\ominus p}(t, s)v(s)\Delta s, \ v \in S_{F,y}\} \\ &= A(y). \end{aligned}$$

Hence $A(x) \leq A(y)$. Similarly by (A2), for each $t \in [0, b]$, $B(x) \leq B(y)$.
Thus A and B are isotone increasing on $[\alpha, \beta]$. Finally we prove that $A(y) + B(y) \subset [\alpha, \beta]$ for each $y \in [\alpha, \beta]$. Let $h \in A(y) + B(y)$ be any element. Then, there exists $v \in S_{v,y}$ such that for each $t \in [0, b]$,

$$h(t) = e_{\ominus p}(t, 0)\eta + \int_0^t e_{\ominus p}(t, s)v(s)\Delta s + \sum_{0<t_k<t} e_{\ominus p}(t, t_k)I_k(y(t_k^-)).$$

Let

$$t_i = \max\{t_k : t_k < t\}.$$

By definition of the upper solution and the conditions (A1)-(A3), we get for each $t \in [0,b]$,

$$\begin{aligned} h(t) &\leq e_{\ominus p}(t,0)\beta(0) + \int_0^{t_1} e_{\ominus p}(t,s)[\beta^{\Delta}(s) + p(t)\beta^{\sigma}(s)]\Delta s \\ &+ \int_{t_1^+}^{t_2} e_{\ominus p}(t,s)[\beta^{\Delta}(s) + p(t)\beta^{\sigma}(s)]\Delta s \\ &+ \ldots + \int_{t_i}^{t} e_{\ominus p}(t,s)[\beta^{\Delta}(s) + p(t)\beta^{\sigma}(s))]\Delta s \\ &+ \sum_{k=1}^{k=i} e_{\ominus p}(t,t_k) I_k(\beta(t_k^-)). \end{aligned}$$

Thus, for each $t \in [0,b]$

$$\begin{aligned} e_p(t,0)h(t) &\leq \beta(0) + \int_0^{t_1} [e_p(s,0)\beta(s)]^{\Delta}\Delta s + \int_{t_1^+}^{t_2} [e_p(s,0)\beta(s)]^{\Delta}\Delta s \\ &+ \ldots + \int_{t_i^+}^{t} [e_p(s,0)\beta(s)]^{\Delta}\Delta s + \sum_{k=1}^{k=i} e_p(t_k,0) I_k(\beta(t_k^-)) \\ &= \beta(0) + e_p(t_1^-,0)\beta(t_1^-) - e_p(0,0)\beta(0) + e_p(t_2^-,0)\beta(t_2^-) \\ &- e_p(t_1^+,0)\beta(t_1^+) + \ldots + e_p(t,0)\beta(t) - e_p(t_i^+,0)\beta(t_i^+) \\ &+ \sum_{k=1}^{k=i} e_p(t_k,0) I_k(\beta(t_k^-)) \\ &= -e_p(t_1,0)[\beta(t_1^+) - \beta(t_1^-)] - e_p(t_2,0)[\beta(t_2^+) - \beta(t_2^-)] \\ &- \ldots - e_p(t_i,0)[\beta(t_i^+) - \beta(t_i^-)] + e_p(t,0)\beta(t) \\ &+ \sum_{k=1}^{k=i} e_p(t_k,0) I_k(\beta(t_k^-)) \\ &\leq -e_p(t_1,0) I_1(\beta(t_1^-)) - e_p(t_2,0) I_2(\beta(t_2^-)) \\ &- \ldots - e_p(t_i,0) I_i(\beta(t_i^-)) + e_p(t,0)\beta(t) \\ &+ \sum_{k=1}^{k=i} e_p(t_k,0) I_k(\beta(t_k^-)) \\ &= e_p(t,0)\beta(t). \end{aligned}$$

Hence

$$h(t) \leq \beta(t) \text{ for each } t \in [0,b].$$

Similarly, by replacing β with α and reversing the order, we can prove that

$$h(t) \geq \alpha(t), \text{ for each } t \in [0,b].$$

Then

$$\alpha \leq N(y) \leq \beta \text{ for all } y \in [\alpha,\beta].$$

As a consequence of Theorem 4.2, we deduce that N has least and greatest fixed point in $[\alpha, \beta]$. This further implies that the problem (1)-(3) has minimal and maximal solutions on $[0, b]$.

5 Example:

Suppose $\mathbb{T} = [0,1] \cup [2,3] \cup [4,5]$ and p a regressive function. We consider the equation

$$y^{\Delta}(t) = p(t)y(t), \ y(0) = 1.$$

We can easily show that the unique solution of the above equation is given by

$$y(t) = e_p(t,0) = \begin{cases} e^{\int_0^t p(s)\Delta s}, & \text{if } t \in [0,1], \\ \exp(\int_0^1 p(s)\Delta s + \int_2^t p(s)\Delta s), & \text{if } t \in [2,3], \\ \exp(\int_0^1 p(s)\Delta s + \int_2^3 p(s)\Delta s + \int_4^t p(s)\Delta s), & \text{if } t \in [4,5]. \end{cases}$$

Also we consider the following dynamic inclusion of the form

$$y^{\Delta}(t) + p(t)y^{\sigma}(t) \in F(t, y(t)), \ t \in [0,1], \ t \neq \frac{1}{2}, \tag{8}$$

$$y\Big(\frac{1}{2}^+\Big) - y\Big(\frac{1}{2}^-\Big) = I_1\Big(y\Big(\frac{1}{2}^-\Big)\Big), \tag{9}$$

$$y(0) = 0, \tag{10}$$

where $F : [0,1] \times \mathrm{I\!R} \to \mathcal{P}(\mathrm{I\!R})$ is the multivalued map defined by

$$(t,x) \to F(t,x) := \Big[\frac{x^2}{x^2+2} + t, \frac{x^2}{x^2+1} + t + 1\Big].$$

It is clear that F is a compact convex valued multivalued map and of Carathéodory. Let $v \in [\frac{x^2}{x^2+2} + t, \frac{x^2}{x^2+1} + t + 1]$, then we have

$$|v| \leq \max\left(\frac{x^2}{x^2+2} + |t|, \frac{x^2}{x^2+1} + |t| + 1\right) \leq 3, \ \text{for each} \ \ (t,x) \in [0,1] \times \mathrm{I\!R}.$$

Hence

$$\|F(t,x)\| := \sup\Big\{|v| : v \in \Big[\frac{x^2}{x^2+1} + t, \frac{x^2}{x^2+1} + t + 1\Big]\Big\} \leq 3 := p(t)\psi(x),$$

where $p(t) = 1$ and $\psi(x) = 3$. Assume that there exists $c > 0$ such that

$$|I_1(x)| \leq c, \quad \text{for each } x \in \mathrm{I\!R}.$$

We can find $M>0$ such that

$$\frac{M}{c\sup\limits_{t\in[0,1]}e_{\ominus p}\Big(t,\dfrac{1}{2}\Big)+3\sup\limits_{(t,s)\in[0,1]\times[0,1]}e_{\ominus p}(t,s)}>1.$$

Consider the operator $N: PC\longrightarrow\mathcal{P}(PC)$ defined by

$$N(y)\ \ =\{h\in PC:\ h(t)=\int_0^t e_{\ominus p}(t,s)v(s)\Delta s+e_{\ominus p}(t,\frac{1}{2})I_1\Big(y\Big(\frac{1}{2}^-\Big)\Big),\ v\in S_{F,y}\}.$$

Let y be such that $y\in\lambda N(y)$ for some $\lambda\in(0,1)$. Then, there exists $v\in S_{F,y}$ such that for each $t\in[0,1]$ we have

$$y(t)=\lambda\int_0^t e_{\ominus p}(t,s)v(s)\Delta s+\lambda e_{\ominus p}\Big(t,\frac{1}{2}\Big)I_1\Big(y\Big(\frac{1}{2}^-\Big)\Big).$$

This implies that for each $t\in[0,1]$ we have

$$|y(t)|\ \ \leq\ \ c\sup_{t\in[0,1]}e_{\ominus p}\Big(t,\frac{1}{2}\Big)+3\sup_{(t,s)\in[0,1]\times[0,1]}e_{\ominus p}(t,s)$$

Thus

$$\|y\|_{PC}\leq c\sup_{t\in[0,1]}e_{\ominus p}\Big(t,\frac{1}{2}\Big)+3\sup_{(t,s)\in[0,1]\times[0,1]}e_{\ominus p}(t,s).$$

Then all the conditions of Theorem 3.3 hold and thus, the problem (8)-(10) has at least one solution on [0,1].

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