Electronic Journal of Qualitative Theory of Differential Equations
2005, No. **9**, 1-16; **http://www.math.u-szeged.hu/ejqtde/**

ON THE EXPONENTIAL CONVERGENCE TO A LIMIT OF SOLUTIONS OF PERTURBED LINEAR VOLTERRA EQUATIONS

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Abstract. We consider a system of perturbed Volterra integro-differential equations for which the solution approaches a nontrivial limit and the difference between the solution and its limit is integrable. Under the condition that the second moment of the kernel is integrable we show that the solution decays exponentially to its limit if and only if the kernel is exponentially integrable and the tail of the perturbation decays exponentially.

1. Introduction

In this paper we study the exponential decay of the solution of

$$x'(t) = Ax(t) + \int_0^t K(t-s)x(s)\,ds + f(t), \quad t > 0, \tag{1.1a}$$

$$x(0) = x_0, \tag{1.1b}$$

to a constant vector. Here the solution x is a vector-valued function on $[0,\infty)$, A is a real matrix, K is a continuous and integrable matrix-valued function on $[0,\infty)$ and f is a continuous and integrable vector-valued function on $[0,\infty)$.

The solution of (1.1) can be written in terms of the solution of an unperturbed version of the equation. This unperturbed equation is given by

$$R'(t) = AR(t) + \int_0^t K(t-s)R(s)\,ds, \quad t > 0, \tag{1.2a}$$

$$R(0) = I, \tag{1.2b}$$

where the matrix–valued function R is known as the resolvent or fundamental solution of (1.1). The representation of solutions of (1.1) in

1991 *Mathematics Subject Classification.* Primary 45J05, 45D05, 34K20; Secondary 60K05.

Key words and phrases. Volterra integro–differential equations, Volterra integral equations, exponential stability, exponential asymptotic stability,

John Appleby was partially funded by an Albert College Fellowship, awarded by Dublin City University's Research Advisory Panel. Siobhán Devin was funded by The Embark Initiative operated by the Irish Research Council for Science, Engineering and Technology (IRCSET).

terms of R is given by the variation of constants formula

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \geq 0.$$

For this and other reasons, the asymptotic behaviour of R has long been a topic of study, and it is well known that uniform asymptotic stability for (1.1a) is associated with the solution R of (1.2) being integrable. In this case it is interesting to understand the relationship between the rate of decay of the kernel, and the rate of decay of solutions. Authors who have shown that some sort of exponential decay in the kernel can be identified with exponential decay of the resolvent include Murakami [8, 9] and Appleby and Reynolds [2]. Murakami shows that the exponential decay of the solution of (1.2) is equivalent to an exponential decay property on the kernel K under the restriction that none of the elements of K change sign on $[0,\infty)$. A condition of this type will be employed in this paper to identify exponential convergence. In a similar spirit, various authors have identified decay conditions on K which give rise to particular decay properties in the resolvent. For example Burton, Huang and Mahfoud [3] have shown that the existence of the "moments" of the kernel can be identified with the existence of the moments of the solution. Appleby and Reynolds [1] have studied a type of non-exponential decay of solutions (called subexponential decay) which can in certain circumstances be identified with the subexponential decay of the kernel. Jordan and Wheeler [5] and Shea and Wainger [10] have studied the relationship between the existence of the kernel in a certain weighted L^p-space and the existence of the solution in such spaces.

The case where the solutions of (1.2) are neither integrable, nor unstable, has also been considered. Krisztin and Terjéki [6] studied this case and determined conditions under which $R(t)$ converges to a limit R_∞, which need not be trivial, as $t \to \infty$. In addition to determining a formula for R_∞, they showed that the condition $\int_0^\infty t^2\|K(t)\|\,dt < \infty$ is crucial. MacCamy and Wong [7] dealt with a nonlinear version of (1.1). They showed that if the kernel and the perturbation satisfy an exponential decay constraint, then x converges to a nontrivial limit x_∞ exponentially fast.

In this paper we consider the case where the resolvent of (1.1) is not integrable. In the first instance, we find an equivalence between the exponential decay property of $t \mapsto R(t) - R_\infty$ and an exponential decay property of the kernel; we also show for solutions of (1.1) that the exponential decay of $t \mapsto x(t) - x_\infty$ can be identified with exponential decay in the kernel and the perturbation.

2. Mathematical Preliminaries

We introduce some standard notation. We denote by $\mathbb{R}$ the set of real numbers. Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real entries, and I be the identity matrix. We denote by $\mathrm{diag}(a_1, a_2, ..., a_n)$ the $n \times n$ matrix with the scalar entries $a_1, a_2, ..., a_n$ on the diagonal and 0 elsewhere. For $B = (b_{ij}) \in M_n(\mathbb{R})$ we define $\|B\| = \sum_{i=1}^n \sum_{j=1}^n |b_{ij}|$. All other norms on $M_n(\mathbb{R})$ are equivalent to $\| \cdot \|$. If J is an interval in $\mathbb{R}$ and V a finite dimensional normed space, we denote by $C(J, V)$ the family of continuous functions $\phi : J \to V$. The space of Lebesgue integrable functions $\phi : (0, \infty) \to V$ will be denoted by $L^1((0, \infty), V)$. The convolution of F and G is denoted by $F * G$ and defined by

$$(F * G)(t) = \int_0^t F(t-s)G(s)ds, \qquad t \geq 0.$$

We denote by $\mathbb{N}$ the set of natural numbers. We denote by $\mathbb{C}$ the set of complex numbers; the real part of z in $\mathbb{C}$ being denoted by $\mathrm{Re}\, z$ and the imaginary part by $\mathrm{Im}\, z$. If $B : [0, \infty) \to M_n(\mathbb{R})$ then the Laplace transform of B is formally defined to be

$$\hat{B}(z) = \int_0^\infty B(t)e^{-zt}dt.$$

If $\epsilon \in \mathbb{R}$ and $\int_0^\infty \|B(s)\|e^{-\epsilon s}ds < \infty$ then $\hat{B}(z)$ exists for $\mathrm{Re}\, z \geq \epsilon$ and is analytic for $\mathrm{Re}\, z > \epsilon$. If B is a continuous function which satisfies $\|B(t)\| \leq ce^{\beta t}$ for $t > 0$ then the inversion formula

$$B(t) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\epsilon - iT}^{\epsilon + iT} \hat{B}(z)e^{zt}dz = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \hat{B}(z)e^{zt}dz$$

holds for all $\epsilon > \beta$.

We now make our problem precise. Throughout the paper we assume that the function $K : [0, \infty) \to M_n(\mathbb{R})$ satisfies

$$K \in C([0, \infty), M_n(\mathbb{R})) \cap L^1((0, \infty), M_n(\mathbb{R})), \tag{2.1}$$

and the function $f : [0, \infty) \to \mathbb{R}^n$ satisfies

$$f \in C([0, \infty), \mathbb{R}^n) \cap L^1((0, \infty), \mathbb{R}^n). \tag{2.2}$$

It is convenient to define the tail of the kernel K as

$$K_1(t) = \int_t^\infty K(s)\, ds, \quad t \geq 0, \tag{2.3}$$

and the tail of the perturbation f as

$$f_1(t) = \int_t^\infty f(s)\, ds, \quad t \geq 0. \tag{2.4}$$

The existence of K_1 and f_1 is assured by the integrability of K and f respectively. We define the function $t \mapsto x(t; x_0, f)$ to be the unique solution of the initial value problem (1.1). Under the hypothesis (2.1), it is well-known that (1.2) has a unique continuous solution R, which is continuously differentiable. Moreover the solution of (1.1) for any initial condition x_0 is given by

$$x(t; x_0, f) = R(t)x_0 + (R * f)(t), \quad t \geq 0. \tag{2.5}$$

Where x_0 and f are clear from the context we omit them from the notation.

A fundamental result on the asymptotic behaviour of the solution of (1.1) is the following theorem due to Grossman and Miller [4]; under (2.1) the resolvent R of (1.2) is integrable if and only if

$$\det[zI - A - \hat{K}(z)] \neq 0, \quad \text{for } \operatorname{Re} z \geq 0. \tag{2.6}$$

In this paper we consider the case where the solution of (1.1) approaches a constant vector x_∞ which need not be trivial, and so (2.6) does not necessarily hold.

3. Discussion of Results

In this section we explain the connection between the results on exponential decay presented by Murakami in [8, 9] and those here. Murakami obtained the following result in the case where the solutions of (1.2) are integrable.

Theorem 3.1. *Let K satisfy* (2.1). *Suppose the resolvent R of* (1.2) *satisfies*

$$R \in L^1((0, \infty), M_n(\mathbb{R})). \tag{3.1}$$

If

$$\text{each entry of } K \text{ does not change sign on } [0, \infty), \tag{3.2}$$

then the following are equivalent;

(i) *There exists a constant $\alpha > 0$ such that*

$$\int_0^\infty \|K(s)\| e^{\alpha s} ds < \infty. \tag{3.3}$$

(ii) *There exist constants $c_1 > 0$ and $\beta_1 > 0$ such that*

$$\|R(t)\| \leq c_1 e^{-\beta_1 t}, \quad t \geq 0. \tag{3.4}$$

In this paper we begin by considering the case where the solution of (1.2) approaches a constant matrix.

Theorem 3.2. *Let K satisfy* (2.1) *and*

$$\int_0^\infty t^2\|K(t)\|dt < \infty. \tag{3.5}$$

Suppose there exists a constant matrix R_∞ such that the solution R of (1.2) *satisfies*

$$R(\cdot) - R_\infty \in L^1((0,\infty), M_n(\mathbb{R})). \tag{3.6}$$

If

$$\text{each entry of } K \text{ does not change sign on } [0,\infty), \tag{3.7}$$

then the following are equivalent;

(i) *There exists a constant $\alpha > 0$ such that*

$$\int_0^\infty \|K(s)\|e^{\alpha s}\, ds < \infty. \tag{3.8}$$

(ii) *There exist constants $\beta_2 > 0$ and $c_2 > 0$ such that*

$$\|R(t) - R_\infty\| \le c_2 e^{-\beta_2 t}, \quad t \ge 0. \tag{3.9}$$

We can readily see the similarities between Theorem 3.1 and Theorem 3.2: the hypotheses (3.1) and (3.2) in Theorem 3.1 are identical to (3.6) and (3.7) in Theorem 3.2; moreover, the equivalence between (3.3) and (3.4) in Theorem 3.1 is mirrored by the equivalence between (3.8) and (3.9). The hypothesis in Theorem 3.2 which has no counterpart in Theorem 3.1 is (3.5); however, as we mention later, this hypothesis is natural and sometimes indispensible in the case $R_\infty \ne 0$.

It is possible to obtain results comparable to Theorem 3.2 for the solution of the perturbed equation (1.1). More precisely, it is possible to show that the exponential decay of $x - x_\infty$ is equivalent to the exponential decay of the tail of the perturbation and the exponential integrability of the kernel. The following theorem makes this precise.

Theorem 3.3. *Let K satisfy* (2.1) *and* (3.5)*, f satisfy* (2.2)*, and f_1 be defined by* (2.4)*. Suppose that for all x_0 there is a constant vector $x_\infty(x_0, f)$ such that the solution $t \mapsto x(t; x_0, f)$ of* (1.1) *satisfies*

$$x(\cdot\,; x_0, f) - x_\infty(x_0, f) \in L^1((0,\infty), \mathbb{R}^n). \tag{3.10}$$

If K satisfies (3.7) *the following are equivalent;*

(i) *There exists $\alpha > 0$ such that statement (i) of Theorem 3.2 holds and there exist constants $\gamma > 0$, $c_3 > 0$ independent of x_0 such that*

$$\|f_1(t)\| \le c_3 e^{-\gamma t}, \quad t \ge 0. \tag{3.11}$$

(ii) *For each* x_0 *the solution* $t \mapsto x(t; x_0, f)$ *satisfies*

$$\|x(t) - x_\infty\| \le c_4 e^{-\beta_3 t}, \quad t \ge 0, \tag{3.12}$$

for some $\beta_3 > 0$ *independent of* x_0*, and* $c_4 = c_4(x_0) > 0$.

Murakami considered the case where the resolvent of (1.2) is integrable, which forces $R(t) \to 0$ as $t \to \infty$. In this paper, we consider the case where the solutions of (1.1) approach a constant vector, which may not necessarily be trivial, in which case the solution is not integrable. As a result it is not possible to apply Murakami's method of proof directly to our equation. Instead, we find it is necessary to appeal to a result of Krisztin and Terjéki [6] to obtain appropriate hypotheses for Theorem 3.2 and Theorem 3.3.

Before citing the relevant results from [6], we introduce some notation used there and adopted hereinafter. We let $M = A + \int_0^\infty K(s)ds$ and T be an invertible matrix such that $T^{-1}MT$ has Jordan canonical form. Let $e_i = 1$ if all the elements of the i^{th} row of $T^{-1}MT$ are zero, and $e_i = 0$ otherwise. Put $P = T\mathrm{diag}(e_1, e_2, ..., e_n)T^{-1}$ and $Q = I - P$. We now state the relevant theorem.

Proposition 3.4. *If* K *satisfies* (3.5) *and the resolvent* R *of* (1.2) *satisfies* (3.6) *then*

$$\det[zI - A - \hat{K}(z)] \neq 0 \quad \textit{for } \operatorname{Re} z \ge 0 \textit{ and } z \neq 0$$

and

$$\det\left[P - M - \int_0^\infty \int_s^\infty PK(u)duds\right] \neq 0; \tag{3.13}$$

moreover

$$R_\infty = \left[P - M - \int_0^\infty \int_s^\infty PK(u)duds\right]^{-1} P. \tag{3.14}$$

Krisztin and Terjéki's result not only suggests the appropriate hypotheses for our theorems, but guarantees the existence of the constant matrix R_∞ as well as giving a formula for it. We note that under assumptions (3.5) and (3.6) that (2.6) fails at $z = 0$ if $R_\infty \neq 0$.

4. Preparatory Work

In order to prove Theorem 3.2 and Theorem 3.3 we must reformulate (1.2) as was done in Theorem 2 of [6]. In order to make this reformulation precise we state the following lemma.

Lemma 4.1. *Suppose that* (2.1), (3.5), *and* (3.6) *hold. Then*

$$\widehat{Y}(z) + \hat{F}(z)\widehat{Y}(z) = \hat{G}(z) \qquad \operatorname{Re} z \ge 0, \tag{4.1}$$

where $\hat{F}(z)$ *is defined for* $\operatorname{Re} z \geq 0$ *and* $z \neq 0$ *by*

$$\hat{F}(z) = \frac{1}{z} P(\hat{K}(0) - \hat{K}(z)) - \frac{1}{z+1} Q(I + A + \hat{K}(z)) \tag{4.2}$$

and

$$\hat{F}(0) = -Q(I + A + \hat{K}(0)) - P\hat{K}'(0); \tag{4.3}$$

and $\hat{G}(z)$ *is defined for* $\operatorname{Re} z \geq 0$ *and* $z \neq 0$ *by*

$$\begin{aligned}\hat{G}(z) = {} & \frac{1}{(z+1)} Q\left(I - (I + A + \hat{K}(z))R_\infty\right) \\ & - \frac{1}{z}\left(P\hat{K}'(0) + Q\hat{K}(0) - Q\hat{K}(z)\right) R_\infty - \frac{1}{z^2} P(\hat{K}(0) - \hat{K}(z))R_\infty,\end{aligned} \tag{4.4}$$

and

$$\hat{G}(0) = Q - \left(Q(I + A) - \frac{1}{2} P\hat{K}''(0) - Q\hat{K}'(0) + Q\hat{K}(0)\right) R_\infty. \tag{4.5}$$

Proof. As conditions (3.5) and (3.6) hold we know from Proposition 3.4 that (3.14) holds. We now employ an idea used in [6, Theorem 2]. Define the function Φ by $\Phi(t) = P + e^{-t}Q$ for $t \geq 0$. Taking the convolution of each side of (1.2) with Φ, we get $\Phi * R' = \Phi * (AR) + (\Phi * K) * R$, which after integration by parts becomes

$$R(t) + (F * R)(t) = \Phi(t), \quad t \geq 0, \tag{4.6}$$

where

$$F(t) = -e^{-t}(Q + QA) - (e * QK)(t) + P \int_t^\infty K(u)du, \quad t \geq 0,$$

and the function e is defined by $e(t) = e^{-t}$, $t \geq 0$. A further calculation yields

$$Y(t) + (F * Y)(t) = G(t), \quad t \geq 0, \tag{4.7}$$

where $Y(t) = R(t) - R_\infty$ and

$$\begin{aligned}G(t) = e^{-t}Q - e^{-t}(QR_\infty + QAR_\infty) + \int_t^\infty \int_u^\infty PK(s)R_\infty \, ds \, du \\ - \int_t^\infty QK(u)R_\infty du - (e * QKR_\infty)(t), \quad t \geq 0.\end{aligned}$$

Since (2.1) holds we can take the Laplace transform of (4.7) to obtain (4.1) where $\hat{F}$ and $\hat{G}$ are given by (4.2) and (4.4) respectively for $\operatorname{Re} z > 0$ and $z \neq 0$ and are given by (4.3) and (4.5) when $z = 0$. □

Remark 4.2. If we assume that there exists a constant $\alpha > 0$ such that (3.8) of Theorem 3.2 holds then the functions $\hat{F}$ and $\hat{G}$ defined by (4.2) and (4.4) respectively when can be extended into the negative half plane.

The following lemma may be extracted from [6, Theorem 2] and is necessary in the proof of Theorem 5.1.

Lemma 4.3. *If* (2.1), (3.5) *and* (3.6) *hold, then*

$$\det[I + \hat{F}(z)] \neq 0, \quad \operatorname{Re} z \geq 0. \tag{4.8}$$

The following proposition may extracted from [8, 9] and used is later in the proof of Theorem 5.2.

Proposition 4.4. *Let K be a continuous integrable function such that no entry of K changes sign on $[0, \infty)$. Suppose that there is a continuous function $z \mapsto B(z)$ defined for $|\operatorname{Re} z| \leq \alpha_1$ and analytic for $|\operatorname{Re} z| < \alpha_1$, where $\alpha_1 > 0$. If $\hat{K}(z) = B(z)$ for all $0 \leq \operatorname{Re} z < \alpha_1$, then*

$$\int_0^\infty \|K(s)\| e^{\alpha_1 s}\, ds < \infty.$$

The proof is identical in all important details to that of Theorem 2 in [8].

5. Proof of Theorem 3.2

Theorem 3.2 is a consequence of the following results.

Theorem 5.1. *Let K satisfy* (2.1) *and* (3.5), *and R be the solution of* (1.2). *Suppose there exists a constant matrix R_∞ such that* (3.6) *holds. If there exists a constant $\alpha > 0$ such that K obeys* (3.8) *in Theorem 3.2, then there exist constants $\beta_2 > 0$ and $c_2 > 0$ such that R obeys* (3.9) *of Theorem 3.2.*

Theorem 5.2. *Let K satisfy* (2.1), (3.5) *and* (3.7), *and let R be the solution of* (1.2). *Suppose there exists a constant matrix R_∞ such that* (3.6) *holds. If there exist constants $\beta_2 > 0$ and $c_2 > 0$ such that R obeys* (3.9) *in Theorem 3.2, then there exists a constant $\alpha > 0$ such that K obeys* (3.8) *in Theorem 3.2.*

Proof of Theorem 5.1. Since (3.6) holds the inversion formula for the Laplace Transform of Y is well-defined when $\epsilon > 0$;

$$Y(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\epsilon - iT}^{\epsilon + iT} \hat{Y}(z) e^{zt} dz.$$

¿From Lemma 4.3 we know that $\det[I+\hat{F}(z)] \neq 0$ for $\operatorname{Re} z \geq 0$ so we can write

$$\hat{Y}(z) = H_1(z), \quad \operatorname{Re} z \geq 0,$$

where

$$H_1(z) = (I+\hat{F}(z))^{-1}\hat{G}(z), \quad \operatorname{Re} z \geq 0.$$

We begin by showing that

$$Y(t) = \frac{1}{2\pi i}\int_{-\beta_2-i\infty}^{-\beta_2+i\infty} H_1(z)e^{zt}dz, \quad t>0, \tag{5.1}$$

for some $\beta_2 > 0$.

Observe that since $\det[I+\hat{F}(0)] \neq 0$, $H_1(0)$ exists. Using (3.8) and the Riemann–Lebesgue Lemma we see that $\hat{K}(z) \to 0$ as $|z| \to \infty$ for $\operatorname{Re} z \geq -\alpha$, thus we can see from (4.2) that $\hat{F}(z) \to 0$ as $|z| \to \infty$ for $\operatorname{Re} z \geq -\alpha$. Therefore we can find $T_0 > 0$ such that for $|\operatorname{Im} z| > T_0$ we have that $\det[I+\hat{F}(z)] \neq 0$ when $\operatorname{Re} z \geq -\alpha$, $|\operatorname{Im} z| > T_0$. Hence $H_1(z)$ exists when $|\operatorname{Im} z| > T_0$ and $\operatorname{Re} z \geq -\alpha$. Let

$$D = \{z : -\frac{\alpha}{2} \leq \operatorname{Re} z \leq 0, \quad |\operatorname{Im} z| \leq T_0\}$$

and

$$c_0 = \max\{\operatorname{Re} z : z \in D, \quad \det[I+\hat{F}(z)] = 0\}.$$

Since $z \mapsto (I+\hat{F}(z))$ is analytic on the domain $\operatorname{Re} z > -\alpha$, and its determinant is a continuous function of its entries then $z \mapsto \det[I+\hat{F}(z)]$ is analytic on the domain $\operatorname{Re} z > -\alpha$. Thus it has at most a finite number of zeros in the set D, and so $c_0 < 0$. Take a constant $\beta_2 > 0$ so that $\beta_2 < -c_0$. Consider the integration of the function $H_1(z)e^{-zt}$ around the boundary of the box:

$$\{\lambda + i\tau : \ -\beta_2 \leq \lambda \leq \beta_2, \ -T \leq \tau \leq T\}.$$

Since $H_1(z)$ exists and is analytic in this box it follows that the integral over the boundary is zero, that is:

$$\left(\int_{\beta_2-iT}^{\beta_2+iT} + \int_{\beta_2+iT}^{-\beta_2+iT} + \int_{-\beta_2+iT}^{-\beta_2-iT} + \int_{-\beta_2-iT}^{\beta_2-iT}\right) H_1(z)e^{zt}dz = 0.$$

Our claim will be verified if

$$\lim_{T\to\infty}\int_{-\beta_2+iT}^{\beta_2+iT} H_1(z)e^{zt}\,dz = 0, \quad \lim_{T\to\infty}\int_{-\beta_2-iT}^{\beta_2-iT} H_1(z)e^{zt}\,dz = 0.$$

Consider $\|H_1(z)e^{zt}\|$ for $z = \lambda + iT$ and $-\beta_2 \le \lambda \le \beta_2$: then

$$\begin{aligned}
\|H_1(z)e^{zt}\| &= \|(I+\hat{F}(z))^{-1}\hat{G}(z)e^{zt}\| \\
&\le e^{\beta_2 t}\|\hat{G}(z)\|\|(I+\hat{F}(z))^{-1}\| \\
&\le \frac{e^{\beta_2 t}}{T}\cdot\Bigg(\|Q\| + \left\|\left(Q + QA + P\hat{K}'(0) + Q\hat{K}(0)\right)R_\infty\right\| \\
&\quad + \frac{\|(P\hat{K}'(0)+\hat{K}(0))R_\infty\|}{T} + \frac{\|P\hat{K}(0)R_\infty\|}{T^2} \\
&\quad + \frac{\|P\hat{K}(z)R_\infty\|}{T^2} + \frac{\|\hat{K}(z)R_\infty\|}{T}\Bigg)
\end{aligned}$$

¿From the Riemann–Lebesgue Lemma we see that as $|T| \to \infty$, $\hat{K}(z) \to 0$ and $\hat{F}(z) \to 0$, uniformly for $\operatorname{Re} z \ge -\alpha$. Thus $\|H_1(z)e^{zt}\| \to 0$ as $T \to \infty$ with $\operatorname{Re} z \ge -\beta_2$. Using the continuity of $H_1(z)e^{zt}$ and the above we can find constant $m < \infty$ such that

$$\|H_1(z)e^{zt}\| \le \frac{m}{T}e^{\beta_2 t}$$

for $|\operatorname{Re} z| \le \beta_2$, $z = \lambda + iT$, $t > 0$. Also

$$\begin{aligned}
\left\|\int_{-\beta_2+iT}^{\beta_2+iT} H_1(z)e^{zt}dz\right\| &\le \int_{-\beta_2}^{\beta_2} \|H_1(\lambda+iT)e^{(\lambda+iT)t}\|d\lambda \\
&\le \int_{-\beta_2}^{\beta_2} \frac{m}{T}e^{\beta_2 t}d\lambda \le 2\beta_2\frac{m}{T}e^{\beta_2 t}.
\end{aligned}$$

Thus $\int_{-\beta_2+iT}^{\beta_2+iT} H_1(z)e^{zt}dz \to 0$ as $T \to \infty$. A similar argument shows that $\int_{-\beta_2-iT}^{\beta_2-iT} H_1(z)e^{zt}dz \to 0$ as $T \to \infty$. Thus

$$\int_{\beta_2-i\infty}^{\beta_2+i\infty} H_1(z)e^{zt}dz = \int_{-\beta_2-i\infty}^{-\beta_2+i\infty} H_1(z)e^{zt}dz,$$

finishing the demonstration of (5.1).

It is necessary to choose an integrable function H_2 in order to obtain (3.9). We define the function $H_2(z)$ as follows:

$$\begin{aligned}
H_2(z) &= H_1(z) - (z-c_0)^{-1}L \\
&= \frac{1}{z-c_0}\cdot\frac{1}{z+1}\cdot(I+\hat{F}(z))^{-1} \\
&\quad \times\left[(z-c_0)(z+1)\hat{G}(z) - (z+1)(I+\hat{F}(z))L\right]
\end{aligned}$$

where

$$L := Q(I - (I + A + \hat{K}(0))R_\infty) - P\hat{K}'(0)R_\infty.$$

Thus

$$
\begin{aligned}
z^2H_2(z) &= \frac{z}{z-c_0}\cdot\frac{z}{z+1}\cdot(I+\hat{F}(z))^{-1}\\
&\quad\times\Big[\Big(z(z+1)\hat{G}(z)-zL\Big)-c_0(z+1)\hat{G}(z)-L-(z+1)\hat{F}(z)L\Big].
\end{aligned}
$$

Clearly $\frac{z}{z-c_0}$ and $\frac{z}{z+1}\to 1$ as $|z|\to\infty$. As (3.8) holds we know from the Riemann–Lebesgue lemma that $\hat{K}(z)\to 0$ as $|z|\to\infty$ with $\operatorname{Re} z\geq -\alpha$ thus $(I+\hat{F}(z))\to I$ also $\Big(z(z+1)\hat{G}(z)-zL\Big)$, $c_0(z+1)\hat{G}(z)$ and $(z+1)\hat{F}(z)L$ are bounded for $\operatorname{Re} z\geq -\alpha$. Now we have that $\|z^2H_2(z)\|<\infty$ as $|z|\to\infty$ with $\operatorname{Re} z\geq -\beta_2$ that is

$$
\sup_{\tau\in\mathbb{R}}\tau^2\|H_2(\beta_2+i\tau)\|<\infty,
$$

consequently

$$
\int_{-\beta_2-i\infty}^{-\beta_2+i\infty}\|H_2(z)\|ds := c.
$$

Therefore we obtain

$$
\begin{aligned}
\|Y(t)\| &\leq \frac{1}{2\pi}\Big\|\int_{-\beta_2-i\infty}^{-\beta_2+i\infty}H_1(z)e^{zt}\,dz\Big\|\\
&\leq \frac{1}{2\pi}\Big\|\int_{-\beta_2-i\infty}^{-\beta_2+i\infty}H_2(z)e^{zt}\,dz\Big\|+\frac{1}{2\pi}\Big\|\int_{-\beta_2-i\infty}^{-\beta_2+i\infty}\frac{L}{z-c_0}\,dz\Big\|\\
&\leq \frac{1}{2\pi}e^{-\beta_2 t}\int_{-\beta_2-i\infty}^{-\beta_2+i\infty}\|H_2(z)\|\,dz+\frac{1}{2\pi}\|L\|e^{c_0t}\\
&\leq c_2e^{-\beta_2 t},
\end{aligned}
$$

completing our proof. □

Proof of Theorem 5.2. Note that from our hypothesis $\hat{Y}$ exists and is continuous for $\operatorname{Re} z\geq-\beta_2$ and is analytic for $\operatorname{Re} z>-\beta_2$. Due to (3.5) and (3.6) so we can apply Proposition 3.4 to get (3.13). We see that $\det[R_\infty+z\hat{Y}(z)]$ is non-zero at $z=0$. From the continuity of $\hat{Y}(z)$ at zero there exists an open neighbourhood centred at zero on which $\det[R_\infty+z\hat{Y}(z)]\neq 0$. Also $\det[zI+P]$ is non-zero except at zero in an open neighbourhood centred at zero with radius less than one. Choose $\alpha>0$ such that $\det[R_\infty+z\hat{Y}(z)]$ and $\det[zI+P]$ are non zero for $0<|\operatorname{Re} z|<\alpha$. Define the function B as follows

$$
\begin{aligned}
B(z) = (P+zI)^{-1}\big[&-z^2Q(I-(I+A)(\hat{Y}(z)-R_\infty))\\
&+z(z+1)P\hat{K}'(0)R_\infty+(z+1)(P+zQ)\hat{K}(0)R_\infty\\
&+z(z+1)P\hat{K}(0)\hat{Y}(z)+z^2(z+1)\hat{Y}(z)\big](z\hat{Y}(z)+R_\infty)^{-1}
\end{aligned}
$$

for $0 < |\mathrm{Re}\, z| < \alpha$ and $B(0) := \hat{K}(0)$. The function B has been consructed so that $B(z) = \hat{K}(z)$ for $0 \le \mathrm{Re}\, z < \alpha$. Hence Proposition 4.4 can be applied to yield Theorem 5.2. □

6. Proof of Theorem 3.3

Theorem 3.3 is a consequence of the following results:

Theorem 6.1. *Let K satisfy* (2.1) *and* (3.5) *and let f satisfy* (2.2). *Suppose that for all x_0 there is a constant vector $x_\infty(x_0, f)$ such that the solution $t \mapsto x(t; x_0, f)$ of* (1.1) *satisfies* (3.10). *If there exists a constant $\alpha > 0$ such that statement (i) of Theorem 3.2 holds, and there exist constants $\gamma > 0$ and $c_3 > 0$ such that statement (i) of Theorem 3.3 holds, then there exist constants $\beta_3 > 0$, independent of x_0, and $c_4 = c_4(x_0) > 0$, such that statement (ii) of Theorem 3.3 holds.*

Theorem 6.2. *Let K satisfy* (2.1), (3.5) *and* (3.7) *and let f satisfy* (2.2). *Suppose that for all x_0 there is a constant vector $x_\infty(x_0, f)$ such that the solution $t \mapsto x(t; x_0, f)$ of* (1.1) *satisfies* (3.10). *If there exist constants $\beta_3 > 0$, independent of x_0, and $c_4 = c_4(x_0) > 0$ such that statement (ii) of Theorem 3.3 holds, then there exists a constant $\alpha > 0$ such that statement (i) of Theorem 3.2 holds, and moreover there exist constants $\gamma > 0$ and $c_3 > 0$ such that statement (i) of Theorem 3.3 holds.*

Remark 6.3. If we impose a weaker condition, that is if (3.12) of Theorem 3.3(ii) only holds for a basis of initial values, then the same result holds.

Proof of Theorem 6.1. Using (2.2) and (2.5) we have that

$$x_\infty = R_\infty \left(x_0 + \int_0^\infty f(s)\, ds \right),$$

thus

$$\begin{aligned} x(t) - x_\infty &= R(t)x_0 + (R * f)(t) - R_\infty \left(x_0 + \int_0^\infty f(s)\, ds \right) \\ &= (R(t) - R_\infty)x_0 + \int_0^t (R(t-s) - R_\infty) f(s)\, ds - R_\infty f_1(t). \end{aligned}$$

Integrating $((R - R_\infty) * f)(t)$ by parts we obtain

$$\begin{aligned} x(t) - x_\infty &= (R(t) - R_\infty)x_0 - (R(0) - R_\infty) f_1(t) \\ &\quad + (R(t) - R_\infty) f_1(0) - \int_0^t R'(t-s) f_1(s)\, ds - R_\infty f_1(t). \end{aligned} \tag{6.1}$$

Due to the fact that K obeys (3.8), by Theorem 3.2, it follows that $R - R_\infty$ decays exponentially. We prove in the sequel that R' decays exponentially, f_1 also decays exponentially therefore the convolution of R' and f_1 decays exponentially. By use of the above facts and the hypothesis (3.11) on f_1, we have that each term on the right hand side of (6.1) decays exponentially, which yields (3.12).

We now show that R' decays exponentially. We can rewrite the resolvent equation (1.2) as

$$\begin{aligned}(6.2)\quad R'(t) = A(R(t)-R_\infty) + \int_0^t K(t-s)(R(s)-R_\infty)\,ds \\ - K_1(t)R_\infty + \left(A + \int_0^\infty K(s)ds\right)R_\infty.\end{aligned}$$

The first term on the right-hand side of (6.2) decays exponentially since (3.8) holds. We now provide an argument to show that the second term decays exponentially; since $R(t)-R_\infty$ decays exponentially and (3.8) holds we can choose μ such that $e^{\mu t}K(t)$ and $e^{\mu t}(R(t)-R_\infty) \in L^1((0,\infty), M_n(\mathbb{R}))$. Because the convolution of two integrable functions is itself integrable,

$$\begin{aligned}e^{\mu t}\left\|\int_0^t K(t-s)(R(s)-R_\infty)\,ds\right\| \\ = \left\|\int_0^t e^{\mu(t-s)}K(t-s)e^{\mu s}(R(s)-R_\infty)\,ds\right\| \le c_6,\end{aligned}$$

so that the second term on the right-hand side of (6.2) decays exponentially. Note that

$$\begin{aligned}(6.3)\quad c_7 := \int_0^\infty \|K(s)\|e^{\alpha s}ds \ge \int_t^\infty \|K(s)\|e^{\alpha s}ds \\ \ge e^{\alpha t}\int_t^\infty \|K(s)\|ds \ge e^{\alpha t}\|K_1(t)\|,\end{aligned}$$

showing that the last term on the right-hand side of (6.2) also decays exponentially.

Finally we show that $(A + \int_0^\infty K(s)\,ds))R_\infty = 0$. Integrating (6.2) and rearranging the terms yields

$$\begin{aligned}-\left(A + \int_0^\infty K(s)\,ds\right)R_\infty t = \int_0^t A(R(s)-R_\infty)ds \\ + \int_0^t \left\{\int_0^s K(s-u)(R(u)-R_\infty)\,du - K_1(s)R_\infty\right\}ds - (R(t)-R_0).\end{aligned}$$

Each term on the right-hand side of the equation is integrable thus $(A + \int_0^\infty K(s)\,ds)R_\infty = 0$. From the above we see that R' decays exponentially. $\square$

Proof of Theorem 6.2. We begin by proving (3.8). Let $\{\xi_1, \dots, \xi_n\}$ be the standard basis for $\mathbb{R}^n$. As (3.12) holds for all initial values x_0, we can obtain $n+1$ solutions $x_j(t)_{j=1,\cdots,n+1}$ of (1.1) by setting

$$x_j(0) = \xi_j \quad \text{for } j = 1, \cdots, n, \quad x_{n+1}(0) = 0.$$

We know that $x_j(t)$ approaches $x_j(\infty)$ exponentially fast. Introduce

$$s_j(t) = x_j(t) - x_{n+1}(t),$$

and notice $s_j(0) = \xi_j$. Define $S(t)$ in $M_n(\mathbb{R})$ by $S = [s_1(t), \dots, s_n(t)]$. Then $S(0) = I$ and

$$S'(t) = AS(t) + (K * S)(t), \quad t > 0.$$

We see that $S(t) \to S_\infty = [s_1(\infty), \dots, s_n(\infty)]$ exponentially fast, if we put $s_j(\infty) = x_j(\infty) - x_{n+1}(\infty)$. Theorem 5.2 can be applied to obtain (3.8). Note that the rate of convergence of K is independent of x_0.

Using the above we can now prove (3.11). Choose $x_0 = 0$. Integrating (1.1) we obtain

$$x(t) = \int_0^t Ax(s)\,ds + \int_0^t (K * x)(s)\,ds + \int_0^t f(s)\,ds.$$

By rearranging terms,

$$\begin{aligned} x(t) = &\int_0^t A(x(s) - x_\infty)\,ds + \int_0^t \int_0^s K(s-u)(x(u) - x_\infty)\,du\,ds \\ &+ \int_0^t Ax_\infty\,ds + \int_0^t \int_0^s K(s-u)x_\infty\,du\,ds + \int_0^t f(s)\,ds. \end{aligned}$$

By changing the order of integration and changing variable, we see that

$$\begin{aligned} x(t) - x_\infty = &-x_\infty + \int_0^t \left(A + \int_0^{t-u} K(v)\,dv\right)(x(u) - x_\infty)\,du \\ &+ \int_0^t \left(A + \int_0^\infty K(v)\,dv\right) x_\infty\,du - \int_0^t K_1(u)\,du + \int_0^t f(s)\,ds. \end{aligned}$$

It is shown later that $(A + \int_0^\infty K(u)\,du)x_\infty = 0$. Hence

$$\begin{aligned} x(t) - x_\infty = &-x_\infty + \int_0^t \left(A + \int_0^{t-u} K(v)\,dv\right)(x(u) - x_\infty)\,du \\ &- \int_0^t K_1(u)\,du + \int_0^t f(s)\,ds. \end{aligned}$$

Allowing $t \to \infty$ we see that

$$
\begin{aligned}
-x_\infty = &-\int_0^\infty \left(A + \int_0^\infty K(v)\,dv\right)(x(u) - x_\infty)\,du \\
&\qquad + \int_0^\infty K_1(u) x_\infty\,du - \int_0^\infty f(u)\,du,
\end{aligned}
$$

and thus we obtain

$$
\begin{aligned}
x(t) - x_\infty = &-\left(A + \int_0^\infty K(v)\,dv\right)\int_t^\infty (x(u) - x_\infty)\,du \\
&+ \int_t^\infty K_1(u)\,du - f_1(t) - \int_0^t K_1(t-u)(x(u) - x_\infty)\,du,
\end{aligned}
$$

giving

$$
\begin{aligned}
f_1(t) = &\left(A + \int_0^\infty K(u)\,du\right)\int_t^\infty (x(s) - x_\infty)\,ds - \int_t^\infty K_1(u)\,du \\
&- (x(t) - x_\infty) + \int_0^t K_1(t-u)(x(u) - x_\infty)\,du.
\end{aligned} \tag{6.4}
$$

¿From hypothesis (3.12) we can now show that f_1 decays exponentially. First, as K has been shown to obey (3.8) we see from (6.3) that $\int_t^\infty \|K_1(s)\| \le c_8 e^{-\alpha t}$, $t \ge 0$. By (3.12), (3.8) and the last estimate, each term on the right hand side of (6.4) decays exponentially to zero. Note that since K is independent of x_0 and $\|x(t) - x_\infty\| \le c_4(0)e^{-\beta_4 t} = c_4 e^{-\beta_4 t}$ the rate of decay of f_1 is independent of x_0.

Earlier in this proof, we postponed showing that the equality $(A + \int_0^\infty K(u)du)x_\infty = 0$ held. To see this, consider

$$
\begin{aligned}
x'(t) = A(x(t) - x_\infty) &+ \int_0^t K(t-s)(x(s) - x_\infty)\,ds \\
&+ f(t) - K_1(t) + \left(A + \int_0^\infty K(s)\,ds\right)x_\infty.
\end{aligned} \tag{6.5}
$$

Clearly the first four terms on the right-hand side of (6.5) are integrable. Integrating and rearranging terms we obtain

$$
\begin{aligned}
-\left(A + \int_0^\infty K(s)\,ds\right)x_\infty t = &\int_0^t A(x(s) - x_\infty)\,ds - (x(t) - x_0) \\
&+ \int_0^t \left\{\int_0^s K(s-u)(x(u) - x_\infty)\,du + f(s) - K_1(s)\right\}ds.
\end{aligned}
$$

Each of the terms on the right-hand side approach a finite limit which implies $\left(A + \int_0^\infty K(u)du\right)x_\infty = 0$. $\square$

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(Received January 16, 2005)

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