Electronic Journal of Qualitative Theory of Differential Equations
2005, No. **6**, 1-28; **http://www.math.u-szeged.hu/ejqtde/**

Solvability for a Nonlinear Coupled System of Kirchhoff type for the Beam Equations with Nonlocal Boundary Conditions

M. L. Santos *, M. P. C. Rocha & D. C. Pereira
Departamento de Matemática, Universidade Federal do Pará
Campus Universitario do Guamá,
Rua Augusto Corrêa 01, CEP 66075-110, Pará, Brazil
Instituto de Estudos Superiores da Amazônia (IESAM)
Av. Gov. José Malcher 1148, CEP 66.055-260, Belém-Pa., Brazil
FACI-Faculdade Ideal
J. Ferreira
Departamento de Estatística-DE, Universidade Federal do Pará-UFPA
Campus Universitario do Guamá,
Rua Augusto Corrêa 01, CEP 66075-110, Pará, Brazil.

Abstract. In this paper, we investigate a mathematical model for a nonlinear coupled system of Kirchhoff type of beam equations with nonlocal boundary conditions. We establish existence, regularity and uniqueness of strong solutions. Furthermore, we prove the uniform rate of exponential decay. The uniform rate of polynomial decay is considered.

2000 Mathematical Subject Classifications: 11D61, 35J25

Keywords: Kirchhoff beam equation; nonlocal boundary condition; asymptotic behavior.

1. INTRODUCTION

The aim of this article is to study the asymptotic behavior of the solution of a Kirchhoff plates equations system with nonlinear coupled and nonlocal boundary condition. For this, we consider

*Correspondence to: M. L. Santos, Departamento de Matemática, Universidade Federal do Pará, Campus Universitario do Guamá, Rua Augusto Corrêa 01, CEP 66075-110, Pará, Brazil

the following initial boundary-value problem:

$$u_{tt} + u_{xxxx} - M(\int_0^L (u_x^2 + v_x^2)dx)u_{xx} + f(u - v) = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.1}$$

$$v_{tt} + v_{xxxx} - M(\int_0^L (u_x^2 + v_x^2)dx)v_{xx} - f(u - v) = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.2}$$

$$u(0, t) = v(0, t) = u_x(0, t) = v_x(0, t) = 0 \quad \forall t > 0, \tag{1.3}$$

$$u_x(L, t) + \int_0^t g_1(t - s)(u_{xx}(L, s) + \rho_1 u_x(L, s))ds = 0, \quad \forall t > 0, \tag{1.4}$$

$$v_x(L, t) + \int_0^t g_2(t - s)(v_{xx}(L, s) + \rho_2 v_x(L, s))ds = 0, \quad \forall t > 0, \tag{1.5}$$

$$\begin{aligned} &u(L, t) + \int_0^t g_3(t - s)(u_{xxx} - M(\int_0^L (u_x^2 + v_x^2)dx)u_x(L, s) \\ &-\rho_3 u(L, s))ds = 0, \quad \forall t > 0 \end{aligned} \tag{1.6}$$

$$\begin{aligned} &v(L, t) + \int_0^t g_4(t - s)(v_{xxx} - M(\int_0^L (u_x^2 + v_x^2)dx)v_x(L, s) \\ &-\rho_4 v(L, s))ds = 0, \quad \forall t > 0 \end{aligned} \tag{1.7}$$

$$(u(0, x), u(0, x)) = (u_0(x), v_0(x)) \quad \text{in} \quad (0, L), \tag{1.8}$$

$$(u_t(0, x), v_t(0, x)) = (u_1(x), v_1(x)) \quad \text{in} \quad (0, L). \tag{1.9}$$

The equations (1.4)-(1.7) are nonlocal boundary conditions responsible for the memory effect and ρ_1, ρ_2, ρ_3 and ρ_4 are positive constants. Considering the history conditions, we must add to conditions (1.4)-(1.7) the one given by

$$u(L, t) = v(L, t) = u_x(L, t) = v_x(L, t) = 0, \quad \forall t \leq 0.$$

We observe that in problem (1.1)-(1.9) u and v represents the transverse displacement, the relaxation functions g_i, $i = 1, \ldots, 4$, are positive and non increasing belonging to $W^{1,2}(0, \infty)$ and the function $f \in C^1(\mathbb{R})$ satisfies

$$f(s)s \geq 0 \quad \forall s \in \mathbb{R}. \tag{1.10}$$

Additionally, we suppose that f is superlinear, that is

$$f(s)s \geq (2 + \delta)F(s), \quad F(z) = \int_0^z f(s)ds \quad \forall s \in \mathbb{R}, \tag{1.11}$$

for some $\delta > 0$ with the following growth conditions

$$|f(x) - f(y)| \leq C(1 + |x|^{\rho-1} + |y|^{\rho-1})|x - y|, \quad \forall x, y \in \mathbb{R}, \tag{1.12}$$

for some $C > 0$ and $\rho \geq 1$. We shall assume that the function $M \in C^1([0, \infty[)$ satisfy

$$M(\lambda) \geq m_0 > 0, \quad M(\lambda)\lambda \geq \widehat{M}(\lambda), \quad \forall \lambda \geq 0, \tag{1.13}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(s)ds$. Note that because of condition (1.3) the solution of the system (1.1)-(1.9) must belong to the following space

$$W := \{v \in H^2(0,1) \ : \ v(0) = v_x(0) = 0\}.$$

This problem is based on the equation

$$u_{tt} + u_{xxxx} - (\alpha + \beta \int_0^L |u_x(s,t)|^2 ds) u_{xx} = 0 \tag{1.14}$$

which was proposed by Woinowsky-Krieger [15] as a model for vibrating beams with hinged ends.

The equation (1.14) was studied by Dickey [3] by using Fourier sines series, and also by Bernstein [2] and Ball [1] by introducing terms to account for the effects of internal and external damping. A larger class of nonlinear beam equations was studied by Rivera [9], which considered the question regarding viscoelastic effects and regularizing properties. Tucsnak [14] considered the beam equation

$$u_{tt} + \Delta^2 u - M(||\nabla u||_2^2)\Delta u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{1.15}$$

with clamped boundary. He obtained the exponential decay of the energy when a damping of the type $a(x)u_t$ is effective near the boundary. In the same direction, Kouemou Patcheu [5] obtained the exponential decay of the energy for (1.15) when a nonlinear damping $g(u_t)$ was effective in Ω. Also see Pazoto-Menzala [10], To Fu Ma [7] and the references therein. As far as we know there is no result concerning the asymptotic stability of solutions for the system (1.1)-(1.9) where the coupledness is nonlinear and boundary conditions are of memory type. So with the intention to fill this gap we consider here this problem.

Remark: The results obtained in the paper are not valid when $M(s) = s$, being like this the case M(s)=s is an important open problem.

As we have said before we study the asymptotic behavior of the solutions of system (1.1)-(1.9). We show that the energy decays to zero with the same rate of decay as g_i. That is, when the relaxation functions g_i decays exponentially then the energy decays exponentially. But if g_i decays polynomially then the energy will also decay polynomially with the same rate. This means that the memory effect produces strong dissipation capable of making a uniform rate of decay for the energy. The method used here is based on the construction of a suitable functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -C_1\mathcal{L}(t) + C_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \leq -C_1\mathcal{L}(t)^{1+\frac{1}{\alpha}} + \frac{C_2}{(1+t)^{\alpha+1}}$$

for some positive constants C_1, C_2, γ and α. The notation we use in this paper are standard and can be found in Lion's book [6]. In the sequel by C (sometimes $C_1, C_2, \dots$) we denote various positive constants which do not depend on t or on the initial data. The organization of this paper is as follows: In section 2 we prove a basic result on existence, regularity and uniqueness of strong solutions for the system(1.1)-(1.9). We use the Galerkin approximation, Aubin-Lions theorem, energy method introduced by Lions [6] and technical ideas to show existence, regularity and uniqueness of strong solutions for the problem (1.1)-(1.9). Finally in the sections 3 and 4 we study the stability of solutions for the system (1.1)-(1.9). We show that the dissipation is strong enough to produce exponential decay of solution, provided the relaxation function also decays exponentially. When the relaxation function decays polynomially, we show that the solution decays polynomially and with the same rate. We use the technique of the multipliers introduced by Komornik [4], Lions [6] and Rivera [8] coupled with some technical lemmas and some technical ideas.

2. EXISTENCE AND REGULARITY OF GLOBAL SOLUTIONS

In this section we shall study the existence and regularity of solutions for the system (1.1)-(1.9). First, we shall use equations (1.4)-(1.7) to estimate the terms $u_{xx}(L,t) + \rho_1 u_x(L,t)$, $v_{xx}(L,t) + \rho_2 v_x(L,t)$, $u_{xxx}(L,t) - M(\int_0^L (u_x^2 + v_x^2)dx)u_x(L,t) - \rho_3 u(L,t)$ and $v_{xxx}(L,t) - M(\int_0^L (u_x^2 + v_x^2)dx)v_x(L,t) - \rho_4 v(L,t)$. Denoting by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds,$$

the convolution product operator and differentiating the equations (1.4) and (1.7) we arrive at the following Volterra equations

$$
\begin{aligned}
&u_{xx}(L,t)+\rho_1 u_x(L,t)+\frac{1}{g_1(0)}g_1'*(u_{xx}(L,t)+\rho_1 u_x(L,t))\\
&=-\frac{1}{g_1(0)}u_{xt}(L,t),\\
&v_{xx}(L,t)+\rho_1 v_x(L,t)+\frac{1}{g_2(0)}g_2'*(v_{xx}(L,t)+\rho_2 v_x(L,t))\\
&=-\frac{1}{g_2(0)}u_{xt}(L,t),\\
&u_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)u_x(L,t)-\rho_3 u(L,t)\\
&+\frac{1}{g_3(0)}g_3'*(u_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)u_x(L,t)-\rho_3 u(L,t))\\
&=-\frac{1}{g_3(0)}u_t(L,t),\\
&v_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)v_x(L,t)-\rho_4 v(L,t)\\
&+\frac{1}{g_4(0)}g_4'*(v_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)v_x(L,t)-\rho_4 v(L,t))\\
&=-\frac{1}{g_4(0)}v_t(L,t).
\end{aligned}
$$

Applying the Volterra's inverse operator, we get

$$
\begin{aligned}
&u_{xx}(L,t)+\rho_1 u_x(L,t)=-\frac{1}{g_1(0)}\{u_{xt}(L,t)+k_1*u_{xt}(L,t)\},\\
&v_{xx}(L,t)+\rho_2 v_x(L,t)=-\frac{1}{g_2(0)}\{v_{xt}(L,t)+k_2*v_{xt}(L,t)\},\\
&u_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)u_x(L,t)-\rho_3 u(L,t)\\
&=-\frac{1}{g_3(0)}\{u_t(L,t)+k_3*u_t(L,t)\},\\
&v_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)v_x(L,t)-\rho_4 v(L,t)\\
&=-\frac{1}{g_4(0)}\{v_t(L,t)+k_4*v_t(L,t)\}
\end{aligned}
$$

where the resolvent kernels satisfy

$$
k_i+\frac{1}{g_i(0)}g_i'*k_i=-\frac{1}{g_i(0)}g_i',\quad \forall i=1,2,3,4.
$$

Denoting by $\eta_1 = \frac{1}{g_1(0)}$ and $\eta_2 = \frac{1}{g_2(0)}$ we obtain

$$\begin{aligned} & u_{xx}(L,t) + \rho_1 u_x(L,t) = -\eta_1\{u_{xt}(L,t) + k_1(0)u_x(L,t) \\ & -k_1(t)u_x(L,0) + k_1' * u_x(L,t)\} \end{aligned} \tag{2.1}$$

$$\begin{aligned} & v_{xx}(L,t) + \rho_2 v_x(L,t) = -\eta_2\{v_{xt}(L,t) + k_2(0)v_x(L,t) \\ & -k_2(t)v_x(L,0) + k_2' * v_x(L,t)\} \end{aligned} \tag{2.2}$$

$$\begin{aligned} & u_{xxx}(L,t) - M(\int_0^L (u_x^2 + v_x^2)dx)u_x(L,t) - \rho_3 u(L,t) \\ & = \eta_3\{u_t(L,t) + k_3(0)u(L,t) - k_3(t)u(L,0) + k_4' * u(L,t)\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & v_{xxx}(L,t) - M(\int_0^L (u_x^2 + v_x^2)dx)v_x(L,t) - \rho_4 v(L,t) \\ & = \eta_4\{v_t(1,t) + k_4(0)(L,t) - k_4(t)u(L,0) + k_4' * v(L,t)\} \end{aligned} \tag{2.4}$$

Note that taking initial data such that $u(L,0) = v(L,0) = u_x(L,0) = v_x(L,0) = 0$, the identities (2.1)-(2.4) imply (1.4)-(1.7). Since we are interested in relaxation functions of exponential or polynomial type and the identities (2.1)-(2.4) involve the resolvent kernels k_i, we want to know if k_i has the same properties. The following Lemma answers this question. Let h be a relaxation function and k its resolvent kernel, that is

$$k(t) - k * h(t) = h(t). \tag{2.5}$$

Lemma 2.1 *If h is a positive continuous function, then k also is a positive continuous function. Moreover,*

1. *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

   $$h(t) \le c_0 e^{-\gamma t},$$

   then, the function k satisfies

   $$k(t) \le \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$

   for all $0 < \epsilon < \gamma - c_0$.

2. *Given $p > 1$, let us denote by $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p}\, ds$. If there exists a positive constant c_0 with $c_0 c_p < 1$ such that*

   $$h(t) \le c_0 (1+t)^{-p},$$

   then, the function k satisfies

   $$k(t) \le \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}.$$

Proof. Note that $k(0) = h(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}$, so $k(t) > 0$ for all $t \in [0, t_0[$. If $t_0 \in \mathbb{R}^+$, from equation (2.5) we get that $-k * h(t_0) = h(t_0)$ but this is contradictory. Therefore $k(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, let us fix ϵ, such that $0 < \epsilon < \gamma - c_0$ and denote by

$$k_\epsilon(t) := e^{\epsilon t} k(t), \quad h_\epsilon(t) := e^{\epsilon t} h(t).$$

Multiplying equation (2.5) by $e^{\epsilon t}$ we get $k_\epsilon(t) = h_\epsilon(t) + k_\epsilon * h_\epsilon(t)$, hence

$$\sup_{s\in[0,t]} k_\epsilon(s) \;\leq\; \sup_{s\in[0,t]} h_\epsilon(s) + \left(\int_0^\infty c_0 e^{(\epsilon-\gamma)s}\, ds\right) \sup_{s\in[0,t]} k_\epsilon(s) \;\leq\; c_0 + \frac{c_0}{(\gamma-\epsilon)} \sup_{s\in[0,t]} k_\epsilon(s).$$

Therefore

$$k_\epsilon(t) \leq \frac{c_0(\gamma-\epsilon)}{\gamma-\epsilon-c_0},$$

which implies our first assertion. To show the second part let us consider the following notations

$$k_p(t) := (1+t)^p k(t), \quad h_p(t) := (1+t)^p h(t).$$

Multiplying equation (2.5) by $(1+t)^p$ we get $k_p(t) = h_p(t) + \int_0^t k_p(t-s)(1+t-s)^{-p}(1+t)^p h(s)\, ds$, hence

$$\sup_{s\in[0,t]} k_p(s) \;\leq\; \sup_{s\in[0,t]} h_p(s) + c_0 c_p \sup_{s\in[0,t]} k_p(s) \;\leq\; c_0 + c_0 c_p \sup_{s\in[0,t]} k_p(s).$$

Therefore

$$k_p(t) \leq \frac{c_0}{1 - c_0 c_p},$$

which proves our second assertion. ■

Remark: The finiteness of the constant c_p can be found in [13, Lemma 7.4]. Due to this Lemma, in the remainder of this paper, we shall use (2.1)-(2.4) instead of (1.4)-(1.7). Let us denote by

$$(g\Box\varphi)(t) := \int_0^t g(t-s)|\varphi(t) - \varphi(s)|^2 ds.$$

The following lemma states an important property of the convolution operator.

Lemma 2.2 *For $g, \varphi \in C^1([0,\infty[: \mathbb{R})$ we have*

$$(g * \varphi)\varphi_t \;=\; -\frac{1}{2} g(t)|\varphi(t)|^2 + \frac{1}{2} g'\Box\varphi - \frac{1}{2}\frac{d}{dt}\left[g\Box\varphi - \left(\int_0^t g(s)ds\right)|\varphi|^2\right].$$

The proof of this lemma follows by differentiating the term $g\Box\varphi$.

The first order energy of system (1.1)-(1.9) is given by

$$\begin{aligned}
E(t)=E(t;u,v): \;=\;& \frac{1}{2}\int_0^L(|u_t|^2+|v_t|^2+|u_{xx}|^2+|v_{xx}|^2)dx \\
&+\frac{1}{2}\widehat{M}(\int_0^L(u_x^2+v_x^2)dx)+\frac{\rho_1}{2}|u_x(L,t)|^2 \\
&+\frac{\rho_2}{2}|v_x(L,t)|^2+\frac{\rho_3}{2}|u(L,t)|^2+\frac{\rho_4}{2}|v(L,t)|^2 \\
&+\frac{\eta_1}{2}k_1|u_x(L,t)|^2-\frac{\eta_1}{2}k_1'\Box u_x(L,t) \\
&+\frac{\eta_2}{2}k_2|v_x(L,t)|^2-\frac{\eta_2}{2}k_2'\Box v_x(L,t) \\
&+\frac{\eta_3}{2}k_3|u(L,t)|^2-\frac{\eta_3}{2}k_3'\Box u(L,t) \\
&+\frac{\eta_4}{2}k_4|v(L,t)|^2-\frac{\eta_4}{2}k_4'\Box v(L,t) \\
&+\int_0^L F(u-v)dx.
\end{aligned}$$

The well-posedness of system (1.1)-(1.9) is given by the following Theorem.

Theorem 2.1 *Let $k_i\in C^2(\mathbb{R}^+)$ be such that*

$$k_i,-k_i',k_i''\geq 0,\quad \forall i=1,\ldots,4.$$

If (1.10)-(1.13) hold, $(u_0,v_0)\in(H^4(0,L)\cap W)^2$ and $(u_1,v_1)\in W^2$ satisfy the compatibility conditions

$$u_{0,xx}(L)+\rho_1u_{0,x}(L)=-\eta_1u_{1,x}(L) \tag{2.6}$$

$$v_{0,xx}(L)+\rho_2v_{0,x}(L)=-\eta_2v_{1,x}(L) \tag{2.7}$$

$$u_{0,xxx}(L)-M(\int_0^L(u_{0,x}^2+v_{0,x}^2)dx)u_{0,x}(L)-\rho_3u_0(L)=-\eta_3u_1(L) \tag{2.8}$$

$$v_{0,xxx}(L)-M(\int_0^L(u_{0,x}^2+v_{0,x}^2)dx)v_{0,x}(L)-\rho_4v_0(L)=-\eta_4v_1(L) \tag{2.9}$$

then there exists only one solution (u,v) of the coupled system (1.1)-(1.9) satisfying

$$u,v\in L^\infty_{loc}(0,\infty:H^4(0,L)\cap W),$$

$$u_t,v_t\in L^\infty_{loc}(0,\infty:W),$$

$$u_{tt},v_{tt}\in L^\infty_{loc}(0,\infty:L^2(0,L)).$$

Proof. Let us solve the variational problem associated with (1.1)-(1.9), which is given by: find $(u(t), v(t)) \in W \times W$ such that

$$\begin{aligned}
&\int_0^L u_{tt}(t) w dx + \int_0^L u_{xx}(t) w_{xx} dx + M(\|u_x(t)\|^2_{L^2(0,L)} + \|v_x(t)\|^2_{L^2(0,L)}) \int_0^L u_x(t) w_x dx \\
&+ \int_0^L f(u-v) w dx + u_{xxx}(L,t) w(L) \\
&- M(\|u_x(t)\|^2_{L^2(0,L)} + \|v_x(t)\|^2_{L^2(0,L)}) u_x(L,t) w(L) - u_{xx}(L,t) w_x(L) = 0 \\
&\int_0^L v_{tt}(t) w dx + \int_0^L v_{xx}(t) w_{xx} dx + M(\|u_x(t)\|^2_{L^2(0,L)} + \|v_x(t)\|^2_{L^2(0,L)}) \int_0^L v_x(t) w_x dx \\
&- \int_0^L f(u-v) w dx + v_{xxx}(L,t) w(L) \\
&- M(\|u_x(t)\|^2_{L^2(0,L)} + \|v_x(t)\|^2_{L^2(0,L)}) v_x(L,t) w(L) - v_{xx}(L,t) w_x(L) = 0,
\end{aligned}$$

for all $w \in W$. This is done with the Galerkin approximations. Let $\{w^j\}$ be a complete orthogonal system of W for which

$$\{u^0, u^1\} \in \mathrm{Span}\{w^1, w^2\}, \quad \{v^0, v^1\} \in \mathrm{Span}\{w^1, w^2\}.$$

For each $m \in \mathbb{N}$, let us put $W_m = Span\{w^1, w^2, \cdots, w^m\}$. We search for the functions

$$u^m(t) = \sum_{j=1}^m h^j w^j, \quad v^m(t) = \sum_{j=1}^m p^j w^j$$

such that for any $w \in W$, it satisfies the approximate equations

$$\begin{aligned}
&\int_0^L u^m_{tt}(t) w dx + \int_0^L u^m_{xx}(t) w_{xx} dx + M(\|u^m_x(t)\|^2_{L^2(0,L)} + \|v^m_x(t)\|^2_{L^2(0,L)}) \int_0^L u^m_x(t) w_x dx \\
&+ \int_0^L f(u^m - v^m) w dx + u^m_{xxx}(L,t) w(L) \\
&- M(\|u^m_x(t)\|^2_{L^2(0,L)} + \|v^m_x(t)\|^2_{L^2(0,L)}) u^m_x(L,t) w(L) - u^m_{xx}(L,t) w_x(L) = 0 \qquad (2.10) \\
&\int_0^L v^m_{tt}(t) w dx + \int_0^L v^m_{xx}(t) w_{xx} dx + M(\|u^m_x(t)\|^2_{L^2(0,L)} + \|v^m_x(t)\|^2_{L^2(0,L)}) \int_0^L v^m_x(t) w_x dx \\
&- \int_0^L f(u^m - v^m) w dx + v^m_{xxx}(L,t) w(L) \\
&- M(\|u^m_x(t)\|^2_{L^2(0,L)} + \|v^m_x(t)\|^2_{L^2(0,L)}) v^m_x(L,t) w(L) - v^m_{xx}(L,t) w_x(L) = 0 \qquad (2.11)
\end{aligned}$$

with initial conditions

$$(u^m(0), v^m(0)) = (u^0, v^0), \quad (u^m_t(0), v^m_t(0)) = (u^1, v^1) \tag{2.12}$$

We note that (2.10)-(2.12) are in fact an $m \times n$ system of ODEs in the variable t, which is known to have a local solution $(u^m(t), v^m(t))$ in an interval $[0, t_m[$. After the estimate below the approximate solution $(u^m(t), v^m(t))$ will be extended to the interval $[0, T]$, for any given $T > 0$.

Estimate 1

By substituting of (2.10) and (2.11) with $w = u_t^m(t)$ and $w = v_t^m(t)$, respectively, we see that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|u_t^m(t)\|^2_{L^2(0,L)} + \|u_{xx}^m(t)\|^2_{L^2(0,L)}\} \\
&+\frac{1}{2}M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})\frac{d}{dt}\|u_x^m(t)\|^2_{L^2(0,L)} \\
&+\int_0^L f(u^m(t)-v^m(t))u_t^m(t)dx + u_{xxx}^m(L,t)u_t^m(L,t) - u_{xx}^m(L,t)u_{tx}^m(L,t) \\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})u_x^m(L,t)u_t^m(L,t) = 0, \qquad (2.13)\\
&\frac{1}{2}\frac{d}{dt}\{\|v_t^m(t)\|^2_{L^2(0,L)} + \|v_{xx}^m(t)\|^2_{L^2(0,L)}\} \\
&+\frac{1}{2}M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})\frac{d}{dt}\|v_x^m(t)\|^2_{L^2(0,L)} \\
&-\int_0^L f(u^m(t)-v^m(t))v_t^m(t)dx + v_{xxx}^m(L,t)v_t^m(L,t) - v_{xx}^m(L,t)v_{tx}^m(L,t) \\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})v_x^m(L,t)v_t^m(L,t) = 0. \qquad (2.14)
\end{aligned}
$$

Substituting the boundary conditions (2.1)-(2.4) into (2.13) and (2.14), respectively, and using the lemma 2.2 we get, after some calculations

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|u_t^m(t)\|^2_{L^2(0,L)} + \|u_{xx}^m(t)\|^2_{L^2(0,L)}\} + \eta_1 k_1|u_x^m(L,t)|^2 \\
&-\eta_1 k_1'\Box u_x^m(L,t) + \eta_3 k_3|u^m(L,t)|^2 - \eta_3 k_3'\Box u^m(L,t)\} \\
&+\frac{1}{2}M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})\frac{d}{dt}\|u_x^m(t)\|^2_{L^2(0,L)} \\
&+\int_0^L f(u^m(t)-v^m(t))u_t^m(t)dx \\
&= -\rho_1 u_x^m(L,t)u_{xt}^m(L,t) - \eta_1|u_{xt}^m(L,t)|^2 - k_1(t)u_x^m(L,0)u_{xt}^m(L,t) \\
&+\rho_3 u^m(L,t)u_t^m(L,t) - \eta_3|u_t^m(L,t)|^2 - \eta_3 k_3(t)u^m(L,0)u_t^m(L,t) \\
&-\frac{\eta_1}{2}k_1(t)|u_x^m(L,t)|^2 + \frac{\eta_1}{2}k_1'\Box u_x^m(L,t) \\
&-\frac{\eta_3}{2}k_3(t)|u^m(L,t)|^2 + \frac{\eta_3}{2}k_3'\Box u^m(L,t), \qquad (2.15)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|v_t^m(t)\|^2_{L^2(0,L)} + \|v_{xx}^m(t)\|^2_{L^2(0,L)}\} + \eta_2 k_2 |v_x^m(L,t)|^2 \\
&-\eta_2 k_2' \Box v_x^m(L,t) + \eta_4 k_4 |v^m(L,t)|^2 - \eta_4 k_4' \Box v^m(L,t)\} \\
&+\frac{1}{2}M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})\frac{d}{dt}\|v_x^m(t)\|^2_{L^2(0,L)} \\
&-\int_0^L f(u^m(t)-v^m(t))v_t^m(t)dx \\
&= -\rho_2 v_x^m(L,t)v_{xt}^m(L,t) - \eta_2|v_{xt}^m(L,t)|^2 - k_2(t)v_x^m(L,0)v_{xt}^m(L,t) \\
&+\rho_4 v^m(L,t)v_t^m(L,t) - \eta_4|v_t^m(L,t)|^2 - \eta_4 k_4(t)v^m(L,0)v_t^m(L,t) \\
&-\frac{\eta_2}{2}k_2(t)|v_x^m(L,t)|^2 + \frac{\eta_2}{2}k_2'\Box v_x^m(L,t) \\
&-\frac{\eta_4}{2}k_4(t)|v^m(L,t)|^2 + \frac{\eta_4}{2}k_4'\Box v^m(L,t).
\end{aligned}
\tag{2.16}
$$

Summing the equations (2.15) and (2.16) and using the hypotheses on f and (1.13) we arrive at

$$
\begin{aligned}
&\frac{d}{dt}E(t,u^m(t),v^m(t)) = -\rho_1 u_x^m(L,t)u_{xt}^m(L,t) - \eta_1|u_{xt}^m(L,t)|^2 \\
&-k_1(t)u_x^m(L,0)u_{xt}^m(L,t) + \rho_3 u^m(L,t)u_t^m(L,t) - \eta_3|u_t^m(L,t)|^2 \\
&-\eta_3 k_3(t)u^m(L,0)u_t^m(L,t) - \frac{\eta_1}{2}k_1(t)|u_x^m(L,t)|^2 + \frac{\eta_1}{2}k_1'\Box u_x^m(L,t) \\
&-\frac{\eta_3}{2}k_3(t)|u^m(L,t)|^2 + \frac{\eta_3}{2}k_3'\Box u^m(L,t) \\
&-\rho_2 v_x^m(L,t)v_{xt}^m(L,t) - \eta_2|v_{xt}^m(L,t)|^2 - k_2(t)v_x^m(L,0)v_{xt}^m(L,t) \\
&+\rho_4 v^m(L,t)v_t^m(L,t) - \eta_4|v_t^m(L,t)|^2 - \eta_4 k_4(t)v^m(L,0)v_t^m(L,t) \\
&-\frac{\eta_2}{2}k_2(t)|v_x^m(L,t)|^2 + \frac{\eta_2}{2}k_2'\Box v_x^m(L,t) \\
&-\frac{\eta_4}{2}k_4(t)|v^m(L,t)|^2 + \frac{\eta_4}{2}k_4'\Box v^m(L,t).
\end{aligned}
$$

Using the Young and Poincare's inequalities we get

$$
\frac{d}{dt}E(t,u^m(t),v^m(t)) \le C_0 E(0,u^m(t),v^m(t)) + C_1 E(t,u^m(t),v^m(t)),
$$

where $E(0;u^m(t),v^m(t))$ is the energy in $t=0$. Integrating from 0 to $t<t_m$ and using Gronwall's inequality we obtain

$$
E(t,u^m(t),v^m(t)) \le C_2,
$$

where C_2 is a positive constant independent of m and t. Therefore, the approximate solution $(u^m(t),v^m(t))$ can be extended to the whole interval $[0,T]$. In particular, there exist $M_1>0$ such that

$$
E(t,u^m(t),v^m(t)) \le M_1, \tag{2.17}
$$

for all $t\in[0,T]$ and for all $m\in\mathbb{N}$.

Estimate 2

Let us now obtain an estimate for $u_{tt}^m(0)$ and $v_{tt}^m(0)$ in the L^2-norm. Integrating the equation (2.10) with $w = u_{tt}^m(0)$ we get

$$\begin{aligned}
&\int_0^L |u_{tt}^m(0)|^2 dx + \int_0^L u_{xxxx}^0 u_{tt}^m(0) dx \\
&-M(\|u_x^0\|_{L^2(0,L)}^2 + \|v_x^0\|_{L^2(0,L)}^2) \int_0^L u_{xx}^0 u_{tt}^m(0) dx \\
&+ \int_0^L f(u^0 - v^0) u_{tt}^m(0) dx \\
&+M(\|u_x^0\|_{L^2(0,L)}^2 + \|v_x^0\|_{L^2(0,L)}^2) u_x^0(L) u_{tt}^m(L,0) \\
&+u_{xxx}^0(L) u_{tt}^m(L,0) - u_{xx}^m(L,0) u_{tt}^m(L,0) = 0.
\end{aligned}$$

Using the compatibility conditions (2.6) and (2.8) and Poincare inequality we obtain

$$\begin{aligned}
&\|u_{tt}^m(0)\|_{L^2(0,L)}^2 \leq C(\|u_{xx}^0\|_{L^2(0,L)} + \|u_{xxxx}^0\|_{L^2(0,L)}) \|u_{tt}^m(0)\|_{L^2(0,L)} \\
&+M(\|u_x^0\|_{L^2(0,L)}^2 + \|v_x^0\|_{L^2(0,L)}^2) \int_0^L |u_{xx}^0| |u_{tt}^m(0)| dx \\
&+ \int_0^L |f(u^0 - v^0)| |u_{tt}^m(0)| dx.
\end{aligned}$$

Using the hypothesis on f and Sobolev imbedding we arrive at

$$\begin{aligned}
&\|u_{tt}^m(0)\|_{L^2(0,L)}^2 \leq C(\|u_{xx}^0\|_{L^2(0,L)} + \|u_{xxxx}^0\|_{L^2(0,L)} \\
&+M(\|u_x^0\|_{L^2(0,L)}^2 + \|v_x^0\|_{L^2(0,L)}^2) \|u_{xx}^0\|_{L^2(0,L)} \\
&+\|u_x^0\|_{L^2(0,L)} + \|v_x^0\|_{L^2(0,L)}) \|u_{tt}^m(0)\|_{L^2(0,L)}
\end{aligned}$$

and therefore there exists $M_2 > 0$ such that

$$\|u_{tt}^m(0)\|_{L^2(0,L)} \leq M_2, \quad \forall m \in \mathbb{N}. \tag{2.18}$$

Similarly we get

$$\|v_{tt}^m(0)\|_{L^2(0,L)} \leq M_2, \quad \forall m \in \mathbb{N}. \tag{2.19}$$

Estimate 3

Now we are going to obtain an estimate for u_{tt}, v_{tt}, and u_{txx} and v_{txx} in L^2-norm. To avoid differentiating the equations (2.10) and (2.11), we employ a difference argument as done in reference [5].

Let us fix $t > 0$, $\varepsilon > 0$ such that $\varepsilon = T - t$. Taking the difference of (2.10) with $t = t + \varepsilon$ and

$t = t$, and replacing w by $u_t^m(t+\varepsilon) - u_t^m(t)$ we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|u_t^m(t+\varepsilon) - u_t^m(t)\|^2_{L^2(0,L)} + \|u_{xx}^m(t+\varepsilon) - u_{xx}^m(t)\|^2_{L^2(0,L)}\} \\
&\int_0^L \Big[M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)})u_x^m(t+\varepsilon) \\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})u_x^m(t)\Big](u_{tx}^m(t+\varepsilon) - u_{tx}^m(t))dx \\
&-\Big[M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)})u_x^m(L,t+\varepsilon) \\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})u_x^m(L,t)\Big](u_t^m(L,t+\varepsilon) - u_t^m(L,t)) \\
&\int_0^L [f(u^m(t+\varepsilon) - v^m(t+\varepsilon)) - f(u^m(t) - v^m(t))](u_t^m(t+\varepsilon) - u_t^m(t))dx \\
&+(u_{xxx}^m(L,t+\varepsilon) - u_{xxx}^m(L,t))(u_t^m(L,t+\varepsilon) - u_t^m(L,t)) \\
&-(u_{xx}^m(L,t+\varepsilon) - u_{xx}^m(L,t))(u_{tx}^m(L,t+\varepsilon) - u_{tx}^m(L,t)) = 0
\end{aligned}$$

Substituting the boundary conditions (2.1) and (2.2) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|u_t^m(t+\varepsilon) - u_t^m(t)\|^2_{L^2(0,L)} + \|u_{xx}^m(t+\varepsilon) - u_{xx}^m(t)\|^2_{L^2(0,L)} \\
&+\rho_1|u_x(L,t+\varepsilon) - u_x(L,t)|^2 + \rho_3|u(L,t+\varepsilon) - u(L,t)|^2\} \\
&+\int_0^L \Big[M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)})u_x^m(t+\varepsilon) \\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})u_x^m(t)\Big](u_{tx}^m(t+\varepsilon) - u_{tx}^m(t))dx \\
&+\int_0^L [f(u^m(t+\varepsilon) - v^m(t+\varepsilon)) - f(u^m(t) - v^m(t))](u_t^m(t+\varepsilon) - u_t^m(t))dx \\
&= -\eta_1|u_{xt}(L,t+\varepsilon) - u_{xt}(t)|^2 \\
&+\eta_1(k_1(t+\varepsilon) - k_1(t))u_x(L,0)(u_{xt}(L,t+\varepsilon) - u_{xt}(L,t)) \\
&-\eta_1 k_1(0)(u_x(L,t+\varepsilon) - u_x(L,t))(u_{xt}(L,t+\varepsilon) - u_{xt}(L,t)) \\
&-\eta_1\left(\int_0^{t+\varepsilon} k_1'(t-s)u_x(s)ds - \int_0^t k_1'(t-s)u_x(s)ds\right)(u_{xt}(L,t+\varepsilon) - u_{xt}(L,t)) \\
&-\eta_3|u_t(L,t+\varepsilon) - u_t(t)|^2 \\
&+\eta_3(k_3(t+\varepsilon) - k_3(t))u(L,0)(u_t(L,t+\varepsilon) - u_t(L,t)) \\
&-\eta_3 k_3(0)(u(L,t+\varepsilon) - u(L,t))(u_t(L,t+\varepsilon) - u_t(L,t)) \\
&-\eta_3\left(\int_0^{t+\varepsilon} k_3'(t-s)u(s)ds - \int_0^t k_3'(t-s)u(s)ds\right)(u_t(L,t+\varepsilon) - u_t(L,t)) \qquad (2.20)
\end{aligned}$$

Similarly, using (2.11) instead of (2.10) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|v_t^m(t+\varepsilon)-v_t^m(t)\|^2_{L^2(0,L)}+\|v_{xx}^m(t+\varepsilon)-v_{xx}^m(t)\|^2_{L^2(0,L)}\\
&+\rho_2|v_x(L,t+\varepsilon)-v_x(L,t)|^2+\rho_4|v(L,t+\varepsilon)-v(L,t)|^2\}\\
&+\int_0^L\Big[M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)}+\|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)})v_x^m(t+\varepsilon)\\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)}+\|v_x^m(t)\|^2_{L^2(0,L)})v_x^m(t)\Big]\,(v_{tx}^m(t+\varepsilon)-v_{tx}^m(t))dx\\
&-\int_0^L\left[f(u^m(t+\varepsilon)-v^m(t+\varepsilon))-f(u^m(t)-v^m(t))\right](v_t^m(t+\varepsilon)-v_t^m(t))dx\\
&=-\eta_2|v_{xt}(L,t+\varepsilon)-v_{xt}(t)|^2\\
&+\eta_2(k_2(t+\varepsilon)-k_2(t))v_x(L,0)(v_{xt}(L,t+\varepsilon)-v_{xt}(L,t))\\
&-\eta_2k_2(0)(v_x(L,t+\varepsilon)-v_x(L,t))(v_{xt}(L,t+\varepsilon)-v_{xt}(L,t))\\
&-\eta_2\left(\int_0^{t+\varepsilon}k_2'(t-s)v_x(s)ds-\int_0^t k_2'(t-s)v_x(s)ds\right)(v_{xt}(L,t+\varepsilon)-v_{xt}(L,t))\\
&-\eta_4|v_t(L,t+\varepsilon)-v_t(t)|^2\\
&+\eta_4(k_4(t+\varepsilon)-k_4(t))v(L,0)(v_t(L,t+\varepsilon)-v_t(L,t))\\
&-\eta_4k_4(0)(v(L,t+\varepsilon)-v(L,t))(v_t(L,t+\varepsilon)-v_t(L,t))\\
&-\eta_4\left(\int_0^{t+\varepsilon}k_4'(t-s)v(s)ds-\int_0^t k_4'(t-s)v(s)ds\right)(v_t(L,t+\varepsilon)-v_t(L,t))
\end{aligned}
\tag{2.21}
$$

Summing the equations (2.20) and (2.21) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Phi^m(t,\varepsilon)+\int_0^L\Big[M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)}+\|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)})u_x^m(t+\varepsilon)\\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)}+\|v_x^m(t)\|^2_{L^2(0,L)})u_x^m(t)\Big]\,(u_{tx}^m(t+\varepsilon)-u_{tx}^m(t))dx\\
&+\int_0^L\Big[M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)}+\|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)})v_x^m(t+\varepsilon)\\
&-M(\|u_x^m(t)\|^2_{L^2(0,L)}+\|v_x^m(t)\|^2_{L^2(0,L)})v_x^m(t)\Big]\,(v_{tx}^m(t+\varepsilon)-v_{tx}^m(t))dx\\
&+\int_0^L\left[f(u^m(t+\varepsilon)-v^m(t+\varepsilon))-f(u^m(t)-v^m(t))\right](u_t^m(t+\varepsilon)-u_t^m(t))dx\\
&-\int_0^L\left[f(u^m(t+\varepsilon)-v^m(t+\varepsilon))-f(u^m(t)-v^m(t))\right](v_t^m(t+\varepsilon)-v_t^m(t))dx\\
&=-\eta_1|u_{xt}(L,t+\varepsilon)-u_{xt}(t)|^2\\
&+\eta_1(k_1(t+\varepsilon)-k_1(t))u_x(L,0)(u_{xt}(L,t+\varepsilon)-u_{xt}(L,t))\\
&-\eta_1\left(\int_0^{t+\varepsilon}k_1'(t-s)u_x(s)ds-\int_0^t k_1'(t-s)u_x(s)ds\right)(u_{xt}(L,t+\varepsilon)-u_{xt}(L,t))\\
&-\eta_3|u_t(L,t+\varepsilon)-u_t(t)|^2\\
&+\eta_3(k_3(t+\varepsilon)-k_3(t))u(L,0)(u_t(L,t+\varepsilon)-u_t(L,t))\\
&-\eta_3k_3(0)(u(L,t+\varepsilon)-u(L,t))(u_t(L,t+\varepsilon)-u_t(L,t))\\
&-\eta_3\left(\int_0^{t+\varepsilon}k_3'(t-s)u(s)ds-\int_0^t k_3'(t-s)u(s)ds\right)(u_t(L,t+\varepsilon)-u_t(L,t))\\
&=-\eta_2|v_{xt}(L,t+\varepsilon)-v_{xt}(t)|^2\\
&+\eta_2(k_2(t+\varepsilon)-k_2(t))v_x(L,0)(v_{xt}(L,t+\varepsilon)-v_{xt}(L,t))\\
&-\eta_2\left(\int_0^{t+\varepsilon}k_2'(t-s)v_x(s)ds-\int_0^t k_2'(t-s)v_x(s)ds\right)(v_{xt}(L,t+\varepsilon)-v_{xt}(L,t))\\
&-\eta_4|v_t(L,t+\varepsilon)-v_t(t)|^2\\
&+\eta_4(k_4(t+\varepsilon)-k_4(t))v(L,0)(v_t(L,t+\varepsilon)-v_t(L,t))\\
&-\eta_4k_4(0)(v(L,t+\varepsilon)-v(L,t))(v_t(L,t+\varepsilon)-v_t(L,t))\\
&-\eta_4\left(\int_0^{t+\varepsilon}k_4'(t-s)v(s)ds-\int_0^t k_4'(t-s)v(s)ds\right)(v_t(L,t+\varepsilon)-v_t(L,t)),
\end{aligned}
\tag{2.22}
$$

where

$$
\begin{aligned}
\Phi(t,\varepsilon)&=\|u_t^m(t+\varepsilon)-u_t^m(t)\|^2_{L^2(0,L)}+\|u_{xx}^m(t+\varepsilon)-u_{xx}^m(t)\|^2_{L^2(0,L)}\\
&+\|v_t^m(t+\varepsilon)-v_t^m(t)\|^2_{L^2(0,L)}+\|v_{xx}^m(t+\varepsilon)-v_{xx}^m(t)\|^2_{L^2(0,L)}\\
&+\rho_1|u_x(L,t+\varepsilon)-u_x(L,t)|^2+\rho_3|u(L,t+\varepsilon)-u(L,t)|^2\\
&+\rho_2|v_x(L,t+\varepsilon)-v_x(L,t)|^2+\rho_4|v(L,t+\varepsilon)-v(L,t)|^2\\
&+\eta_1k_1(0)|u_x^m(L,t+\varepsilon)-u_x^m(L,t)|^2+\eta_3k_3(0)|u^m(L,t+\varepsilon)-u^m(t)|^2\\
&+\eta_2k_2(0)|v_x^m(L,t+\varepsilon)-v_x^m(L,t)|^2+\eta_4k_4(0)|v^m(L,t+\varepsilon)-v^m(t)|^2.
\end{aligned}
$$

We analyzed some terms of (2.22). Let us denote by

$$
\begin{aligned}
& I_1 = M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)}) \cdot \\
& \int_0^L (u_x^m(t+\varepsilon) - u_x^m(t))(u_{xt}^m(t+\varepsilon) - u_{xt}^m(t))dx \\
& +\Delta M \int_0^L u_x^m(t)(u_{xt}^m(t+\varepsilon) - u_{xt}^m(t))dx \\
& +M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)}) \cdot \\
& \int_0^L (v_x^m(t+\varepsilon) - v_x^m(t))(v_{xt}^m(t+\varepsilon) - v_{xt}^m(t))dx \\
& +\Delta M \int_0^L v_x^m(t)(v_{xt}^m(t+\varepsilon) - v_{xt}^m(t))dx,
\end{aligned}
$$

where

$$
\begin{aligned}
& \Delta M = [M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)}) \\
& -M(\|u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t)\|^2_{L^2(0,L)})].
\end{aligned}
$$

Integrating by parts we have that

$$
\begin{aligned}
& I_1 = M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)}) \cdot \\
& (u_x^m(L,t+\varepsilon) - u_x^m(L,t))(u_t^m(L,t+\varepsilon) - u_t^m(L,t)) \\
& -M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)}) \cdot \\
& \int_0^L (u_{xx}^m(t+\varepsilon) - u_{xx}^m(t))(u_t^m(t+\varepsilon) - u_t^m(t))dx \\
& +\Delta M u_x^m(L,t)(u_t^m(L,t+\varepsilon) - u_t^m(L,t)) \\
& -\Delta M \int_0^L u_{xx}^m(t+\varepsilon)(u_t^m(t+\varepsilon) - u_t^m(t))dx \\
& +M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)}) \cdot \\
& (v_x^m(L,t+\varepsilon) - v_x^m(L,t))(v_t^m(L,t+\varepsilon) - v_t^m(L,t)) \\
& -M(\|u_x^m(t+\varepsilon)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon)\|^2_{L^2(0,L)}) \cdot \\
& \int_0^L (v_{xx}^m(t+\varepsilon) - v_{xx}^m(t))(v_t^m(t+\varepsilon) - v_t^m(t))dx \\
& +\Delta M v_x^m(L,t)(v_t^m(L,t+\varepsilon) - v_t^m(L,t)) \\
& -\Delta M \int_0^L v_{xx}^m(t+\varepsilon)(v_t^m(t+\varepsilon) - v_t^m(t))dx. \qquad (2.23)
\end{aligned}
$$

Now, from the Mean Value Theorem and estimate (2.17), there exist a constant $C_1 > 0$ such that

$$
|\Delta M| \leq C_1 \left(\|u_x^m(t+\varepsilon) - u_x^m(t)\|^2_{L^2(0,L)} + \|v_x^m(t+\varepsilon) - v_x^m(t)\|^2_{L^2(0,L)} \right). \qquad (2.24)
$$

Since $u(0) = v(0) = u_x(0) = v_x(0) = 0$ for $u, v \in V$, we have

$$\|u^m\|_{L^\infty(0,L)} \leq \sqrt{L}\|u^m\|_{L^2(0,L)}, \quad \|v^m\|_{L^\infty(0,L)} \leq \sqrt{L}\|v^m\|_{L^2(0,L)} \tag{2.25}$$

$$\|u^m_x\|_{L^\infty(0,L)} \leq \sqrt{L}\|u^m_{xx}\|_{L^2(0,L)}, \quad \|v^m_x\|_{L^\infty(0,L)} \leq \sqrt{L}\|v^m_{xx}\|_{L^2(0,L)}. \tag{2.26}$$

Then using (2.24)-(2.26) and Young inequality, there exists a positive constant C_2 such that

$$\begin{aligned}
|I_1| \leq C_2 &\Big(\|u^m_{xx}(t+\varepsilon) - u^m_{xx}(t)\|^2_{L^2(0,L)} + \|v^m_{xx}(t+\varepsilon) - v^m_{xx}(t)\|^2_{L^2(0,L)} \\
&+\|u^m_t(t+\varepsilon) - u^m_t(t)\|^2_{L^2(0,L)} + \|v^m_t(t+\varepsilon) - v^m_t(t)\|^2_{L^2(0,L)}\Big) \\
&+\frac{\eta_3}{4}|u^m_t(L,t+\varepsilon) - u^m_t(L,t)|^2 + \frac{\eta_4}{4}|v^m_t(L,t+\varepsilon) - v^m_t(L,t)|^2.
\end{aligned} \tag{2.27}$$

Putting

$$\begin{aligned}
I_2 = &\int_0^L \left[f(u^m(t+\varepsilon) - v^m(t+\varepsilon)) - f(u^m(t) - v^m(t))\right](u^m_t(t+\varepsilon) - u^m_t(t))dx \\
- &\int_0^L \left[f(u^m(t+\varepsilon) - v^m(t+\varepsilon)) - f(u^m(t) - v^m(t))\right](v^m_t(t+\varepsilon) - v^m_t(t))dx,
\end{aligned}$$

and using a similar arguments as above yields

$$|I_2| \leq C_3\left(\|u^m_{xx}(t+\varepsilon) - u^m_{xx}(t)\|^2_{L^2(0,L)} + \|v^m_{xx}(t+\varepsilon) - v^m_{xx}(t)\|^2_{L^2(0,L)}\right). \tag{2.28}$$

Using Young and Poincaré inequalities and hypothesis on k_i we, after some calculations

$$\begin{aligned}
&\left|-\eta_1\left(\int_0^{t+\varepsilon} k_1'(t+\varepsilon-s)u^m_x(L,s)ds - \int_0^t k_1'(t-s)u^m_x(L,s)ds\right)\cdot(u^m_{xt}(L,t+\varepsilon) - u^m_{xt}(L,t))\right| \\
&\leq \frac{\eta_1}{8}|u^m_{xt}(L,t+\varepsilon) - u^m_{xt}(L,t)|^2 + C_0\|k_1'\|_{L^1(0,\infty)}\int_0^t \|u^m_{xx}(s)\|^2_{L^2(0,L)}ds.
\end{aligned} \tag{2.29}$$

similarly we obtain

$$\begin{aligned}
&\left|-\eta_2\left(\int_0^{t+\varepsilon} k_2'(t+\varepsilon-s)v^m_x(L,s)ds - \int_0^t k_2'(t-s)v^m_x(L,s)ds\right)\cdot(v^m_{xt}(L,t+\varepsilon) - v^m_{xt}(L,t))\right| \\
&\leq \frac{\eta_2}{8}|v^m_{xt}(L,t+\varepsilon) - v^m_{xt}(L,t)|^2 + C_0\|k_2'\|_{L^1(0,\infty)}\int_0^t \|v^m_{xx}(s)\|^2_{L^2(0,L)}ds,
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
&\left|-\eta_3\left(\int_0^{t+\varepsilon} k_3'(t+\varepsilon-s)u^m(L,s)ds - \int_0^t k_3'(t-s)u^m(L,s)ds\right)\cdot(u^m_t(L,t+\varepsilon) - u^m_t(L,t))\right| \\
&\leq \frac{\eta_3}{8}|u^m_t(L,t+\varepsilon) - u^m_t(L,t)|^2 + C_0\|k_3'\|_{L^1(0,\infty)}\int_0^t \|u^m_{xx}(s)\|^2_{L^2(0,L)}ds,
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\left|-\eta_4\left(\int_0^{t+\varepsilon} k_4'(t+\varepsilon-s)v^m(L,s)ds - \int_0^t k_4'(t-s)v^m(L,s)ds\right)\cdot(v^m_t(L,t+\varepsilon) - v^m_t(L,t))\right| \\
&\leq \frac{\eta_4}{8}|v^m_t(L,t+\varepsilon) - v^m_t(L,t)|^2 + C_0\|k_2'\|_{L^1(0,\infty)}\int_0^t \|v^m_{xx}(s)\|^2_{L^2(0,L)}ds.
\end{aligned} \tag{2.32}$$

Substituting (2.27)-(2.32) into (2.22) we obtain

$$\frac{d}{dt}\Phi^m(t,\varepsilon) \leq (C_2 + C_3)\Phi^m(t,\varepsilon),$$

and therefore

$$\Phi^m(t,\varepsilon) \le \Phi^m(0,\varepsilon)\exp((C_1+C_2)T), \quad \forall t \in [0,T].$$

Dividing the above inequality by ε^2 and letting $\varepsilon \to 0$ gives

$$\begin{aligned}
&\|u^m_{txx}(t)\|^2_{L^2(0,L)} + \|v^m_{txx}(t)\|^2_{L^2(0,L)} + \|u^m_{tt}(t)\|^2_{L^2(0,L)} + \|v^m_{tt}(t)\|^2_{L^2(0,L)} \\
&\le (\|u^1_{xx}\|^2_{L^2(0,L)} + \|v^1_{xx}(t)\|^2_{L^2(0,L)} + \|u^0_{tt}(t)\|^2_{L^2(0,L)} + \|v^0_{tt}(t)\|^2_{L^2(0,L)}
\end{aligned} \tag{2.33}$$

and from estimate (2.18) and (2.19) we find a constant M_3, depending only in T, such that

$$\begin{aligned}
&\|u^m_{txx}(t)\|^2_{L^2(0,L)} + \|v^m_{txx}(t)\|^2_{L^2(0,L)} + \|u^m_{tt}(t)\|^2_{L^2(0,L)} + \|v^m_{tt}(t)\|^2_{L^2(0,L)} \le M_3 \\
&\forall m \in \mathbb{N}, \quad \forall t \in [0,T].
\end{aligned} \tag{2.34}$$

With the estimate (2.17) and (2.34) we can use Lions-Aubin lemma to get the necessary compactness in order to pass (2.10)-(2.11) to the limit. That concludes the proof of the existence of global solutions in $[0,T]$. To prove the uniqueness of solutions of the problem (1.1)-(1.9) we use the method of the energy introduced by Lions [6], coupled with Gronwall's inequality and the hypothesis introduced in the paper about the functions M, f, k_i and the obtained estimate.

3. EXPONENTIAL DECAY

In this section we shall study the asymptotic behavior of the solutions of system (1.1)-(1.9) when the resolvent kernels k_1 and k_2 are exponentially decreasing, that is, there exist positive constants b_1, b_2 such that

$$k_i(0) > 0, \quad k_i'(t) \le -b_1 k_i(t), \quad k_i''(t) \ge -b_2 k_i'(t) \quad \text{for} \quad i = 1, \dots, 4. \tag{3.1}$$

Note that this conditions implies that

$$k_i(t) \le k_i(0)e^{-b_1 t} \quad \text{for} \quad i = 1, \dots, 4.$$

Our point of departure will be to establish some inequalities for the solution of system (1.1)-(1.9).

Lemma 3.1 *Any strong solution (u,v) of the system (1.1)-(1.9) satisfies:*

$$\begin{aligned}
\frac{d}{dt}E(t) \quad \le \quad & -\frac{\eta_1}{2}|u_{xt}(L,t)|^2 - \frac{\eta_2}{2}|v_{xt}(L,t)|^2 - \frac{\eta_3}{2}|u_t(L,t)|^2 - \frac{\eta_4}{2}|v_t(L,t)|^2 \\
& + \frac{\eta_1}{2}k_1^2(t)|u_x(L,0)|^2 + \frac{\eta_2}{2}k_2^2(t)|v_x(L,0)|^2 + \frac{\eta_3}{2}k_3^2(t)|u(L,0)|^2 \\
& + \frac{\eta_4}{2}k_4^2(t)|v(L,0)|^2 + \frac{\eta_1}{2}k_1'(t)|u_x(L,t)|^2 + \frac{\eta_2}{2}k_2'(t)|v_x(L,t)|^2 \\
& + \frac{\eta_3}{2}k_3'(t)|u(L,t)|^2 + \frac{\eta_4}{2}k_4'(t)|v(L,t)|^2 - \frac{\eta_1}{2}k_1''\Box u_x(L,t) \\
& - \frac{\eta_2}{2}k_2''\Box v_x(L,t) - \frac{\eta_3}{2}k_3''\Box u(L,t) - \frac{\eta_4}{2}k_4''\Box v(L,t).
\end{aligned}$$

Proof. Multiplying the equation (1.1) by u_t and integrating by parts over $(0,L)$ we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L|u_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_0^L|u_{xx}|^2dx+\frac{1}{2}M(\int_0^L(u_x^2+v_x^2)dx)\frac{d}{dx}\int_0^L|u_x|^2dx\\&+\int_0^L f(u-v)u_tdx=-(u_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)u_x(L,t))u_t(L,t)\\&+u_{xx}(L,t)u_{xt}(L,t).\end{aligned}$$

Similarly, using the equation (1.2) instead of (1.1) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L|v_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_0^L|v_{xx}|^2dx+\frac{1}{2}M(\int_0^L(u_x^2+v_x^2)dx)\frac{d}{dx}\int_0^L|v_x|^2dx\\&-\int_0^L f(u-v)v_tdx=-(v_{xxx}(L,t)-M(\int_0^L(u_x^2+v_x^2)dx)v_x(L,t))v_t(L,t)\\&+v_{xx}(L,t)v_{xt}(L,t).\end{aligned}$$

Summing the two last equalities, substituting the boundary terms by (2.1)-(2.4) and using Lemma 2.2 our conclusion follows. ■

Let us consider the following binary operator

$$(k\diamond\varphi)(t):=\int_0^t k(t-s)(\varphi(t)-\varphi(s))ds.$$

Then applying the Hölder's inequality for $0\le\omega\le1$ we have

$$|(k\diamond\varphi)(t)|^2\le\left[\int_0^t|k(s)|^{2(1-\omega)}ds\right](|k|^{2\omega}\Box\varphi)(t). \tag{3.2}$$

Let us introduce the following functionals

$$\mathcal{N}(t):=\int_0^L(|u_t|^2+|v_t|^2+|u_{xx}|^2+|v_{xx}|^2+F(u-v))dx+\widehat{M}(\int_0^L(u_x^2+v_x^2)dx),$$

$$\psi(t)=\int_0^L\left\{xu_x+\left(\frac{1}{2}-\theta\right)u\right\}u_tdx+\int_0^L\left\{xv_x+\left(\frac{1}{2}-\theta\right)v\right\}v_tdx,$$

where θ is a small positive constant. The following Lemma plays an important role for the construction of the Lyapunov functional.

Lemma 3.2 *For any strong solution (u,v) of the system (1.1)-(1.9) we get*

$$\begin{aligned}\frac{d}{dt}\psi(t)\quad\le\quad&-\frac{\theta}{2}\mathcal{N}(t)+C(|u_{xt}(L,t)|^2+|k_1(t)u_x(L,t)|^2+|k_1'\diamond u_x(L,t)|^2\\&+|k_1(t)u_x(L,0)|^2)+C(|v_{xt}(L,t)|^2+|k_2(t)v_x(L,t)|^2\\&+|k_2'\diamond v_x(L,t)|^2+|k_2(t)v_x(L,0)|^2)+C(|u_t(L,t)|^2\\&|k_3(t)u(L,t)|^2+|k_3'\diamond u(L,t)|^2+|k_3(t)u(L,0)|^2)\\&C(|v_t(L,t)|^2|k_4(t)v(L,t)|^2+|k_4'\diamond v(L,t)|^2+|k_4(t)v(L,0)|^2).\end{aligned}$$

for some positive constant C.

Proof. Differentiating the functional ψ with respect to the time and substituting the equations (1.1) and (1.2) we obtain

$$\begin{aligned}\frac{d}{dt}\psi(t) &= -\theta\int_0^L(|u_t|^2+|v_t|^2)dx+\frac{1}{2}(|u_t(L,t)|^2+|v_t(L,t)|^2)\\ &-\int_0^L xu_{xxxx}u_xdx-(\frac{1}{2}-\theta)\int_0^L uu_{xxxx}dx-\int_0^L xv_{xxxx}v_xdx\\ &-(\frac{1}{2}-\theta)\int_0^L vv_{xxxx}dx+\int_0^L\{xu_x+(\frac{1}{2}-\theta)u\}M(\int_0^L(u_x^2+v_x^2)dx)u_{xx}dx\\ &+\int_0^L\{xv_x+(\frac{1}{2}-\theta)v\}M(\int_0^L(u_x^2+v_x^2)dx)v_{xx}dx-\int_0^L x(u_x-v_x)f(u-v)dx\\ &-(\frac{1}{2}-\theta)\int_0^L(u-v)f(u-v)dx. \end{aligned}\tag{3.3}$$

We analyze some of the terms of the equality (3.3). Integrating by parts we get

$$\begin{aligned}\int_0^L xu_{xxxx}u_xdx &= u_{xxx}(L,t)u_x(L,t)-u_{xx}(L,t)u_x(L,t)\\ &\quad+2\int_0^L|u_{xx}|^2dx-|u_{xx}(L,t)|^2, \end{aligned}\tag{3.4}$$

$$\begin{aligned}\int_0^L xv_{xxxx}v_xdx &= v_{xxx}(L,t)v_x(L,t)-v_{xx}(L,t)v_x(L,t)\\ &\quad+2\int_0^L|v_{xx}|^2dx-|v_{xx}(L,t)|^2, \end{aligned}\tag{3.5}$$

$$\int_0^L uu_{xxxx}dx = u_{xxx}(L,t)u(L,t)-u_{xx}(L,t)u_x(L,t)+\int_0^L|u_{xx}|^2dx, \tag{3.6}$$

$$\int_0^L vv_{xxxx}dx = v_{xxx}(L,t)v(L,t)-v_{xx}(L,t)u_x(L,t)+\int_0^L|v_{xx}|^2dx, \tag{3.7}$$

$$\int_0^L x(u_x-v_x)f(u-v)dx = F(u(L,t)-v(L,t))-\int_0^L F(u-v)dx. \tag{3.8}$$

Since f is superlinear we have

$$(\frac{1}{2}-\theta)\int_0^L(u-v)f(u-v)dx\geq(2+\delta)(\frac{1}{2}-\theta)\int_0^L F(u-v)dx. \tag{3.9}$$

Using the hypothesis (1.13) we have

$$\begin{aligned}&\int_0^L x(u_x+v_x)M(\int_0^L(u_x^2+v_x^2)dx)(u_x+v_x)_xdx\leq-\frac{1}{2}\widehat{M}(\int_0^L(u_x^2+v_x^2)dx)\\ &+LM(\int_0^L(u_x^2+v_x^2)dx)(u_x^2(L,t)+v_x^2(L,t)), \end{aligned}\tag{3.10}$$

$$\begin{aligned}&(\frac{1}{2}-\theta)\{\int_0^L uM(\int_0^L(u_x^2+v_x^2)dx)u_{xx}dx+\int_0^L vM(\int_0^L(u_x^2+v_x^2)dx)v_{xx}dx\}\\ &\leq(\frac{1}{2}-\theta)M(\int_0^L(u_x^2+v_x^2)dx)(u(L,t)u_x(L,t)+v(L,t)v_x(L,t))\\ &-(\frac{1}{2}-\theta)\widehat{M}(\int_0^L(u_x^2+v_x^2)dx). \end{aligned}\tag{3.11}$$

Substituting (3.4)-(3.11) into (3.3) we arrive at

$$\begin{aligned}
\frac{d}{dt}\psi(t) \quad \leq \quad & -\theta\int_0^L(|u_t|^2+|v_t|^2)dx+\frac{1}{2}(|u_t(L,t)|^2+|v_t(L,t)|^2)\\
& -u_{xxx}(L,t)u_x(L,t)+u_{xx}(L,t)u_x(L,t)-2\int_0^L|u_{xx}|^2dx\\
& -(\frac{1}{2}-\theta)u_{xxx}(L,t)u(L,t)+(\frac{1}{2}-\theta)u_{xx}(L,t)u_x(L,t)\\
& -v_{xxx}(L,t)v_x(L,t)+v_{xx}(L,t)v_x(L,t)-2\int_0^L|u_{xx}|^2dx\\
& -(\frac{1}{2}-\theta)v_{xxx}(L,t)v(L,t)+(\frac{1}{2}-\theta)v_{xx}(L,t)v_x(L,t)\\
& -|u_{xx}(L,t)|^2-|v_{xx}(L,t)|^2-(\frac{1}{2}-\theta)\int_0^L|u_{xx}|^2dx\\
& -(\frac{1}{2}-\theta)\int_0^L|v_{xx}|^2dx+LM(\int_0^L(u_x^2+v_x^2)dx)u_x^2(L,t)+v_x^2(L,t))\\
& -M(\int_0^L(u_x^2+v_x^2)dx)(u(L,t)u_x(L,t)+v(L,t)v_x(L,t))\\
& -(\frac{1}{2}-\theta)\widehat{M}(\int_0^L(u_x^2+v_x^2)dx)-(\frac{\delta}{2}-\theta(2+\delta))\int_0^L F(u-v)dx.
\end{aligned}$$

Noting that the boundary conditions(2.1)-(2.4) were written as

$$\begin{aligned}
u_{xx}(L,t) &= -\rho_1 u_x(L,t)-\eta_1\{u_{xt}(L,t)+k_1(t)u_x(L,t)\\
&\quad -k_1'\diamond u_x(L,t)-k_1(t)u_x(L,0)\},\\
v_{xx}(L,t) &= -\rho_2 v_x(L,t)-\eta_2\{v_{xt}(L,t)+k_2(t)v_x(L,t)\\
&\quad -k_2'\diamond v_x(L,t)-k_2(t)v_x(L,0)\},\\
u_{xxx}(L,t) &= \rho_3 u(L,t)+\eta_3\{u_t(L,t)+k_3(t)u(L,t)\\
&\quad -k_3'\diamond u(L,t)-k_3(t)u(L,0)\}\\
v_{xxx}(L,t) &= \rho_4 v(L,t)+\eta_3\{v_t(L,t)+k_4(t)v(L,t)\\
&\quad -k_4'\diamond v(L,t)-k_4(t)v(L,0)\}
\end{aligned}$$

and taking into account that ρ_1, ρ_2, ρ_3, ρ_4 and θ are small, our conclusion follows. ■

To show that the energy decays exponentially we shall need the following Lemma.

Lemma 3.3 *Let f be a real positive function of class C^1. If there exist positive constants γ_0, γ_1 and c_0 such that*

$$f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and c such that

$$f(t) \leq (f(0)+c)e^{-\gamma t}.$$

Proof. See e. g. [12]

Finally, we shall show the main result of this section.

Theorem 3.1 *Let us take* $(u_0, v_0) \in W^2$ *and* $(u_1, v_1) \in (L^2(0, L))^2$. *If the resolvent kernels* k_1, k_2, k_3 *and* k_4 *satisfy (3.1) then there exist positive constants* α_1 *and* α_2 *such that*

$$E(t) \leq \alpha_1 e^{-\alpha_2 t} E(0),$$

for all $t \geq 0$.

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in (H^4(0, L) \cap W)^2$ and $(u_1, v_1) \in W^2$ satisfying the compatibility conditions. Our conclusion follows by standard density arguments. Using hypothesis (3.1) in Lemma 3.1 we get

$$\begin{aligned}
\frac{d}{dt}E(t) \quad \leq \quad & -\frac{\eta_1}{2}(|u_{xt}(L,t)|^2 - b_1 k_1' \Box u_x(L,t) + k_1(t)|u_x(L,t)|^2 - k_1^2(t)|u_x(L,0)|^2) \\
& -\frac{\eta_2}{2}(|v_{xt}(L,t)|^2 - b_1 k_2' \Box v_x(L,t) + k_2(t)|v_x(L,t)|^2 - k_2^2(t)|v_x(L,0)|^2) \\
& -\frac{\eta_3}{2}(|u_t(L,t)|^2 - k_3' \Box u(L,t) + k_3(t)|u(L,t)|^2 - k_3^2(t)|u(L,0)|^2) \\
& -\frac{\eta_4}{2}\left(|v_t(L,t)|^2 - k_4' \Box v(L,t) + k_4(t)|v(L,t)|^2 - k_2^4(t)|v(L,0)|^2\right)
\end{aligned}$$

On the other hand applying inequality (3.2) with $\omega = 1/2$ in Lemma 3.2 we obtain

$$\begin{aligned}
\frac{d}{dt}\psi(t) \leq & -\frac{\theta}{2}\mathcal{N}(t) + C\left(|u_{xt}(L,t)|^2 + k_1(t)|u_x(L,t)|^2 - k_1' \Box u_x(L,t) + |k_1(t)u_x(L,0)|^2\right) \\
& + C\left(|v_{xt}(L,t)|^2 + k_2(t)|v_x(L,t)|^2 - k_2' \Box v_x(L,t) + |k_2(t)v_x(L,0)|^2\right) \\
& + C\left(|u_t(L,t)|^2 + k_3(t)|u(L,t)|^2 - k_3' \Box u(L,t) + |k_3(t)u(L,0)|^2\right) \\
& + C\left(|v_t(L,t)|^2 + k_4(t)|v(L,t)|^2 - k_4' \Box v(L,t) + |k_4(t)v(L,0)|^2\right).
\end{aligned}$$

Let us introduce the following functional

$$\mathcal{L}(t) := NE(t) + \psi(t), \tag{3.12}$$

with $N > 0$. Taking N large, the previous inequalities imply that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\theta}{2}E(t) + 2NR^2(t)E(0),$$

where $R(t) = k_1(t) + k_2(t) + k_3(t) + k_4(t)$. Moreover, using Young's inequality and taking N large we find that

$$\frac{N}{2}E(t) \leq \mathcal{L}(t) \leq 2NE(t). \tag{3.13}$$

From this inequality we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\theta}{2}\mathcal{L}(t) + 2NR^2(t)E(0),$$

from where follows, in view of Lemma 3.3 and of the exponential decay of k_i, that

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + C\} e^{-\zeta t},$$

for some positive constants C and ζ. From the inequality (3.13) our conclusion follows. ■

4. POLYNOMIAL RATE OF DECAY

Here our attention will be focused on the uniform rate of decay when the resolvent kernels k_1, k_2, k_3 and k_4 decay polynomially like $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Therefore, we will assume that the resolvent kernels k_i satisfy

$$k_i(0) > 0, \quad k_i'(t) \leq -b_1 k_i(t)^{1+\frac{1}{p}}, \quad k_i''(t) \geq b_2[-k_i'(t)]^{1+\frac{1}{p+1}} \quad \text{for} \quad i = 1, \ldots, 4 \tag{4.1}$$

for some $p > 1$ and some positive constants b_1 and b_2. The following lemmas will play an important role in the sequel.

Lemma 4.1 *Let (u, v) be a solution of system (1.1)-(1.9) and let us denote by $(\phi_1, \phi_3) = (u_x(L,t), u(L,t))$ and $(\psi_2, \psi_4) = (v_x(L,t), v(L,t))$. Then, for $p > 1$, $0 < r < 1$ and $t \geq 0$, we have*

$$\left(|k_i'|\Box\phi_i\right)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}} \leq \left(2\int_0^t |k_i'(s)|^r ds ||u||^2_{L^\infty(0,t;H^2(0,L))}{}^{\frac{1}{(1-r)(p+1)}}\right) |k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma_1,$$
$$\forall i = 1, 3$$
$$\left(|k_i'|\Box\psi_i\right)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}} \leq \left(2\int_0^t |k_i'(s)|^r ds ||v||^2_{L^\infty(0,t;H^2(0,L))}{}^{\frac{1}{(1-r)(p+1)}}\right) |k_i'|^{1+\frac{1}{p+1}}\Box\psi_i d\Gamma_1,$$
$$\forall i = 2, 4$$

while for $r = 0$ we get

$$\left(|k_i'|\Box\phi_i d\Gamma_1\right)^{\frac{p+2}{p+1}} \leq 2\left(\int_0^t ||u(s,.)||^2_{H^2(0,1)} ds + t||u(s,.)||^2_{H^2(0,L)}\right)^{p+1} |k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma_1,$$
$$\forall i = 1, 3$$
$$\left(|k_i'|\Box\psi_i d\Gamma_1\right)^{\frac{p+2}{p+1}} \leq 2\left(\int_0^t ||v(s,.)||^2_{H^2(0,1)} ds + t||v(s,.)||^2_{H^2(0,L)}\right)^{p+1} |k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma_1,$$
$$\forall i = 2, 4.$$

Proof. See e. g. [11]

Lemma 4.2 *Let $f \geq 0$ be a differentiable function satisfying*

$$f'(t) \leq -\frac{c_1}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^\beta} f(0) \quad \textit{for} \quad t \geq 0,$$

for some positive constants c_1, c_2, α and β such that

$$\beta \geq \alpha + 1.$$

Then there exists a constant $c > 0$ such that

$$f(t) \leq \frac{c}{(1+t)^{\alpha}} f(0) \quad for \quad t \geq 0.$$

Proof. See e. g. [12]

Theorem 4.1 *Let us take $(u_0, v_0) \in W$ and $(u_1, v_1) \in L^2(0, L)$. If the resolvent kernels k_i satisfy the conditions (4.1), then there exists a positive constant C such that*

$$E(t) \leq \frac{C}{(1+t)^{p+1}} E(0).$$

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in (H^4(0, L) \cap W)^2$ and $(u_1, v_1) \in W^2$ satisfying the compatibility conditions. Our conclusion will follow by standard density arguments. We use some estimates of the previous section which are independent of the behavior of the resolvent kernels k_1, k_2, k_3, k_4. Using hypothesis (4.1) in Lemma 3.1 yields

$$\begin{aligned}
\frac{d}{dt}E(t) \quad \leq \quad & -\frac{\eta_1}{2}\{|u_{xt}(L,t)|^2 + b_2[-k_1']^{1+\frac{1}{p+1}} \square u_x(L,t) \\
& + b_1 k_1^{1+\frac{1}{p}}(t)|u_x(L,t)|^2 - |k_1(t)u_x(L,0)|^2\} \\
& -\frac{\eta_2}{2}\{|v_{xt}(L,t)|^2 + b_2[-k_2']^{1+\frac{1}{p+1}} \square v_x(L,t) \\
& + b_1 k_2^{1+\frac{1}{p}}(t)|v_x(L,t)|^2 - |k_2(t)v_x(L,0)|^2\} \\
& -\frac{\eta_3}{2}\{|u_t(L,t)|^2 + b_2[-k_3']^{1+\frac{1}{p+1}} \square u(L,t) \\
& + b_1 k_3^{1+\frac{1}{p}}(t)|u(L,t)|^2 - |k_3(t)u(L,0)|^2\} \\
& -\frac{\eta_4}{2}\{|v_t(L,t)|^2 + b_2[-k_4']^{1+\frac{1}{p+1}} \square v(L,t) \\
& + b_1 k_4^{1+\frac{1}{p}}(t)|v(L,t)|^2 - |k_4(t)v(L,0)|^2\}.
\end{aligned}$$

Applying inequality (3.2) with $\omega = \frac{p+2}{2(p+1)}$ and using hypothesis (4.1) we obtain the following estimates

$$\begin{aligned}
&|k_1' \diamond u_x(L,t)|^2 \leq C[-k_1']^{1+\frac{1}{p+1}} \square u_x(L,t), \\
&|k_2' \diamond v_x(L,t)|^2 \leq C[-k_2']^{1+\frac{1}{p+1}} \square v_x(L,t), \\
&|k_3' \diamond u(L,t)|^2 \leq C[-k_3']^{1+\frac{1}{p+1}} \square u(L,t), \\
&|k_4' \diamond v(L,t)|^2 \leq C[-k_4']^{1+\frac{1}{p+1}} \square v(L,t).
\end{aligned}$$

Using the above inequalities in Lemma 3.2 yields

$$
\begin{aligned}
\frac{d}{dt}\psi(t) \leq & -\frac{\theta}{2}\mathcal{N}(t) + C\{|u_{xt}(L,t)|^2 + k_1^{1+\frac{1}{p}}(t)|u_x(1,t)|^2 \\
& + [-k_1']^{1+\frac{1}{p+1}}\Box u_x(L,t) + |k_1(t)u_x(L,0)|^2\} \\
& + C\{|v_{xt}(L,t)|^2 + k_2^{1+\frac{1}{p}}(t)|v_x(L,t)|^2 \\
& + [-k_2']^{1+\frac{1}{p+1}}\Box v_x(L,t) + |k_2(t)v_x(L,0)|^2\} \\
& + C\{|u_t(L,t)|^2 + k_3^{1+\frac{1}{p}}(t)|u(L,t)|^2 \\
& + [-k_3']^{1+\frac{1}{p+1}}\Box u(L,t) + |k_3(t)u(L,t)|^2\} \\
& + C\{|v_t(L,t)|^2 + k_4^{1+\frac{1}{p}}(t)|v(L,t)|^2 \\
& + [-k_4']^{1+\frac{1}{p+1}}\Box v(L,t) + |k_4(t)v(L,t)|^2\}.
\end{aligned}
$$

With this conditions, taking N large the Lyapunov functional defined in (3.12) satisfies

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -\frac{\theta}{2}\mathcal{N}(t) + 2NR^2(t)E(0) \\
& -\frac{NC_2}{2}\{[-k_1']^{1+\frac{1}{p+1}}\Box u_x(L,t) - \frac{NC_2}{2}\{[-k_2']^{1+\frac{1}{p+1}}\Box v_x(L,t) \\
& +[-k_3']^{1+\frac{1}{p+1}}\Box u(L,t) + [-k_{43}']^{1+\frac{1}{p+1}}\Box v(L,t)\}. \qquad (4.2)
\end{aligned}
$$

Let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. From (4.1) we have that

$$
\int_0^\infty |k_i'|^r \leq C\int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for} \quad i = 1, \ldots, 4.
$$

Using this estimate in Lemma 4.1 we get

$$
[-k_1']^{1+\frac{1}{p+1}}\Box u_x(L,t) \geq CE(0)^{-\frac{1}{(1-r)(p+1)}}\left([-k_1']\Box u_x(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}}, \qquad (4.3)
$$

$$
[-k_2']^{1+\frac{1}{p+1}}\Box v_x(L,t) \geq CE(0)^{-\frac{1}{(1-r)(p+1)}}\left([-k_2']\Box v_x(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}}, \qquad (4.4)
$$

$$
[-k_3']^{1+\frac{1}{p+1}}\Box u(L,t) \geq CE(0)^{-\frac{1}{(1-r)(p+1)}}\left([-k_3']\Box u(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}}, \qquad (4.5)
$$

$$
[-k_4']^{1+\frac{1}{p+1}}\Box v(L,t) \geq CE(0)^{-\frac{1}{(1-r)(p+1)}}\left([-k_4']\Box v(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}}. \qquad (4.6)
$$

On the other hand, from the Trace theorem we have

$$
E(t)^{1+\frac{1}{(1-r)(p+1)}} \leq CE(0)^{\frac{1}{(1-r)(p+1)}}\mathcal{N}(t). \qquad (4.7)
$$

Substitution of (4.3)-(4.7) into (4.2) results in

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq & -CE(0)^{-\frac{1}{(1-r)(p+1)}}E(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0) \\
& - CE(0)^{-\frac{1}{(1-r)(p+1)}}\left\{\left([-k_1']\Box u_x(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}} + \left([-k_2']\Box v_x(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}}\right\} \\
& + \left([-k_3']\Box u(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}} + \left([-k_4']\Box v(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}}.
\end{aligned}
$$

Taking into account the inequality (3.13) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{C}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0),$$

for some $C > 0$, from where follows, applying Lemma 4.2, that

$$\mathcal{L}(t) \le \frac{C}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0).$$

Since $(1-r)(p+1) > 1$ we get, for $t \ge 0$, the following bounds

$$\begin{aligned}
t\|u\|^2_{H^2(0,L)} \le Ct\mathcal{L}(t) &< \infty, \\
t\|v\|^2_{H^2(0,L)} \le Ct\mathcal{L}(t) &< \infty, \\
\int_0^t \|u\|^2_{H^2(0,L)}ds \le C\int_0^t \mathcal{L}(t)ds &< \infty, \\
\int_0^t \|v\|^2_{H^2(0,L)}ds \le C\int_0^t \mathcal{L}(t)ds &< \infty.
\end{aligned}$$

Using the above estimates in Lemma 4.1 with $r = 0$ we get

$$\begin{aligned}
[-k_1']^{1+\frac{1}{p+1}}\Box u_x(L,t) &\ge \frac{C}{E(0)^{\frac{1}{p+1}}}\left([-k_1']\Box u_x(L,t)\right)^{1+\frac{1}{p+1}}, \\
[-k_2']^{1+\frac{1}{p+1}}\Box v_x(L,t) &\ge \frac{C}{E(0)^{\frac{1}{p+1}}}\left([-k_2']\Box v_x(L,t)\right)^{1+\frac{1}{p+1}}, \\
[-k_3']^{1+\frac{1}{p+1}}\Box u(L,t) &\ge \frac{C}{E(0)^{\frac{1}{p+1}}}\left([-k_3']\Box u(L,t)\right)^{1+\frac{1}{p+1}}, \\
[-k_4']^{1+\frac{1}{p+1}}\Box v(L,t) &\ge \frac{C}{E(0)^{\frac{1}{p+1}}}\left([-k_4']\Box v(L,t)\right)^{1+\frac{1}{p+1}}.
\end{aligned}$$

Using these inequalities instead of (4.3)-(4.6) and reasoning in the same way as above it results that

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{C}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + 2NR^2(t)E(0).$$

Applying Lemma 4.2 again, we obtain

$$\mathcal{L}(t) \le \frac{C}{(1+t)^{p+1}}\mathcal{L}(0).$$

Finally, from (3.13) we conclude

$$E(t) \le \frac{C}{(1+t)^{p+1}}E(0),$$

which completes the present proof. ■

Remark: The techniques in this paper may be used to study the problem (1.1)-(1.9) with moving boundary. This is a very important open problem. In this case, we define others appropriate functionals to prove the exponential and polynomial decay rates of the energy of weak solutions for the problem (1.1)-(1.9). Result concerning the above system in domains with moving boundary will appear in a forthcoming paper.

Acknowledgements: The authors are thankful to the referee of this paper for valuable suggestions which improved this paper. The fourth author of this paper was partially supported by CNPq-Brasília under grant No. 301025/2003-7.

References

[1] J. Ball, Stability theory for an extensible beam, *J. Differential Equations, 14, (1973), 399-418.*

[2] I. N. Bernstein, On a class of functional differential equations, *Izv. Akad. Nauk. SSR Ser. Math., 4, (1940), 17-26.*

[3] R. W. Dickey, Infinite systems of nonlinear oscilation equations with linear damping, *SIAM J. Appl. Math., 19, (1970), 208-214.*

[4] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, John Wiley and Sons-Masson, Paris (1994).

[5] S. Kouémou Patcheu, On a global solution and asymptotic behaviour for the generalized damped extensible beam equation, *J. Differential equations, 135, (1997), 299-314.*

[6] J. L. Lions, Quelques Mèthodes de resolution de problèmes aux limites non lineaires. *Dunod Gauthiers Villars, Paris (1969).*

[7] T. F. Ma, Boundary stabilization for a non-linear beam on elastic bearings, *Math. Meth. Appl. Sci., 24, (2001), 583-594.*

[8] J.E. Muñoz Rivera, *Energy decay rates in linear thermoelasticity.* Funkc. Ekvacioj, 1992; 35(1):19-30.

[9] J. E. Muñoz Rivera, Global solution and regularizing properties on a class of nonlinear evolution equation, *J. Differential Equations, 128, (1996), 103-124.*

[10] A. F. Pazoto-G. P. Menzala, Uniform stabilization of a nonlinear beam model with thermal effects and nonlinear boundary dissipation, *Funkcialaj Ekvacioj, 43, 2, (2000), 339-360.*

[11] M. L. Santos, Asymptotic behavior of solutions to wave equations with a memory condition at the boundary, *E. J. Diff. Eqs. 73 (2001) 1-11.*

[12] M. L. Santos, Decay rates for solutions of a system of wave equations with memory, *E. J. Diff. Eqs. 38 (2002) 1-17.*

[13] R. Racke, *Lectures on nonlinear evolution equations. Initial value problems.* Aspect of Mathematics E19. Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden (1992).

[14] M. Tucsnak, Semi-internal stabilization for a nonlinear Euler-Bernoulli equation, *Math. Meth. Appl. Sci., 19, (1996), 897-907.*

[15] S. Woinowsky-Krieger, The effect of axial force on the vibration of hinged bars, *J. Appl. Mech., 17, (1950), 35-36.*

(Received December 3, 2004)