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The Oscillatory Behavior of Second Order Nonlinear Elliptic Equations

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Abstract: Some oscillation criteria are established for the nonlinear damped elliptic differential equation of second order

$$\sum_{i,j=1}^{N} D_i[\, a_{ij}(x) D_j y\,] + \sum_{i=1}^{N} b_i(x) D_i y + p(x) f(y) = 0, \tag{E}$$

which are different from most known ones in the sense that they are based on a new weighted function $H(r,s,l)$ defined in the sequel. Both the cases when $D_i b_i(x)$ exists for all i and when it does not exist for some i are considered.

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1. Introduction and Preliminaries

In this paper, we are concerned with the oscillatory behavior of the general nonlinear damped elliptic differential equation of second order

$$\sum_{i,j=1}^{N} D_i[\, a_{ij}(x) D_j y\,] + \sum_{i=1}^{N} b_i(x) D_i y + p(x) f(y) = 0, \tag{1.1}$$

where $x = (x_1, \cdots, x_N) \in \Omega(a) \subseteq \mathbb{R}^N$, $N \geq 2$, $D_i y = \partial y/\partial x_i$ for all i, $|x| = [\sum_{i=1}^{N} x_i^2\,]^{\frac{1}{2}}$, $\Omega(a) = \{x \in \mathbb{R}^N : |x| \geq a\}$ for some $a > 0$.

In what follows, the solution of Eq.(1.1) is every function of the class $C^{2+\mu}_{loc}(\Omega(a), \mathbb{R})$, $0 < \mu < 1$, which satisfies Eq.(1.1) almost everywhere on $\Omega(a)$. We consider only nontrivial solution of Eq.(1.1) which is defined for all large $|x|$ (cf [1]). The oscillation is considered in the usual sense, i.e., a solution $y(x)$ of Eq.(1.1) is said to be oscillatory if it has zero on $\Omega(b)$ for every $b \geq a$. Equation (1.1) is said to be oscillatory if every solution (if any exists) is oscillatory. Conversely, Equation(1.1) is nonoscillatory if there exists a solution which is not oscillatory.

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Equation (1.1) is an very important type of partial differential equations, which has wide applications in various problems dealing with physics, biology and glaciology, etc., see [1]. In the qualitative theory of nonlinear partial differential equations, one of the important problems is to determine whether or not solutions of the equation under consideration are oscillatory. For the similinear elliptic equation

$$\sum_{i,j=1}^{N} D_i[\, a_{ij}(x) D_j y\,] + p(x) f(y) = 0, \tag{1.2}$$

the oscillation theory is fully developed by [6, 9, 11-13, 15] where further references can be found. In particular, Noussair and Swanson [6] first gave Fite-Leighton type oscillation criteria [2, 5] for Eq.(1.2). For a related study, we refer to [12] in which a classical Kamenev theorem [3] (as extended and improved by Phiols [7] and Yan [14]) is to be extended to Eq.(1.2). However, as far as we know that the equation (1.1) in general form has never been the subject of systematic investigations.

In the case when $N = 1$, $a_{ij}(x) = 1$ for all i, j, $f(y) = y$, Eq.(1.2) reduces second order ordinary differential equation

$$y''(t) + p(t)y(t) = 0, \quad p \in C([t_0, \infty), \mathbb{R}). \tag{1.3}$$

Recently, by using an weighted function, Sun [8] gave an interesting result. More precisely, Sun proved the following theorem.

Theorem 1.1. *Equation (1.3) is oscillatory provided that for each $l \geq t_0$, there exists a constant $\alpha > 1/2$, such that*

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \int_l^t (t-s)^{2\alpha}(s-l)^2 p(s)ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.$$

Early similar results were proved by Kamenev [3], Kong [4], Philos [7], Wintner [10] and Yan [14]. But, Theorem 1.1 is simpler and more sharper than that of previous results. It is noting that Theorem 1.1 is given in [8] for a differential equation which is more general than Eq.(1.3). But, the above particular form of Sun's theorem is the basic one.

In present paper, one main objective is to extend Theorem 1.1 to Eq.(1.1). In section 2, by using an weighted function $H(r, s, l)$, we shall establish some oscillation criteria for Eq.(1.1) for the case when $D_i b_i(x)$ exists for all i . Then in section 3, we deal with the oscillation of Eq.(1.1) for the case when $D_i b_i(x)$ does not exist for some i . Finally in section 4, we will show the application of our oscillation criteria by several examples.

To formulate our results we shall use the following notations.

Following Sun [8], we shall define a class of functions $\mathcal{H}$. For this purpose, we first define the sets.

$$D_0 = \{(r, s, l) : r > s > l \geq a\} \quad \text{and} \quad D = \{(r, s, l) : r \geq s \geq l \geq a\}.$$

An weighted function $H \in C(D, \mathbb{R})$ is said to belong to the class $\mathcal{H}$ defined by $H \in \mathcal{H}$ if

(H_1) $H(r,r,l)=0, H(r,l,l)=0$ for $r>l\geq a$, and $H(r,s,l)\neq 0$ for $(r,s,l)\in D_0$.

(H_2) $H(r,s,l)$ has a continuous partial derivative on D with respect to the second variable, and there is a function $h\in C(D_0,\mathbb{R})$ such that

$$\frac{\partial H}{\partial s}(r,s,l)=h(r,s,l)H(r,s,l) \quad \text{for } (r,s,l)\in D_0. \tag{1.4}$$

Let $H\in\mathcal{H}$ and $\phi\in C([a,\infty),\mathbb{R})$. We now define an integral operator $\mathbf{T}_l^r$ in terms of $H(r,s,l)$ and $\phi(s)$ as

$$\mathbf{T}_l^r(\phi)=\int_l^r H(r,s,l)\phi(s)ds \quad \text{for } r\geq l\geq a. \tag{1.5}$$

It is easily seen that $\mathbf{T}_l^r$ satisfies the following:

(A_1) $\mathbf{T}_l^r(k_1\phi_1+k_2\phi_2)=k_1\mathbf{T}_l^r(\phi_1)+k_2\mathbf{T}_l^r(\phi_2)$;

(A_2) $\mathbf{T}_l^r(\phi')=-\mathbf{T}_l^r(h\phi)$.

Here $\phi_1,\ \phi_2\in C([a,\infty),\mathbb{R})$, $\phi\in C^1([a,\infty),\mathbb{R})$, and k_1,k_2 are real numbers.

By choosing specific functions $H(r,s,l)$, it is possible to derive several oscillation criteria for Eq.(1.1). For instance, for an arbitrary positive function $\xi\in C([a,\infty),\mathbb{R}^+)$, define the kernel function

$$H(r,s,l)=\left[\int_s^r\frac{1}{\xi(\tau)}d\tau\right]^\alpha\left[\int_l^s\frac{1}{\xi(\tau)}d\tau\right]^\beta,\quad \alpha>0,\ \beta>0,\ (r,s,l)\in D, \tag{1.6}$$

where $\xi(\tau)=1, H(r,s,l)=(r-s)^\alpha(s-l)^\beta$, and when $\xi(\tau)=\tau, H(r,s,l)=(\ln r/s)^\alpha(\ln s/l)^\beta$. It is easily verified that the kernel function (1.6) satisfies (H_1) and (H_2).

2. Oscillation results for the case when $D_ib_i(x)$ exists for all i

In this section, we establish oscillation theorems which extend Theorem 1.1 to Eq.(1.1) for the case when $D_ib_i(x)$ exists for all i . For this purpose, we shall impose the following conditions:

(C_1) $f\in C(\mathbb{R},\mathbb{R})\cup C^1(\mathbb{R}-\{0\},\mathbb{R})$, $yf(y)>0$ and $f'(y)\geq k>0$ whenever $y\neq 0$;

(C_2) $p\in C^\mu_{loc}(\Omega(a),\mathbb{R})$, $\mu\in(0,1)$;

(C_3) $b_i\in C^{1+\mu}_{loc}(\Omega(a),\mathbb{R})$ for all i, $\mu\in(0,1)$;

(C_4) $A=(a_{ij})_{N\times N}$ is a real symmetric positive definite matrix function with $a_{ij}\in C^{1+\mu}_{loc}(\Omega(a),\mathbb{R})$ for all i,j, $\mu\in(0,1)$.

Denote by $\lambda_{\max}(x)$ the largest eigenvalue of the matrix A. We suppose that there exists a function $\lambda\in C([a,\infty),\mathbb{R}^+)$ such that

$$\lambda(r)\geq\max_{|x|=r}\lambda_{\max}(x) \quad \text{for } \ r>a.$$

Theorem 2.1. *Let (C_1)-(C_4) hold. Suppose that for each $l\geq a$, there exist functions $\eta\in C([a,\infty),\mathbb{R})$, $H\in\mathcal{H}$, such that*

$$\limsup_{r\to\infty}\mathbf{T}_l^r\left(\theta_1-\frac{1}{4}g_1h^2\right)>0, \tag{2.1}$$

where

$$\theta_1(r) = \rho_1(r) \left\{ \int\limits_{S_r} \left[p(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \sum_{i=1}^{N} D_i b_i \right] d\sigma + \frac{k\,\eta^2(r) r^{1-N}}{\omega_N \lambda(r)} - \eta'(r) \right\},$$

$$g_1(r) = \frac{\omega_N}{k} \lambda(r) \rho_1(r) r^{N-1}, \quad \rho_1(r) = \exp\left[-\int\limits_a^r \frac{2\,k\,\eta(s) s^{1-N}}{\omega_N \lambda(s)} ds \right],$$

and $S_r = \{x \in \mathbb{R}^N : |x| = r\}$ *for* $r > 0$, $B^T = \{b_1(x), \cdots, b_N(x)\}$, σ *denotes the measure on* S_r, ω_N *denotes the surface area of the unit sphere in* $\mathbb{R}^N$, *i.e.,* $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$. *Then Eq.(1.1) is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of Eq.(1.1), and suppose that there exists a $b \geq a$ such that $y = y(x) \neq 0$ for all $x \in \Omega(b)$. Define

$$W(x) = \frac{1}{f(y)} A(x) Dy + \frac{1}{2k} B \quad \text{for } x \in \Omega(b), \tag{2.2}$$

where $Dy = (\,D_1 y, \cdots, D_N y\,)^T$. Differentiation of the $i-$th component of (2.2) with respect to x_i gives

$$D_i W_i(x) = -\frac{f'(y)}{f^2(y)} D_i y \left[\sum_{i=1}^{N} a_{ij} D_j y \right] + \frac{1}{f(y)} D_i \left[\sum_{j=1}^{N} a_{ij} D_j y \right] + \frac{1}{2k} D_i b_i,$$

for all i. Summation over i, using of Eq.(1.1) and (2.2), leads to

$$\begin{aligned} \operatorname{div} W(x) &= -\frac{f'(y)}{f^2(y)} (Dy)^T A Dy - \frac{1}{f(y)} \left[p(x) f(y) + B^T Dy \right] + \frac{1}{2k} \sum_{i=1}^{N} D_i b_i \\ &\leq -k \left[W - \frac{1}{2k} B \right]^T A^{-1} \left[W - \frac{1}{2k} B \right] - p(x) \\ &\quad -B^T A^{-1} \left[W - \frac{1}{2k} B \right] + \frac{1}{2k} \sum_{i=1}^{N} D_i b_i \\ &= -k W^T A^{-1} W - p(x) + \frac{1}{4k} B^T A^{-1} B + \frac{1}{2k} \sum_{i=1}^{N} D_i b_i. \end{aligned} \tag{2.3}$$

Put

$$Z(r) = \rho_1(r) \left[\int_{S_r} W(x) \cdot \nu(x)\, d\sigma + \eta(r) \right] \quad \text{for } \ r \geq b, \tag{2.4}$$

where $\nu(x) = x/r$, $r = |x| \neq 0$, denotes the outward unit normal to S_r. By means of the Green formula and (2.3), we have

$$\begin{aligned} Z'(r) &= \frac{\rho_1'(r)}{\rho_1(r)} Z(r) + \rho_1(r) \left\{ \int_{S_r} \operatorname{div} W(x)\, d\sigma + \eta'(r) \right\} \\ &\leq \frac{\rho_1'(r)}{\rho_1(r)} Z(r) - \rho_1(r) \left\{ k \int_{S_r} (W^T A^{-1} W)(x)\, d\sigma \right. \\ &\quad \left. + \int_{S_r} \left[p(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \sum_{i=1}^{N} D_i b_i \right] d\sigma - \eta'(r) \right\}. \end{aligned} \tag{2.5}$$

In view of (C_4), we have that

$$(W^T A^{-1} W)(x) \geq \lambda_{\max}^{-1}(x)|W(x)|^2. \tag{2.6}$$

The Schwartz inequality yields

$$\int_{S_r} |W(x)|^2 \, d\sigma \geq \frac{r^{1-N}}{\omega_N} \left[\int_{S_r} W(r) \cdot \nu(x) \, d\sigma \right]^2. \tag{2.7}$$

Thus, by (2.5)-(2.7), we obtain

$$\begin{aligned}
Z'(r) &\leq \frac{\rho_1'(r)}{\rho_1(r)} Z(r) - \rho_1(r) \left\{ \frac{k r^{1-N}}{\omega_N \lambda(r)} \left[\int_{S_r} W(x) \cdot \nu(x) \, d\sigma \right]^2 \right. \\
&\quad \left. + \int_{S_r} \left[p(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \sum_{i=1}^{N} D_i b_i \right] d\sigma - \eta'(r) \right\} \\
&= \frac{\rho_1'(r)}{\rho_1(r)} Z(r) - \rho_1(r) \left\{ \frac{k r^{1-N}}{\omega_N \lambda(r)} \left[\frac{Z(r)}{\rho_1(r)} - \eta(r) \right]^2 \right. \\
&\quad \left. + \int_{S_r} \left[p(x) - \frac{1}{4k} B^T A^{-1} B - \frac{1}{2k} \sum_{i=1}^{N} D_i b_i \right] d\sigma - \eta'(r) \right\} \\
&= -\theta_1(r) - \frac{1}{g_1(r)} Z^{\,2}(r),
\end{aligned}$$

that is, for $r \geq b$

$$Z'(r) \leq -\theta_1(r) - \frac{1}{g_1(r)} Z^{\,2}(r). \tag{2.8}$$

Applying the operator $\mathbf{T}_l^{\,r}$ to (2.8), we have the following inequality

$$\mathbf{T}_b^{\,r}(\theta_1) \leq \mathbf{T}_b^{\,r}(Z') - \mathbf{T}_b^{\,r}\left(\frac{1}{g_1} Z^2 \right)$$

In view of (H_2), (A_1) and (A_2), we get that

$$\begin{aligned}
\mathbf{T}_b^{\,r}(\theta_1) &\leq \mathbf{T}_b^{\,r}(hZ) - \mathbf{T}_b^{\,r}\left(\frac{1}{g_1} Z^2 \right) \\
&\leq -\mathbf{T}_b^{\,r}\left(\left[\frac{1}{\sqrt{g_1}} Z - \frac{1}{2}\sqrt{g_1} h \right]^2 \right) + \frac{1}{4} \mathbf{T}_b^{\,r}\left(g_1 h^2 \right).
\end{aligned} \tag{2.9}$$

Clearly, inequality (2.9) contradicts (2.1). $\square$

For the case $H(r,s,l) = H_1(r,s)H_2(s,l)$, by Theorem 2.1, we have the following theorem.

Theorem 2.2. *Let $(C_1) - (C_4)$ hold. Suppose that for each $l \geq a$, there exist functions $\eta \in C([a,\infty),\mathbb{R})$, H_1, $H_2 \in C(D_1,\mathbb{R})$, such that*

$$\limsup_{r\to\infty} \int_l^r H_1(r,s) H_2(s,l) \left\{ \theta_1(s) - \frac{1}{4} g_1(s) \left[h_1(r,s) - h_2(s,l) \right]^2 \right\} ds > 0, \tag{2.10}$$

where $D_1 = \{(r,s) : r \geq l \geq a\}$, h_1, h_2 *are defined as*

$$\frac{\partial}{\partial s} H_1(r,s) = -h_1(r,s) H_1(r,s), \quad \frac{\partial}{\partial s} H_2(s,l) = h_2(s,l) H_2(s,l),$$

and θ_1, g_1 *are defined as in Theorem 2.1. Then Eq.(1.1) is oscillatory.*

Now, define

$$G_1(r) = \int_a^r \frac{1}{g_1(s)} ds, \quad r \geq l \geq a,$$

and

$$H(r,s,l) = [\, G_1(r) - G_1(s)\,]^\alpha [\, G_1(s) - G_1(l)\,]^2, \quad (r,s,l) \in D,$$

for some $\alpha > 1$, then

$$h(r,s,l) = \left[-\frac{\alpha}{G_1(r) - G_1(s)} + \frac{2}{G_1(s) - G_1(l)} \right] \frac{1}{g_1(s)}, \quad (r,s,l) \in D_0.$$

Based on the above results, we obtain the following Kamenev type oscillation criteria.

Theorem 2.3. *Let* $(C_1) - (C_4)$ *hold, and* $\lim_{r\to\infty} G_1(r) = \infty$. *Suppose that for each* $l \geq a$, *there exist a function* $\eta \in C([a,\infty), \mathbb{R})$ *and some* $\alpha > 1$, *such that*

$$\limsup_{r\to\infty} \frac{1}{G_1^{\alpha+1}(r)} \int_l^r [\, G_1(r) - G_1(s)\,]^\alpha [\, G_1(s) - G_1(l)\,]^2 \theta_1(s) ds > \frac{\alpha}{2(\alpha^2 - 1)}, \tag{2.11}$$

where g_1, θ_1 *are defined as in Theorem 2.1. Then Eq.(1.1) is oscillatory.*

Proof. Noting that

$$\begin{aligned}
&\int_l^r H(r,s,l) g_1(s) h^2(r,s,l) ds \\
&= \int_l^r [\, G_1(r) - G_1(s)\,]^\alpha [\, G_1(s) - G_1(l)\,]^2 \left[\frac{\alpha}{G_1(r) - G_1(s)} - \frac{2}{G_1(s) - G_1(l)} \right]^2 \frac{1}{g_1(s)} ds \\
&= \int_l^r [\, G_1(r) - G_1(s)\,]^{\alpha-2} \left\{ \alpha [\, G_1(s) - G_1(l)\,] - 2[\, G_1(r) - G_1(s)] \right\}^2 dG(s) \\
&= \frac{2\alpha}{\alpha^2 - 1} [\, G_1(r) - G_1(l)\,]^{\alpha+1}.
\end{aligned} \tag{2.12}$$

In view of $\lim_{r\to\infty} G_1(r) = \infty$, from (2.11) and (2.12), we have that

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{1}{G_1^{\alpha+1}(r)} \int_l^r H(r,s,l) \left[\theta_1(s) - \frac{1}{4} g_1(s) h^2(r,s,l) \right] ds \\
&= \limsup_{r\to\infty} \frac{1}{G_1^{\alpha+1}(r)} \int_l^r [\, G_1(r) - G_1(s)\,]^\alpha [\, G_1(s) - G_1(l)\,]^2 \theta_1(s) ds - \frac{\alpha}{2(\alpha^2 - 1)} > 0.
\end{aligned}$$

It follows that

$$\limsup_{r\to\infty}\int_l^r H(r,s,l)\left[\theta_1(s)-\frac{1}{4}g_1(s)h^2(r,s,l)\right]ds>0,$$

i.e., (2.1) holds. By Theorem 2.1, Eq.(1.1) is oscillatory. □

Similar to that of the proof of Theorem 2.3, we have

Theorem 2.4. *Let $(C_1)-(C_4)$ hold, and $\lim_{r\to\infty}G_1(r)=\infty$. Suppose that for each $l\geq a$, there exist a function $\eta\in C([a,\infty),\mathbb{R})$ and some $\alpha>1$ such that*

$$\limsup_{r\to\infty}\frac{1}{G_1^{\alpha+1}(r)}\int_l^r[\,G_1(r)-G_1(s)\,]^2[\,G_1(s)-G_1(l)\,]^\alpha\theta_1(s)ds>\frac{\alpha}{2(\alpha^2-1)},\tag{2.13}$$

where g_1, θ_1 are defined as in Theorem 2.1, and $G_1(s)$ is defined in as Theorem 2.3. Then Eq.(1.1) is oscillatory.

3. Oscillation results for the case when $D_ib_i(x)$ does not exist for some i

In this section, we establish oscillation criteria for Eq.(1.1) in case when $D_ib_i(x)$ does not exist for some i. We begin with the following lemma, the proof of this lemma is easy and thus omitted.

Lemma 3.1. *For two n-dimensional vectors u, $v\in\mathbb{R}^N$, and a positive constant c, then*

$$c\,u\,u^T+u\,v^T\geq\frac{c}{2}u\,u^T-\frac{1}{2c}v\,v^T.\tag{3.1}$$

Theorem 3.1. *Let $(C_1),(C_2),(C_4)$, and*

(C_3') $b_i\in C^\mu_{loc}(\Omega(a),\mathbb{R})$ for all i, $\mu\in(0,1)$

hold. Suppose that for each $l\geq a$, there exist functions $\eta\in C([a,\infty),\mathbb{R})$, $H\in\mathcal{H}$, such that

$$\limsup_{r\to\infty}\mathbf{T}_l^r\left(\theta_2-\frac{1}{4}g_2h^2\right)>0,\tag{3.2}$$

where

$$\theta_2(r)=\rho_2(r)\left\{\int_{S_r}\left[p(x)-\frac{1}{2k}\lambda(x)|B^TA^{-1}|^2\right]d\sigma+\frac{k\,\eta^2(r)r^{1-N}}{2\,\omega_N\lambda(r)}-\eta'(r)\right\},$$

$$g_2(r)=\frac{2\,\omega_N}{k}\lambda(r)\rho_2(r)r^{N-1}\quad\rho_2(r)=\exp\left[-\int_a^r\frac{k\,\eta(s)s^{1-N}}{\omega_N\lambda(s)}ds\right],$$

and S_r, $d\sigma,\omega_N$ are defined as in Theorem 2.1. Then Eq.(1.1) is oscillatory.

Proof. Let $y = y(x)$ be a nonoscillatory solution of Eq.(1.1), and suppose that there exists a $b \geq a$ such that $y = y(x) \neq 0$ for all $x \in \Omega(b)$. Define

$$W(x) = \frac{1}{f(y)} A(x) Dy \quad \text{for } x \in \Omega(b). \tag{3.3}$$

Differentiation of the $i-$th component of (3.3) with respect to x_i gives

$$D_i W_i(x) = -\frac{f'(y)}{f^2(y)} D_i y \left[\sum_{i=1}^{N} a_{ij} D_j y \right] + \frac{1}{f(y)} \left[\sum_{j=1}^{N} a_{ij} D_j y \right],$$

for all i. Summation over i, using of Eq.(1.1), leads to

$$\begin{aligned} \operatorname{div} W(x) &= -f'(y)(W^T A^{-1} W)(x) - (B^T A^{-1} W)(x) - p(x) \\ &\leq -k(W^T A^{-1} W)(x) - (B^T A^{-1} W)(x) - p(x) \\ &\leq -\frac{k}{\lambda(x)}(W^T W)(x) - (B^T A^{-1} W)(x) - p(x) \qquad (\text{by Lemma 3.1}) \\ &\leq -\frac{k}{2\lambda(x)} |W(x)|^2 + \frac{1}{2\,k} \lambda(x) |B^T A^{-1}|^2 - p(x). \end{aligned} \tag{3.4}$$

Put

$$Z(r) = \rho_2(r) \left[\int_{S_r} W(x) \cdot \nu(x)\, d\sigma + \eta(r) \right] \quad \text{for } r \geq b. \tag{3.5}$$

By means of the Green formula in (3.5), in view of (3.4), we obtain

$$\begin{aligned} Z'(r) &= \frac{\rho_2'(r)}{\rho_2(r)} Z(r) + \rho_2(r) \left\{ \int\limits_{S_r} \operatorname{div} W(x)\, d\sigma + \eta'(r) \right\} \\ &\leq \frac{\rho_2'(r)}{\rho_2(r)} Z(r) - \rho_2(r) \left\{ \frac{k}{2\lambda(r)} \int\limits_{S_r} |W(x)|^2 d\sigma \right. \\ &\qquad \left. + \int\limits_{S_r} \left[p(x) - \frac{1}{2\,k} \lambda(x) |B^T A^{-1}|^2 \right] d\sigma - \eta'(r) \right\}. \end{aligned} \tag{3.6}$$

Thus, by (2.6), (2.7) and (3.6), we get that

$$\begin{aligned} Z'(r) &\leq \frac{\rho_2'(r)}{\rho_2(r)} Z(r) - \rho_2(r) \left\{ \frac{k\, r^{1-N}}{2\,\omega_N \lambda(r)} \left[\int\limits_{S_r} W(x) \cdot \nu(x)\, d\sigma \right]^2 \right. \\ &\qquad \left. + \int\limits_{S_r} \left[p(x) - \frac{1}{2\,k} \lambda(x) |B^T A^{-1}|^2 \right] d\sigma - \eta'(r) \right\} \\ &= \frac{\rho_2'(r)}{\rho_2(r)} Z(r) - \rho_2(r) \left\{ \frac{k\, r^{1-N}}{2\,\omega_N \lambda(r)} \left[\frac{Z(r)}{\rho_2(r)} - \eta(r) \right]^2 \right. \\ &\qquad \left. + \int\limits_{S_r} \left[p(x) - \frac{1}{2\,k} \lambda(x) |B^T A^{-1}|^2 \right] d\sigma - \eta'(r) \right\} \\ &= -\theta_2(r) - \frac{1}{g_2(r)} Z^2(r). \end{aligned}$$

The rest of proof is similar to that of Theorem 2.1 and hence omitted . $\square$

The following results are analogous to Theorems 2.2-2.4 with the assumption (C_3) replaced by (C_3'), so these proofs are omitted.

Theorem 3.2. *Let $(C_1), (C_2), (C_3')$, and (C_4) hold. Suppose that for each $l \geq a$, there exist functions $\eta \in C([a,\infty),\mathbb{R})$, H_1, $H_2 \in C(D_1,\mathbb{R})$, such that*

$$\limsup_{r\to\infty} \int_l^r H_1(r,s)H_2(s,l)\left\{\theta_2(s) - \frac{1}{4}g_2(s)\left[\,h_1(r,s) - h_2(s,l)\,\right]^2\right\} ds > 0, \tag{3.7}$$

where D_1, h_1, h_2 are defined as in Theorem 2.2, and g_2, θ_2 are defined as in Theorem 3.1. Then Eq.(1.1) is oscillatory.

Theorem 3.3. *Let $(C_1), (C_2), (C_3')$ and (C_4) hold and $\lim\limits_{r\to\infty} G_2(r) = \infty$. Suppose that for each $l \geq a$, there exist a function $\eta \in C([a,\infty),\mathbb{R})$ and some $\alpha > 1$, such that*

$$\limsup_{r\to\infty} \frac{1}{G_2^{\alpha+1}(r)} \int_l^r [\,G_2(r) - G_2(s)\,]^{\alpha}[\,G_2(s) - G_2(l)\,]^2\theta_2(s)ds > \frac{\alpha}{2(\alpha^2-1)}, \tag{3.8}$$

where g_2, θ_2 are defined as in Theorem 3.1, and $G_2(r) = \int\limits_a^r 1/g_2(s)ds$. Then Eq.(1.1) is oscillatory.

Theorem 3.4. *Let $(C_1), (C_2), (C_3')$ and (C_4) hold, and $\lim\limits_{r\to\infty} G_2(r) = \infty$. Suppose that for each $l \geq a$, there exist a function $\eta \in C([a,\infty),\mathbb{R})$ and some $\alpha > 1$, such that*

$$\limsup_{r\to\infty} \frac{1}{G_2^{\alpha+1}(r)} \int_l^r [\,G_2(r) - G_2(s)\,]^2[\,G_2(s) - G_2(l)\,]^{\alpha}\theta_2(s)ds > \frac{\alpha}{2(\alpha^2-1)}, \tag{3.9}$$

where g_2, θ_2 are defined as in Theorem 3.1, and G_2 is defined as in Theorem 3.3. Then Eq.(1.1) is oscillatory.

4. Examples and Remarks

In this section, we will show the applications of our oscillation criteria by two examples. We will see that the equations in these examples are oscillatory based on sections 2 and 3, though the oscillation cannot be demonstrated by the results in [6, 9, 11-13, 15].

Example 4.1. Consider Eq.(1.1) with

$$\begin{aligned} &A(x) = diag\left(\frac{1}{|x|}, \frac{1}{|x|}\right), \quad B(x) = \left(\frac{\sqrt{3}}{|x|^2}, 0\right)^T, \\ &p(x) = \frac{\gamma}{4\pi^2|x|\ln^2|x|}, \quad f(y) = y + y^5, \end{aligned} \tag{4.1}$$

where $|x| \geq 1$, $N = 2$, $\gamma > 0$, $k = 1$, and $\lambda(r) = 1/r$.

Let us apply Theorem 2.4 with $\eta(r) = \pi/r$, so that

$$\rho_1(r) = \frac{1}{r}, \quad g_1(r) = \frac{2\pi}{r}, \quad G_1(r) = 2\pi \ln r, \quad \theta_1(r) = \frac{\gamma}{2\pi r \ln^2 r}.$$

A straightforward computation yields, for some $\alpha > 1$ and $r \geq l \geq 1$,

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{1}{G_1^{\alpha+1}(r)} \int_l^r [\,G_1(r) - G_1(s)\,]^2 [\,G_1(s) - G_1(l)\,]^\alpha \theta_1(s) ds \\
&= \gamma \limsup_{r\to\infty} \frac{1}{(\ln r)^{\alpha+1}} \int_l^r \left(\ln \frac{r}{s}\right)^2 \left(\ln \frac{s}{l}\right)^{\alpha+1} \frac{1}{s \ln^2 s} ds \\
&= \frac{2\gamma}{\alpha(\alpha^2-1)}.
\end{aligned}$$

Then for any $\gamma > \frac{1}{4}$, there exists a constant $\alpha > 1$ such that $\frac{2\gamma}{\alpha(\alpha^2-1)} > \frac{\alpha}{2(\alpha-1)}$. i.e., (2.13) holds. Using Theorem 2.4, we find that Eq.(4.1) is oscillatory if $\gamma > \frac{1}{4}$.

Example 4.2. Consider Eq.(1.1) with

$$\begin{aligned}
&A(x) = diag(1,1), \quad B(x) = \left(\frac{|\sin|x||}{|x|}, \frac{|\sin|x||}{|x|}\right)^T, \\
&p(x) = \frac{\gamma}{8\pi^2|x|^2}, \quad f(y) = y + y^3,
\end{aligned} \tag{4.2}$$

where $|x| \geq 1$, $N = 2$, $\gamma > 0$, $k = 1$, and $\lambda(r) = 1$.

Let us apply Theorem 3.4 with $\eta(r) = 2\pi$, so that

$$\rho_2(r) = \frac{1}{r}, \quad g_2(r) = 4\pi, \quad G_2(r) = 4\pi(r-1), \quad \theta_2(r) = \frac{\gamma}{4\pi r^2}.$$

A straightforward computation yields, for some $\alpha > 1$ and $r \geq l \geq 1$,

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{1}{G_2^{\alpha+1}(r)} \int_l^r [\,G_2(r) - G_2(s)\,]^2 [\,G_2(s) - G_2(l)\,]^\alpha \theta_2(s) ds \\
&= \gamma \limsup_{r\to\infty} \frac{1}{r^{\alpha+1}} \int_l^r (r-s)^2 (s-l)^{\alpha+1} \frac{1}{s^2} ds \\
&= \frac{2\gamma}{\alpha(\alpha^2-1)}.
\end{aligned}$$

Then for any $\gamma > \frac{1}{4}$, there exists a constant $\alpha > 1$ such that $\frac{2\gamma}{\alpha(\alpha^2-1)} > \frac{\alpha}{2(\alpha-1)}$. i.e., (3.9) holds. Using Theorem 3.4, we find that Eq.(4.2) is oscillatory if $\gamma > \frac{1}{4}$.

Remark 4.1. The above results hold true if we replace condition $f'(y) \geq k$ for $y \neq 0$ with following one:

$$\frac{f(y)}{y} \geq k > 0, \quad \text{for } y \neq 0,$$

but the function $p(x)$ should be nonnegative for all $x \in \Omega(a)$.

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