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ASYMPTOTIC ESTIMATES FOR PDE WITH p-LAPLACIAN AND DAMPING.

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Abstract. We study the positive solutions of equation

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + \left\langle \vec{b}(x), ||\nabla u||^{p-2}\nabla u \right\rangle + c(x)|u|^{q-2}u = 0,$$

via the Riccati technique and prove an integral sufficient condition on the potential function $c(x)$ and the damping $\vec{b}(x)$ which ensures that no positive solution of the equation satisfies a lower (if $p > q$) or upper (if $q > p$) bound eventually.

1. Introduction

In the paper we investigate the equation with p-Laplacian and Emden-Fowler type nonlinearity

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + \left\langle \vec{b}(x), ||\nabla u||^{p-2}\nabla u \right\rangle + c(x)|u|^{q-2}u = 0, \tag{1}$$

where p, q are real numbers satisfying $p, q > 1$, $||\cdot||$ is the usual Euclidean norm and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^n$. Equation (1) is sometimes referred as a quasilinear equation. We focus our attention to the case when the equation lacks the homogeneity property, i.e., when $p \neq q$.

The potential function $c(x)$ is supposed to be locally Hölder continuous, there are no other restrictions. Among others, we don't assume anything concerning either the fixed sign or the radial symmetry of the potential $c(x)$.

Equation (1) arises in many important physical and biological problems. The reader is referred to [3] and the references therein for a more detailed list of applications.

An important case $p = q$ has been studied extensively in the last years. It turned out that in this case an oscillation theory similar to theory of linear equations can be established. If $p = q$, then equation (1) has the homogeneity property (a constant multiple of a solution is a solution as well) and from this reason is (1) with $p = q$ called half-linear equation. Following this terminology, we will refer to the case $q < p$ as to the sub-half-linear case and $q > p$ will be the super-half-linear case.

The fact that the properties of a half-linear equation and equation (1) with condition $p \neq q$ are very different is known already from the study of one-dimensional ordinary differential equation

$$(|u'|^{p-2}u')' + b(x)|u'|^{p-2}u' + c(x)|u|^{q-2}u = 0. \tag{2}$$

This equation is usually studied under sign restriction — the function $c(x)$ is supposed to be either positive or negative. For recent results about equation (2) or some its special cases or generalizations see [1, 2, 6, 7, 8, 9]

Important investigations about equation (1) concern the radial solutions of the equation with radial potential, proved often by a moving plane procedure, see e.g. [4, 5] and the references therein.

The aim of this paper if to study the asymptotic behavior of the solutions of (1) on exterior domains. More precisely, we present sufficient conditions which ensure that no solution of the equation possesses a lower or upper bound by the function $\varphi(||x||)$ for $||x||$ sufficiently large.

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Namely, given a function $\varphi(r)$, we derive a sufficient condition which ensure that no positive solution of the equation satisfies inequality

$$|u(x)|^{\delta} \le \varphi(||x||)$$

for $||x|| > r_0$ no matter how large the number r_0 is. The coefficient δ is a real number which is positive in the super-half-linear case and negative in the sub-half-linear case. Taking the function φ to be a suitable power function, this yields also results concerning integrability or quadratic integrability of the solution, which is an important problem from the physicist's point of view.

As far as the author knows, the results are new also in the case of one-dimensional ordinary differential equation, since most of the papers dealing with this equation use a sign restrictions, which are not necessary here.

Notation. We will use the following notation for sets in $\mathbb{R}^n$:

$$\begin{aligned}\Omega(r) &:= \{x \in \mathbb{R}^n : ||x|| \ge r\},\\ S(r) &:= \partial\Omega(r) = \{x \in \mathbb{R}^n : ||x|| = r\},\\ \Omega(a,b) &:= \Omega(a) \setminus \Omega(b) = \{x \in \mathbb{R}^n : a \le ||x|| < b\}.\end{aligned}$$

The conjugate number to the number p will be denoted by p^*. Hence

$$\frac{1}{p} + \frac{1}{p^*} = 1 \quad \text{and} \quad p^* = \frac{p}{p-1}.$$

The number ω_n denotes the surface of the n-dimensional unit sphere in $\mathbb{R}^n$, i.e. $\omega_n = \int_{S(1)} \mathrm{d}\sigma\,$, where $\mathrm{d}\sigma$ is the integral element on the sphere.

Remind that the integration over the sets like $\Omega(a,b)$ is performed using hyperspherical coordinates on the spheres $S(r)$ first and the integration over $r \in (a,b)$ follows.

2. Main results

Theorem 1. *Suppose that there exist positive functions $a(x) \in C^1(\Omega(1), \mathbb{R}^+)$, $\varphi \in C([1,\infty), \mathbb{R}^+)$ and a constant $l > 1$ such that*

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} P_{a,\varphi,l}(x)\,\mathrm{d}x = \infty \tag{3}$$

and

$$\lim_{r\to\infty} \int^{r} \varphi(r)\Big(r^{n-1}\tilde{a}(r)\Big)^{-p^*/p} \mathrm{d}r = \infty, \tag{4}$$

where $\tilde{a}(r)$ is the mean value of the function $a(x)$ on the sphere $S(r)$, i.e.

$$\tilde{a}(r) = \frac{1}{\omega_n r^{n-1}} \int_{S(r)} a(x)\,\mathrm{d}\sigma\,,$$

and the function $P_{a,\varphi,l}(x)$ is defined by the relation

$$P_{a,\varphi,l}(x) = a(x)c(x) - \frac{1}{p}||\vec{b}(x)a(x) - \nabla a(x)||^p \left[\frac{1}{l}(q-1)p^*\varphi(||x||)a(x)\right]^{-p/p^*}. \tag{5}$$

Then equation (1) has no positive solution which satisfies

$$|u(x)|^{\frac{q-p}{p-1}} \ge \varphi(||x||) \tag{6}$$

for large $||x||$.

Remark 1. Let us emphasize that inequality (6) presents either a bound from below or from above, depending on the mutual relationship between p and q. If $p > q$, then (6) is equivalent to

$$u(x) \ge \varphi^{\frac{p-1}{q-p}}(||x||) \tag{7}$$

and if $p < q$, then (6) can be written in the form

$$u(x) \le \varphi^{\frac{p-1}{q-p}}(||x||). \tag{8}$$

Thus, in the super-half-linear case we have a bound from below and in the sub-half-linear case a bound from above.

Proof of Theorem 1. Suppose, by contradiction, that there exits a real number $T > 1$ such that equation (1) possesses a positive solution u which satisfies (6) for $||x|| \ge T$. The n-vector function

$$\vec{w}(x) = \frac{||\nabla u(x)||^{p-2}\nabla u(x)}{u^{q-1}(x)} \tag{9}$$

is well-defined on the domain $\Omega(T)$ and a direct computation shows

$$\operatorname{div}\vec{w} = \frac{\operatorname{div}(||\nabla u||^{p-2}\nabla u)}{u^{q-1}} + (1-q)\frac{||\nabla u||^p}{u^q}.$$

According to (1), the function $\vec{w}$ satisfies the equality

$$\operatorname{div}\vec{w} + c(x) + \left\langle \vec{b}(x), \vec{w} \right\rangle + (q-1)||\vec{w}||^{p^*} u^{\frac{q-p}{p-1}}(x) = 0.$$

In view of (6) we conclude for $x \in \Omega(T)$ the inequality

$$\operatorname{div}\vec{w} + c(x) + \left\langle \vec{b}(x), \vec{w} \right\rangle + (q-1)||\vec{w}||^{p^*} \varphi(||x||) \le 0.$$

Multiplying this relation by the positive function $a(x)$ and splitting the expression $(q-1)||\vec{w}||^{p^*}\varphi(||x||)$ into two parts using identity $\frac{1}{l} + \frac{1}{l^*} = 1$ we get

$$\operatorname{div}(a\vec{w}) - \langle \vec{w}, \nabla a \rangle + ac + \left\langle a\vec{b}, \vec{w} \right\rangle + \frac{q-1}{l} a\varphi||\vec{w}||^{p^*} + \frac{q-1}{l^*} a\varphi||\vec{w}||^{p^*} \le 0, \tag{10}$$

where an explicit dependence on x is suppressed for brevity. Using the well-known Young inequality

$$\frac{||X||^{p^*}}{p^*} \pm \langle X, Y \rangle + \frac{||Y||^p}{p} \ge 0$$

we get

$$\begin{aligned}
&\frac{q-1}{l} a\varphi||\vec{w}||^{p^*} + \left\langle \vec{w}, \vec{b}a - \nabla a \right\rangle \\
&\quad = \frac{\left[||\vec{w}||^{p^*} \left[\frac{q-1}{l} a\varphi p^*\right]^{1/p^*}\right]^{p^*}}{p^*} \\
&\qquad + \left\langle \vec{w}\left[\frac{q-1}{l} a\varphi p^*\right]^{1/p^*}, (\vec{b}a - \nabla a)\left[\frac{q-1}{l} a\varphi p^*\right]^{-1/p^*} \right\rangle \\
&\quad \ge -\frac{||\vec{b}a - \nabla a||^p \left[\frac{q-1}{l} a\varphi p^*\right]^{-p/p^*}}{p}.
\end{aligned}$$

The substitution into (10) yields

$$\operatorname{div}(a\vec{w}) + ac - \frac{||\vec{b}a - \nabla a||^p \left[\frac{q-1}{l} a\varphi p^*\right]^{-p/p^*}}{p} + \frac{q-1}{l^*} a\varphi||\vec{w}||^{p^*} \le 0$$

which, in the notation of this theorem, reads as

$$\operatorname{div}\Big(a(x)\vec{w}(x)\Big) + P_{a,\varphi,l}(x) + \frac{q-1}{l^*} a(x)\varphi(||x||)||\vec{w}(x)||^{p^*} \le 0,$$

where we again wrote the dependence on the variable x. Recall that the last inequality is satisfied for all $x \in \Omega(T)$. Integrating this inequality over the domain $\Omega(T,r)$ and using the Gauss–Ostrogradski theorem we get

$$\int_{S(r)} a(x)\vec{w}(x)\vec{\nu}(x)\,\mathrm{d}\sigma - \int_{S(T)} a(x)\vec{w}(x)\vec{\nu}(x)\,\mathrm{d}\sigma + \int_{\Omega(T,r)} P_{a,\varphi,l}(x)\,\mathrm{d}x$$
$$+ \int_{\Omega(T,r)} \frac{q-1}{l^*} a(x)\varphi(||x||)||\vec{w}(x)||^{p^*}\,\mathrm{d}x \le 0.$$

From (3) and from the fact that the second term in the last inequality is finite we conclude that there is a number $T_0 > T$ such that

$$\int_{\Omega(T,r)} \frac{q-1}{l^*} a(x)\varphi(||x||)||\vec{w}(x)||^{p^*}\,\mathrm{d}x \le - \int_{S(r)} a(x)\vec{w}(x)\vec{\nu}(x)\,\mathrm{d}\sigma \tag{11}$$

holds for all $r \ge T_0$.

Denote

$$Z(r) = \int_{\Omega(T,r)} a(x)\varphi(||x||)||\vec{w}(x)||^{p^*}\,\mathrm{d}x\,. \tag{12}$$

With this definition we have

$$Z'(r) = \varphi(r) \int_{S(r)} a(x)||\vec{w}(x)||^{p^*}\,\mathrm{d}\sigma$$

and the Hölder inequality gives

$$\begin{aligned}
\pm \int_{S(r)} \alpha(x)\vec{w}(x)\vec{\nu}(x)\,\mathrm{d}\sigma &\le \left(\int_{S(r)} \alpha(x)||\vec{w}(x)||^{p^*}\,\mathrm{d}\sigma\right)^{1/p^*} \left(\int_{S(r)} \alpha(x)\,\mathrm{d}\sigma\right)^{1/p} \\
&= \left(\int_{S(r)} \alpha(x)||\vec{w}(x)||^{p^*}\,\mathrm{d}\sigma\right)^{1/p^*} \left(\omega_n r^{n-1}\tilde{a}(r)\right)^{1/p} \\
&= \left(Z'(r)\right)^{1/p^*} \left(\omega_n r^{n-1}\tilde{a}(r)\right)^{1/p}.
\end{aligned}$$

Using this estimate in (11) and using notation (12) we obtain for all $r \ge T_0$ the inequality

$$\frac{q-1}{l^*} Z(r) \le \left(\frac{Z'(r)}{\varphi(r)}\right)^{1/p^*} \left(\omega_n r^{n-1}\tilde{a}(r)\right)^{1/p} \tag{13}$$

and equivalently

$$\left(\frac{q-1}{l^*}\right)^{p^*} \varphi(r) \left(\omega_n r^{n-1}\tilde{a}(r)\right)^{-p^*/p} \le \frac{Z'(r)}{Z^{p^*}(r)}. \tag{14}$$

An integration of this inequality on the interval (T_0, ∞) gives a convergent integral on the right side and the divergent one on the left side (see (4)). This contradiction proves the statement of the theorem. □

We formulate several corollaries of the main Theorem hereinafter. To simplify our further investigations, we focus our attention to the undamped case of the equation (1), i.e. we will consider the equation

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)|u|^{q-2}u = 0. \tag{15}$$

Corollary 1. *Let*

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} c(x)\,\mathrm{d}x = \infty. \tag{16}$$

Then equation (15) *possesses no positive solution satisfying inequality*

$$|u(x)|^{q-p} \geq ||x||^{n-p} \tag{17}$$

on $\Omega(r)$ *for arbitrary large* r.

Proof. Choose $a(x) = 1$ and $\varphi(r) = r^{\frac{n-p}{p-1}}$. With this choice we have $P_{a,\varphi,l}(x) = c(x)$ for any l and $\tilde{a}(r) = 1$. This shows that (3) takes the form (16). Further

$$\begin{aligned} \varphi(r)\Big(r^{n-1}\tilde{a}(r)\Big)^{-p^*/p} &= r^{\frac{n-p}{p-1}} r^{-(n-1)p^*/p} \\ &= r^{\frac{n-p}{p-1} - \frac{n-1}{p-1}} = r^{-1} \end{aligned}$$

and hence (4) holds. Now Theorem 1 can be use to conclude the statement. □

Another simple condition which ensures (3) arises in the case when the functions a and φ are such that the integral of the second part of the definition (5) of the function $P_{a,\varphi,l}(x)$ is convergent. In this case the divergence of the integral $\int_{\Omega(r_0,r)} a(x)c(x)\,\mathrm{d}x$ is a condition which implies (3). In the following we focus our attention to the more delicate case, when integral of both parts of the function $P_{a,\varphi,l}(x)$ diverges.

Corollary 2. *Let* $\alpha \neq 0$ *and* $\beta := \alpha(p-1) - (n-p) \neq 0$ *be real numbers. Suppose that there exists a real number* $l > 1$ *such that*

$$\lim_{r\to\infty} \int_{\Omega(r_0,r)} \left[||x||^{\beta} c(x) - \frac{1}{p}|\beta|^p \Big(\frac{q-1}{l}p^*\Big)^{-p/p^*} \frac{1}{||x||^n} \right] \mathrm{d}x = \infty. \tag{18}$$

Then equation (15) *possesses no positive solution satisfying inequality*

$$|u(x)|^{\frac{q-p}{p-1}} \geq ||x||^{\alpha} \tag{19}$$

on $\Omega(r)$ *for arbitrary large* r.

Proof. Choose $\varphi(r) = r^{\alpha}$, $a(x) = ||x||^{\beta}$. Then $\tilde{a}(r) = r^{\beta}$ and

$$\varphi(r)\Big(r^{n-1}\tilde{a}(r)\Big)^{-p^*/p} = r^{\alpha} r^{-(n-1+\beta)p^*/p} = r^{-1}$$

and hence (4) holds. Further

$$\begin{aligned} P_{a\varphi,l}(x) &= ||x||^{\beta}c(x) - \frac{1}{p}\Big(|\beta|||x||^{\beta-1}\Big)^p \Big(\frac{q-1}{l}p^*||x||^{\alpha+\beta}\Big)^{-p/p^*} \\ &= ||x||^{\beta}c(x) - \frac{1}{p}|\beta|^p\Big(\frac{q-1}{l}p^*\Big)^{-p/p^*} ||x||^{(\beta-1)p-(\alpha+\beta)p/p^*} \end{aligned}$$

and a direct computation shows that

$$(\beta - 1)p - (\alpha+\beta)p/p^* = -n.$$

Hence (3) takes the form (18). The assumptions of Theorem 1 are satisfied and (19) follows from (6). □

The following corollary presents a more effective condition than (18) since the number $l > 1$ is removed from this condition.

Corollary 3. *Replace* (18) *in Corollary 2 by the condition*

$$\liminf_{r\to\infty} \frac{1}{\ln r} \int_{\Omega(r_0,r)} ||x||^{\beta} c(x)\,\mathrm{d}x > \frac{1}{p}|\beta|^p \big[(q-1)p^*\big]^{-p/p^*} \omega_n. \tag{20}$$

Then the conclusion of Corollary 2 remains intact.

Proof. Inequality (20) implies that there exist numbers $l > 1$, $\varepsilon > 0$ and $R > r_0$ such that

$$\int_{\Omega(r_0,r)} ||x||^{\beta} c(x)\,\mathrm{d}x \ \geq \frac{1}{p}|\beta|^p \big[\frac{q-1}{l} p^*\big]^{-p/p^*} \omega_n \ln r + \varepsilon \ln r$$

for all $r > R$. This inequality is equivalent to the inequality

$$\int_{\Omega(r_0,r)} \Big[||x||^{\beta} c(x) - \frac{1}{p}|\beta|^p \big[\frac{q-1}{l} p^*\big]^{-p/p^*} \frac{1}{||x||^n} \Big]\,\mathrm{d}x \ \geq \varepsilon \ln r$$

and the limit process $r \to \infty$ yields (18). □

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