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VISCOUS-INVISCID COUPLED PROBLEM WITH INTERFACIAL DATA

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Abstract. The work presented in this article shows that the viscous/inviscid coupled problem (VIC) has a unique solution when interfacial data are imposed. Domain decomposition techniques and non-uniform relaxation parameters were used to characterize the solution of the new system. Finally, some exact solutions for the VIC problem are provided. These type of solutions are an improvement over those found in recent literatures.

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1. **Introduction.** The Navier-Stokes equation is the primary equation of computational fluid dynamics describing the flow/motion of fluids in $\mathbb{R}^n$, $(n = 2, 3)$. These types of equations are often used in computations of aircraft and ship design, weather prediction, and climate modelling. By appropriate assumptions, it has been generalized to a system of equations known as the incompressible Navier-Stokes equations, see [8]. This important system has been studied for centuries by mathematicians, engineers and other scientists to explain and predict the behavior of the system under consideration, but still the understanding of the solutions to this system remains minimal. The challenge is to make substantial progress toward a mathematical theory which will solve the puzzle behind the Navier-Stokes equations. To make contributions to this mathematical theory, scientists have studied and derived many other systems from it. Among them is the viscous/inviscid coupled problem (VIC) introduced first by Xu Chuanju in his Ph.D dissertation [15].

The work presented in this paper involves Xu's [17] problem and focuses on three main objectives. The first one is to show the existence and uniqueness of the solution for the system, which results from the viscous/inviscid coupled problem when interfacial data (VIC-ID) are imposed. The second objective is to prove that the solution of this system can be obtained as a limit of solutions of two subproblems defined in different subdomains of the domain by using non-uniform relaxation parameters. Xu [17] used a similar techniques for the VIC problem but using lifting operators and uniform relaxation parameter. Finally, the last objective is to provide new exact solutions when all boundary conditions are satisfied in at least one of the subdomains (weaker boundary conditions) of the viscous/inviscid coupled problem, these solutions are an inprovement over those found in recent literatures. The new improvements presented in this paper demostrate progress towards the existing theory of the VIC problem and therefore for the Navier-Stokes equations.

We end this section by introducing some notation, definitions and very well known result from P.D.E that can be found [1] and [14] . Along this article we use boldface letters to denote vectors and vectors functions.

Let Ω be a bounded, connected, open subset of $\mathbb{R}^2$, with Lipschitz continuous boundary $\partial\Omega$, and let Ω_- and Ω_+ be two open subsets of Ω which satisfy the following conditions

(i) $\Omega_- \cap \Omega_+ = \emptyset$.

(ii) $\overline{\Omega}_- \cup \overline{\Omega}_+ = \overline{\Omega}$.

We define the boundaries Γ_+ and Γ_- of the subdomains Ω_+ and Ω_- respectively, as follows

$$\Gamma_+ = \partial\Omega \cap \partial\Omega_+,$$
$$\Gamma_- = \partial\Omega \cap \partial\Omega_-,$$

and the interface is given by

$$\Gamma = \partial\Omega_- \cap \partial\Omega_+,$$

we assume $\Gamma \neq \emptyset$. Let n denote the unit normal on $\partial\Omega$ to Ω, and n_+, n_- are the unit normals to $\partial\Omega_+$, and $\partial\Omega_-$, respectively. See figure 1 for a typical decomposition of the domain Ω.

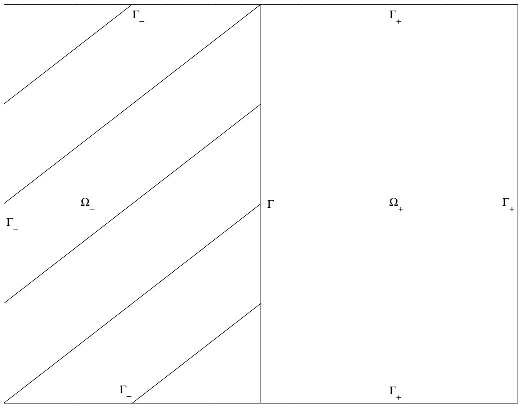

Figure 1: Decomposition of the domain Ω

We denote by $L^2(\Omega)$ the space of real functions defined on Ω that are square-integrable over Ω in the sense of Lebesgue measure $dx = dx_1dx_2$. This is a Hilbert space with the scalar product

$$(u, v) = \int_\Omega u(x)v(x)dx.$$

We then define the constrained space $L^2_0(\Omega)$ as

$$L^2_0(\Omega) = \left\{v \in L^2(\Omega); \int_\Omega vdx = 0\right\}$$

so $L^2_0(\Omega)$ consists of all square integrable functions having zero mean over Ω. Let $\mathcal{D}(\Omega)$ be the linear space of functions infinitely differentiable and with compact support on Ω. Then set

$$\mathcal{D}(\overline{\Omega}) = \{\phi|_\Omega : \phi \in \mathcal{D}(\mathbb{R}^2)\}.$$

For any integer m, we define the Sobolev space $H^m(\Omega)$ to be the set of functions in $L^2(\Omega)$ whose partial derivatives of order less than or equal to m belong to $L^2(\Omega)$; i.e

$$H^m(\Omega) = \{v \in L^2(\Omega) : \partial^\alpha v \in L^2(\Omega) \text{ for all } |\alpha| \leqq m\},$$

where

$$\partial^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}},$$

with $\alpha = (\alpha_1,\ \alpha_2)$, α_i is a non-negative natural number and $|\alpha| = \alpha_1 + \alpha_2$.

The set $H^m(\Omega)$ has the following properties:

(i) $H^m(\Omega)$ is a Banach space with the norm

$$\|u\|_{m,\Omega} = \Big(\sum_{|\alpha|\ \leqq\ m} \int_\Omega |\partial^\alpha u(x)|^2 dx\ \Big)^{1/2}$$

(ii) $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\big(u, v\big)_{m,\ \Omega} = \sum_{|\alpha|\ \leqq\ m} \int_\Omega \partial^\alpha u(x) \partial^\alpha v(x) dx$$

(iii) $H^m(\Omega)$ can be equipped with the seminorm

$$|u\|_{m,\ \Omega} = \Big(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha u(x)|^2 dx\ \Big)^{1/2}.$$

Since we are dealing with 2-dimensional vector functions, we use the following notation

$$\mathbf{L}^2(\Omega) = \{L^2(\Omega)\}^2, \qquad \mathbf{H}^m(\Omega) = \{H^m(\Omega)\}^2,$$

and assume that these product spaces are equipped with the usual product norm (or any equivalent norm).

We defined the space $H^1_0(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ for the norm $\|.\|_{m,\ \Omega}$. In order to study more closely the boundary values of functions of $H^m(\Omega)$, we assume that Γ, the boundary of Ω, is bounded and Lipschitz continuous, i.e. Γ can be represented parametrically by Lipschitz continuous functions. Let $d\sigma$ denote the surface measure on Γ and let $L^2(\Gamma)$ be the space of square integrable functions on Γ with respect to $d\sigma$, equipped with the norm

$$\|v\|_{0,\ \Gamma} = \{\int_\Gamma (v(\sigma))^2 d\sigma\}^{\frac{1}{2}}$$

Theorem 1 (Trace Theorem). *If $\partial\Omega$ is bounded and Lipschitz continuous, then there exist a bounded linear mapping $\gamma : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$, such that,*

(i) $\|\gamma(v)\|_{0,\ \partial\Omega} \leqq\ k\|v\|_{1,\ \Omega}$, *for all* $v \in H^1(\Omega)$.

(ii) $\gamma v = v|_{\partial\Omega}$ *for all* $v \in \mathcal{D}(\overline{\Omega})$.

Theorem 2. *With γ defined as above, we have*

1. $Ker(\gamma) = H_0^1(\Omega)$.
2. *The range space of γ is a proper and dense subspace of $L^2(\partial\Omega)$.*

For proofs see [3] and [11]. Both theorems can be extended to vector-valued functions.
The range space of the mapping γ, denote by $H^{1/2}(\partial\Omega)$, is a Hilbert space with the norm

$$\|\mu\|_{1/2,\partial\Omega} = \inf_{\substack{v\in H^1(\Omega)\\ \gamma v=\mu}} \|v\|_{1,\ \Omega}.$$

Let $H^{-1/2}(\partial\Omega)$ be the corresponding dual space of $H^{1/2}(\partial\Omega)$, with norm given by

$$\|\hat{\mu}\|_{-1/2,\partial\Omega} = \sup_{\substack{\mu\in H^{1/2}(\partial\Omega)\\ \mu\neq 0}} \frac{|\langle\mu,\hat{\mu}\rangle|}{\|\mu\|_{1/2,\partial\Omega}},$$

where $\langle\ ,\ \rangle$ denotes the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. For any vector function $\mathbf{v} \in \mathbf{L}^2(\Omega)$, we consider the pair of functions $\mathbf{v}_- = \mathbf{v}|_{\Omega_-}$ and $\mathbf{v}_+ = \mathbf{v}|_{\Omega_+}$. We define the following inner products in Ω_+ and Ω_- respectively as follows:

$$(\mathbf{u}_+,\ \mathbf{v}_+)_+ = \int_{\Omega_+} \mathbf{u}_+\mathbf{v}_+ dx,$$
$$(\mathbf{u}_-,\ \mathbf{v}_-)_- = \int_{\Omega_-} \mathbf{u}_-\mathbf{v}_- dx,$$

and for any Ψ and $\Phi\ \in \mathbf{L}^2(\Gamma)$ as:

$$(\Psi,\ \Phi)_\Gamma = \int_\Gamma \Psi\Phi d\sigma.$$

The scalar product on $\mathbf{L}^2(\Omega_-) \times \mathbf{L}^2(\Omega_+)$,

$$(\mathbf{u},\ \mathbf{v}) = (\mathbf{u}_-,\ \mathbf{v}_-)_- + (\mathbf{u}_+,\ \mathbf{v}_+)_+,$$

which coincides with the usual one on $\mathbf{L}^2(\Omega)$.
Consider the following space:

$$\mathbf{V} = \{\mathbf{v}\ \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\mathbf{v} = 0\}$$

and we denote $\mathbf{V}^\perp$ the orthogonal complement of $\mathbf{V}$ in $\mathbf{H}_0^1(\Omega)$ for the scalar product $(\nabla\mathbf{u}\ ,\ \nabla\mathbf{v}) = \sum \frac{\partial\mathbf{u}_i}{\partial x^j}\frac{\partial\mathbf{v}_i}{\partial x^j}$. Thus we have the following divergence isomorphism theorem.

Theorem 3 (Divergence Isomorphism Theorem). *The divergence operator is an isomorphism from* $\mathbf{V}^{\perp}$ *onto* $L_0^2(\Omega)$ *and satisfies*

$$\|\mathbf{v}\|_{1,\ \Omega} \leq \frac{1}{\beta}\|\nabla\cdot\mathbf{v}\|_{0,\ \Omega}\ \forall\mathbf{v}\in\ V^{\perp}. \tag{1}$$

For a proof see Girault [7]. This theorem plays an important role in the proof of uniqueness and existence of the VIC-ID problem.

2. **Viscous-Inviscid Coupled Problem with Interfacial Data.** In this article we show that is it is possible to impose further conditions on the interfacial data and still have a solution to the problem. These conditions are expressed in the form of membership to certain function spaces. Specifically, we study the VIC-ID problem. For $\mathbf{f}\in\mathbf{L}^2(\Omega)$, $\hat{g}^0\in H^{-1/2}(\Gamma)$, $\hat{\mathbf{p}}_+^0\in\mathbf{L}^2(\Gamma)$, α and ν positive constants, find two pairs $(\mathbf{u}_-,\mathbf{u}_+)$, $(p_-,\ p_+)$, defined in $(\Omega_-,\ \Omega_+)$, satisfying the following conditions

$$\begin{aligned}
\alpha\mathbf{u}_- - \nu\Delta\mathbf{u}_- + \nabla p_- &= \mathbf{f}_-\ \textit{in}\ \Omega_-\\
\nabla\cdot\mathbf{u}_- &= 0\ \textit{in}\ \Omega_-\\
\mathbf{u}_- &= 0\ \textit{on}\ \Gamma_-\\
\alpha\mathbf{u}_+ + \nabla p_+ &= \mathbf{f}_+\ \textit{in}\ \Omega_+\\
\nabla\cdot\mathbf{u}_+ &= 0\ \textit{in}\ \Omega_+\\
\mathbf{u}_+ n_+ &= 0\ \textit{on}\ \Gamma_+\\
\nu\frac{\partial\mathbf{u}_-}{\partial n_-} - p_- n_- &= p_+ n_+\ \textit{on}\ \Gamma\\
p_+ n_+ &= \hat{\mathbf{p}}_+^0\ \ \textit{on}\ \Gamma\\
\mathbf{u}_+ n_+ &= -\mathbf{u}_- n_-\ \textit{on}\ \Gamma\\
-\mathbf{u}_- n_- &= \hat{g}^0\ \textit{on}\ \Gamma;
\end{aligned} \tag{2}$$

where the domain Ω satisfies the properties mentioned previously, and the function $\hat{g}^0$ satisfies the condition

$$\int_\Gamma \hat{g}^0 = 0.$$

In fluid mechanics this conditions is generally known as compatibility condition, see [5] and [10]. From above, the first three equations correspond to the viscous part, from fourth to sixth to the inviscid part, and the last four corresponds to the interface data. Next we prove existence and uniqueness of the solution. For that we use the saddle point theory which involves: (i) finding the weak or variational formulation of the VIC-ID problem, to describe the spaces where the interfacial data have sense. (ii) rewriting the weak formulation to find the saddle point problem; which involves the definition of two bilinear forms **a**, **b** and must satisfy the following conditions:

(i) Continuity of the bilinear form **a**: $|\ \mathbf{a}(\mathbf{u},\ \mathbf{v})\ | \leqq c\|\mathbf{u}\|_X\|\mathbf{v}\|_X$, for some positive constant c.

(ii) Coerciveness of the bilinear form $\mathbf{a}$: $\mathbf{a}(\mathbf{v},\ \mathbf{v}) \geqq d\|\mathbf{v}\|_X^2$, for some positive constant d.

(iii) Continuity of the bilinear form $\mathbf{b}$: $\mid \mathbf{b}(\mathbf{u},\ q) \mid \leqq c\|\mathbf{u}\|_X\|q\|_M$, for some positive constant c.

(iv) Inf-sup condition for the bilinear form $\mathbf{b}$: $\inf_{q\in M}\sup_{\mathbf{v}\in X}\frac{\mathbf{b}(\mathbf{v},\ q)}{\|\mathbf{v}\|_X\|q\|_M} \geq \beta$, for some positive constant β.

Above conditions guarantee the existence and uniqueness of the solution for the given problem, for more details see [3] and [4]. By showing the following steps we can find the weak formulation of (2).

(i) From the first and fourth equations of (2) the following inner product equations are obtained

$$\alpha(\mathbf{u}_-,\ \mathbf{v}_-)_- - \nu(\Delta\mathbf{u}_-,\ \mathbf{v}_-)_- + (\nabla p_-,\ \mathbf{v}_-)_- = (\mathbf{f}_-,\ \mathbf{v}_-)_- \tag{3}$$

$$\alpha(\mathbf{u}_+,\ \mathbf{v}_+)_+ + (\nabla p_+,\ \mathbf{v}_+)_+ = (\mathbf{f}_+,\ \mathbf{v}_+)_+ \tag{4}$$

(ii) From the second and the fifth equations of (2) we get the two pair of inner product equations

$$(\nabla\cdot\mathbf{u}_-,\ q_-)_- = 0, \tag{5}$$

and

$$(\nabla\cdot\mathbf{u}_+,\ q_+)_+ = 0; \tag{6}$$

adding equations (5) and (6), we have

$$(\nabla\cdot\mathbf{u}_+,\ q_+)_+ + (\nabla\cdot\mathbf{u}_-,\ q_-)_- = 0 \tag{7}$$

Using the following well known identities for vector functions:

$$\begin{aligned}
-(\Delta\mathbf{u}_-,\ \mathbf{v}_-)_- &= (\nabla\mathbf{u}_-,\ \nabla\mathbf{v}_-)_- - \int_\Gamma \frac{\partial\mathbf{u}_-}{\partial n_-}\mathbf{v}_- d\sigma,\\
(\nabla p_-,\ \mathbf{v}_-)_- &= \int_\Gamma p_-\mathbf{v}_- n_- d\sigma - \int_\Omega \nabla\cdot\mathbf{v}_- p_- dx,\\
(\nabla\cdot\mathbf{u}_+,\ q_+)_+ &= \int_\Gamma \mathbf{u}_+ n_+ q_+ d\sigma - (\mathbf{u}_+,\ \nabla q_+)_+,
\end{aligned}$$

adding equations (3) and (4), and by appropriate substitutions, We have the following weak formulation of VIC-ID problem: find $(\mathbf{u}, p) \in X\times M$, such that for all $\mathbf{v}\in X,\ q\in M$,

$$\begin{aligned}
\alpha(\mathbf{u},\ \mathbf{v}) + \nu(\nabla\mathbf{u}_-,\ \nabla\mathbf{v}_-)_- - (p_-,\ \nabla\cdot\mathbf{v}_-)_- + (\nabla p_+,\ \mathbf{v}_+)_+ &= (\mathbf{f},\ \mathbf{v}) + \langle\hat{\mathbf{p}}_+^0,\ \mathbf{v}_-\rangle_\Gamma\\
-(\nabla\cdot\mathbf{u}_-,\ q_-)_- + (\mathbf{u}_+,\ \nabla q_+)_+ &= \langle\hat{g}^0,\ q_+\rangle_\Gamma
\end{aligned} \tag{8}$$

where X, M are the two Hilbert space, defined by

$$X = \{\mathbf{v};\, \mathbf{v}|_{\Omega_-} \in \mathbf{H}^1(\Omega_-),\, \mathbf{v}|_{\Omega_+} \in \mathbf{L}^2(\Omega_+),\, \mathbf{v}|_{\Gamma_-} = 0\}, \tag{9}$$

$$M = \{q;\, q|_{\Omega_-} \in L^2(\Omega_-),\, q|_{\Omega_+} \in H^1(\Omega_+),\, \int_\Omega q dx = 0\}, \tag{10}$$

with respective norms

$$\|\mathbf{v}\|_X = \|\mathbf{v}_-\|_{1,\ \Omega_-} + \|\mathbf{v}_+\|_{0,\ \Omega_+},$$

and

$$\|q\|_M = \|q_-\|_{0,\ \Omega_-} + |q_+|_{1,\ \Omega_+}.$$

The VIC-ID problem is well posed in the sense that its corresponding weak formulation admits a unique solution. The statement of the theorem and proof is given below, which is one of the main results of these work.

Theorem 4. *For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\hat{g}^0 \in H^{-1/2}(\Gamma)$, $\hat{\mathbf{p}}_+^0 \in \mathbf{L}^2(\Gamma), \alpha$ and ν positive constants, problem (8) admits a unique solution; furthermore, its unique solution $(\mathbf{u},\ p)$ satisfies VIC-ID problem.*

Proof. The second part of the theorem is trivial. In order to prove the well posedness of (8), we need to apply the saddle point theory. This can be achieved by reorganizing the terms of the weak formulation by defining two bilinear forms $\mathbf{a}, \mathbf{b}$ as follows:

$$\mathbf{a}(\mathbf{u},\ \mathbf{v}) = \alpha(\mathbf{u},\ \mathbf{v}) + \nu(\nabla\mathbf{u}_-,\ \nabla\mathbf{v}_-)_- \quad \forall\ \mathbf{u} \in X,\ \forall\ \mathbf{v} \in X,$$

$$\mathbf{b}(\mathbf{v},\ q) = (\mathbf{v}_+,\ \nabla q_+)_+ - (\nabla\cdot\mathbf{v}_-,\ q_-)_- \quad \forall\ \mathbf{v} \in X,\ \forall\ q \in M.$$

By appropriate substitutions in (8), the saddle point problem is given as: find $(\mathbf{u},\ p) \in X \times M$ such that,

$$\mathbf{a}(\mathbf{u},\ \mathbf{v}) + \mathbf{b}(\mathbf{v},\ p) = (\mathbf{f},\ \mathbf{v}) + \langle\hat{\mathbf{p}}_+^0,\ \mathbf{v}_-\rangle_\Gamma \quad \forall\mathbf{v} \in X,$$

$$\mathbf{b}(\mathbf{v},\ q) = \langle\hat{g}^0,\ q_+\rangle_\Gamma \qquad \forall q \in M.$$

To show the above forms $\mathbf{a}$, $\mathbf{b}$ satisfy the conditions mentioned previously:

(i) The form $\mathbf{a}$ is continuous, since

$$\begin{aligned} |\mathbf{a}(\mathbf{u},\mathbf{v})| &\leqq \alpha|(\mathbf{u},\mathbf{v})| + \nu|(\nabla\mathbf{u}_-,\nabla\mathbf{v}_-)_-| \\ &\leqq \alpha(|(\mathbf{u}_-,\mathbf{v}_-)_-| + |(\mathbf{u}_+,\mathbf{v}_+)_+|) + \nu|\mathbf{u}_-|_{1,\Omega_-}|\mathbf{v}_-|_{1,\Omega_-} \end{aligned}$$

taking $max(\alpha,\nu) = c$ and applying Schwartz inequality,

$$|\mathbf{a}(\mathbf{u},\mathbf{v})| \leqq c(\|\mathbf{u}_-\|_{1,\Omega_-}\|\mathbf{v}_-\|_{1,\Omega_-} + \|\mathbf{u}_+\|_{0,\Omega_+}\|\mathbf{v}_+\|_{0,\Omega_+}),$$

then

$$| \mathbf{a}(\mathbf{u},\ \mathbf{v}) \mid \leqq c(\|\mathbf{u}_-\|_{1,\Omega_-}\|\mathbf{v}_-\|_{1,\ \Omega_-} + \|\mathbf{u}_+\|_{0,\Omega_+}\|\mathbf{v}_+\|_{0,\ \Omega_+}$$
$$+\|\mathbf{u}_+\|_{0,\Omega_+}\|\mathbf{v}_-\|_{1,\ \Omega_-} + \|\mathbf{u}_-\|_{1,\Omega_+}\|\mathbf{v}_+\|_{0,\ \Omega_+}).$$

Therefore

$$| \mathbf{a}(\mathbf{u},\ \mathbf{v}) \mid \leqq c\|\mathbf{u}\|_X\|\mathbf{v}\|_X.$$

(ii) The form $\mathbf{a}$ is coercive,

$$\mathbf{a}(\mathbf{v},\ \mathbf{v}) = \alpha(\mathbf{v},\ \mathbf{v}) + \nu(\nabla\mathbf{v}_-,\ \nabla\mathbf{v}_-)_-$$
$$= \alpha((\mathbf{v}_+,\ \mathbf{v}_+)_+ + (\mathbf{v}_-,\ \mathbf{v}_-)_-) + \nu(\nabla\mathbf{v}_-,\ \nabla\mathbf{v}_-)_-,$$

by definition of the inner product, we have

$$\mathbf{a}(\mathbf{v},\mathbf{v}) = \alpha(|\mathbf{v}_+|^2_{0,\Omega_+} + |\mathbf{v}_-|^2_{0,\Omega_-}) + \nu|\mathbf{v}_-|^2_{1,\ \Omega_-},$$

choosing $min(\alpha,\nu) = d_0$, and applying Schwartz inequality we get

$$\mathbf{a}(\mathbf{v},\mathbf{v}) \geqq d_0(|\mathbf{v}_+|^2_{0,\Omega_+} + \|\mathbf{v}_-\|^2_{1,\Omega_-}) \geqq d\|\mathbf{v}\|^2_X$$

where $d = \frac{d_0}{2}$.

(iii) The form $\mathbf{b}$ is continuous,

$$|\mathbf{b}(\mathbf{v},\ q)| \leqq \|q_-\|_{0,\ \Omega_-}|\mathbf{v}_-|_{1,\ \Omega_-} + |q_+|_{1,\ \Omega_+}\|\mathbf{v}_+\|_{0,\ \Omega_+},$$

since $|\mathbf{v}_-|_{1,\ \Omega_-} \leqq c_0\ \|\mathbf{v}_-\|_{1,\ \Omega_-}$, and adding appropriate terms to complete the norms for the above form, we obtain

$$|\mathbf{b}(\mathbf{v},\ q)| \leqq c_0\ \|q_-\|_{0,\Omega_-}\|\mathbf{v}_-\|_{1,\Omega_-} + |q_+|_{1,\Omega_+}\|\mathbf{v}_+\|_{0,\Omega_+} + \|q_-\|_{0,\Omega_-}\|\mathbf{v}_+\|_{0,\Omega_+}$$
$$+\ |q_+|_{1,\Omega_+}\|\mathbf{v}_-\|_{1,\Omega_-},$$

choosing $c = max(1,\ c_0)$, it follows that

$$|\mathbf{b}(\mathbf{v},\ q)| \leqq c\ \|\mathbf{v}\|_X\|q\|_M.$$

(iv) The form $\mathbf{b}$ satisfies the *inf-sup* condition in the space $X \times M$: The objective is to decompose the general vector $\mathbf{v}$ into $\mathbf{v}_+$ and $\mathbf{v}_-$ in such a way that *inf-sup* condition is satisfied.
Let $q \in M$, and q_- can be decomposed as

$$q_- = q_-^0 + r_-, \tag{11}$$

where $q_-^0 \in L^2_0(\Omega_-)$ and r_- is a constant. The decomposition of q_- is justified by the following calculation

$$\int_{\Omega_-} q_- dx = k_1,$$

then

$$\int_{\Omega_-} (q_- - \frac{k_1}{|\Omega_-|})\ dx = 0,$$

where k_1 is a constant,and set

$$q^0_- = q_- - r_-, \text{ and } r_- = \frac{k_1}{|\Omega_-|}.$$

Using theorem 3 it follows that there exists a positive constant β_- and a function $\mathbf{v}^0_- \in \mathbf{H}^1_0(\Omega_-)$ such that

$$\nabla\cdot\mathbf{v}^0_- = -q^0_- \quad \text{and} \quad \|\mathbf{v}^0_-\|_{1,\ \Omega_-} \leqq \frac{1}{\beta_-}\|q^0_-\|_{0,\ \Omega_-}. \tag{12}$$

Choosing a function $\mathbf{g} \in X$ such that

$$\int_\Gamma \mathbf{g} n_- d\sigma = |\Omega_-|,$$

where $|\Omega_-|$ is the measure of Ω_-. Let $\mathbf{w}$ be a function in $\mathbf{H}^1_0(\Omega_-)$ which satisfies:

$$(\nabla\cdot\mathbf{w},\ q)_- = (\nabla\cdot\mathbf{g},\ q)_- \quad \forall q \in L^2_o(\Omega_-). \tag{13}$$

To establish the existence of $\mathbf{w}$ we apply a special case of the saddle point theory when only one bilinear form is given. Consider the following bilinear form
$\hat{\mathbf{b}} : \mathbf{H}^1_0(\Omega_-) \times L^2_o(\Omega_-) \longrightarrow \mathbb{R}$, defined as

$$\hat{\mathbf{b}}(\mathbf{v},\ q) = -(\mathbf{v},\ \nabla q)_- \quad \forall\ (\mathbf{v},\ q) \in \mathbf{H}^1_0(\Omega_-) \times L^2_o(\Omega_-),$$

that satisfies

$$\hat{\mathbf{b}}(\mathbf{w}, q) = (\nabla\cdot\mathbf{g}, q)_- \quad \forall q \in\ L^2_o(\Omega_-).$$

The norms for this spaces are $\|\ .\ \|_{\mathbf{H}^1_0} = \|\ .\ \|_{0,\ \Omega_-}$ and $\|\ .\ \|_{L^2_o} = |\ .\ |_{1,\Omega_-}$, respectively. Since the continuity of $\hat{\mathbf{b}}$ is obvious from the definition itself. Next step is to show that the form $\hat{\mathbf{b}}$ satisfies the *inf-sup* condition as follows: taking $\mathbf{v} = -\nabla q$, and substituting in the form $\hat{\mathbf{b}}$ we have the following

$$\begin{aligned}\hat{\mathbf{b}}(\mathbf{v},\ q) &= (\nabla q,\ \nabla q) = \|\ \nabla q\ \|^2_{0,\ \Omega_-} = \|\ \nabla q\ \|_{0,\ \Omega_-}|q|_{1,\ \Omega_-} \\ &= \|\ \mathbf{v}\ \|_{0,\ \Omega_-}|q|_{1,\ \Omega_-}.\end{aligned}$$

Therefore $\hat{\mathbf{b}}$ satisfies the *inf-sup* condition for $\beta = 1$. To guarantee surjectivity of $\hat{\mathbf{b}}$ it is necessary to show that for every $q \in L^2_o(\Omega_-)$ there exists $\mathbf{v} \in \mathbf{H}^1_0(\Omega_-)$ such that $\hat{\mathbf{b}}(\mathbf{v},\ q) \neq 0$. By choosing $\mathbf{v} = -\nabla q$ such a way so that it satisfies the required condition, otherwise $\|\ \nabla q\ \|^2_{0,\ \Omega_-} = 0$ occurs, if and

only if q is zero. Therefore, the existence of $\mathbf{w}$ is proved.
Since we prove the existences of $\mathbf{w}$, then the following statement is true

$$(\nabla\cdot\mathbf{w},\ q)_- = (\nabla\cdot\mathbf{g},\ q)_- \quad \forall q \in L^2_o(\Omega_-).$$

Let $\hat{\mathbf{v}}_- = \mathbf{g} - \mathbf{w}$ which satisfies

$$(\nabla\cdot\hat{\mathbf{v}}_-,\ q)_- = 0 \quad \forall q \in L^2_o(\Omega_-) \quad \text{and} \quad \int_\Gamma \hat{\mathbf{v}}_- n_- d\sigma = |\Omega_-|. \tag{14}$$

By setting $\mathbf{v}_- = \mathbf{v}^0_- - r_-\hat{\mathbf{v}}_-$, and using relationships (11) , (12), and (14) it follows that

$$\mathbf{v}_- \in \mathbf{H}^1(\Omega_-), \quad \mathbf{v}_-|_{\Gamma_-} = 0,$$

and

$$-(q_-,\ \nabla\cdot\mathbf{v}_-)_- = -(q^0_- + r_-, \nabla\cdot(\mathbf{v}^0_- - r_-\hat{\mathbf{v}}_-))_-$$

by linearity of the inner product, we have

$$\begin{aligned} -(q_-,\ \nabla\cdot\mathbf{v}_-)_- = &- (q^0_-,\ \nabla\cdot\mathbf{v}^0_-)_- + (q^0_-,\ r_-\nabla\cdot\hat{\mathbf{v}}_-)_- - (r_-,\ \nabla\cdot\mathbf{v}^0_-)_- \\ &+ r_-(r_-,\ \nabla\cdot\hat{\mathbf{v}}_-)_-. \end{aligned}$$

Using the properties of q^0_-, $\nabla\hat{\mathbf{v}}_-$ and divergence theorem, it follows

$$-(q_-,\ \nabla\cdot\mathbf{v}_-)_- = (q^0_-,\ q^0_-)_- + r^2_- \int_\Gamma \hat{\mathbf{v}}_- n_- d\sigma,$$

which can be equivalently expressed in the following form

$$-(q_-,\ \nabla\cdot\mathbf{v}_-)_- = \|q^0_-\|^2_{0,\ \Omega_-} + r^2_-|\Omega_-|. \tag{15}$$

By applying the same procedure as in equation (11) to decompose q_+ in the subdomain Ω_+ as

$$q_+ = q^0_+ + r_+,$$

where $q^0_+ \in H^1(\Omega_+) \cap L^2_0(\Omega_-)$ and r_+ is a constant.
Let $\mathbf{v}^0_+ = \nabla q^0_+$,
then

$$\frac{(\nabla q_+,\ \mathbf{v}^0_+)_+}{\|\mathbf{v}^0_+\|_{0,\ \Omega_+}} = \|\nabla q^0_+\|_{0,\ \Omega_+} = \|\nabla q_+\|_{0,\ \Omega_+} = |q_+|_{1,\ \Omega_+}.$$

Using the fact that $\nabla q_+ \in \mathbf{L}^2(\Omega_+)$, we can choose $\mathbf{v}_+ = \mathbf{v}^0_+$,
so

$$(\nabla q_+,\ \mathbf{v}_+)_+ = |q_+|^2_{1,\ \Omega_+}, \tag{16}$$

and the vector $\mathbf{v}$ is characterized as follows:

$$\mathbf{v}(x) = \begin{cases} \mathbf{v}_-(x) & \text{if x} \in \Omega_- \\ \mathbf{v}_+(x) & \text{if x} \in \Omega_+ \end{cases}$$

Therefore, $\mathbf{v} \in X$ and from equations (15), and (16), we can write

$$\begin{aligned}\mathbf{b}(\mathbf{v},\ q) &= -(\nabla\cdot\mathbf{v}_-,\ q_-)_- (\mathbf{v}_+,\ \nabla q_+)_+ \\ &= \|q_-^0\|_{0,\ \Omega_-}^2 + r_-^2|\Omega_-| + |q_+|_{1,\ \Omega_+}^2.\end{aligned}$$

By using

$$\|q_-\|_{0,\ \Omega_-}^2 = (q_-,\ q_-)_- = (q_-^0 + r_-,\ q_-^0 + r_-)_-,$$

and applying the linearity of the inner product in the above equation, we obtain

$$\|q_-\|_{0,\ \Omega_-}^2 = (q_-^0,\ q_-^0)_- + 2(q_-^0,\ r_-)_- + r_-^2|\Omega_-|. \tag{17}$$

By using (17) in the bilinear form $\mathbf{b}$, we have

$$\mathbf{b}(\mathbf{v},\ q) = \|q_-\|_{0,\ \Omega_-}^2 - r_-^2|\Omega_-| + r_-^2|\Omega_-| + |q_+|_{1,\ \Omega_+}^2.$$

Therefore, the bilinear form $\mathbf{b}$ satisfies

$$\mathbf{b}(\mathbf{v},\ q) \geq \|q\|_M^2. \tag{18}$$

The next step is to show that the components of the vector $\mathbf{v}$ are bounded. Using the definition of $\mathbf{v}_-$ and the relationships obtained in (12), (13), and (14), we have the following estimates

$$\|\mathbf{v}_-\|_{0,\ \Omega_-} = \|\mathbf{v}_-^0 - r_-\hat{\mathbf{v}}_-\|_{0,\ \Omega_-} \leq \frac{1}{\beta_-}\|q_-^0\|_{0,\ \Omega_-} + r_-\|\hat{\mathbf{v}}_-\|_{1,\ \Omega_-}. \tag{19}$$

Since $\mathbf{w} \in \mathbf{H}_0^1(\Omega_-)$, there exist constants $\bar{c}$ and $\hat{c}$ such that

$$\|\hat{\mathbf{v}}_-\|_{1,\ \Omega_-} \leq \bar{c}\ \|\mathbf{w}_-\|_{1,\ \Omega_-} \leq \hat{c}\ \|\mathbf{g}_-\|_{1,\ \Omega_-}. \tag{20}$$

Therefore,

$$\|\mathbf{v}_-\|_{0,\ \Omega_-} \leq \frac{1}{\beta_-}\|q_-^0\|_{0,\ \Omega_-} + \hat{c}r_-\|\mathbf{g}_-\|_{1,\ \Omega_-}. \tag{21}$$

By knowing that $\|r_-\|_{0,\ \Omega_-} = \|q_- - q_-^0\|_{0,\ \Omega_-}$, we can find a constant c_1 such that

$$\|r_-\|_{0,\ \Omega_-}\|\mathbf{g}\|_{1,\ \Omega_-} \leq \frac{c_1}{\beta_-}\ \|q_-\|_{0,\ \Omega_-}. \tag{22}$$

By combining (19), (20), (21), and (22), it follows that $\mathbf{v}_-$ satisfies the following inequality:

$$\|\mathbf{v}_-\|_{1,\ \Omega_-} \leq \frac{c_2}{\beta_-}\|q_-\|_{0,\ \Omega_-}, \tag{23}$$

where $c_2 = \frac{1+\hat{c}c_1}{\beta_-}$. Using the definition of $\mathbf{v}_+$ we get the following estimates:

$$\|\mathbf{v}_+\|_{1,\ \Omega_+} \leq \|\nabla q_+^0\|_{0,\ \Omega_+} \leq \|\nabla q_+\|_{0,\ \Omega_+} \leq |q_+|_{1,\ \Omega_+} \leq \|q\|_M. \tag{24}$$

By making use of (23) and (24), it follows that

$$\|\mathbf{v}\|_X \leq \frac{c_2}{\beta_-}\|q_-\|_{0,\ \Omega_-} + \|q\|_M,$$

since $\|q_-\|_{0,\ \Omega_-} \leq \|q\|_M$, we have

$$\|\mathbf{v}\|_X \leq \frac{c_2 + \beta_-}{\beta_-}\|q\|_M. \tag{25}$$

Using (18) and (25), we obtain

$$\frac{\mathbf{b}(\mathbf{v},\ q)}{\|\mathbf{v}\|_X} \geq \frac{\|q\|_M^2}{\frac{c_1+\beta_-}{\beta_-}\|q\|_M},$$

and setting $\beta = \frac{\beta_-}{c_1+\beta_-}$, in the above inequality we get

$$\frac{\mathbf{b}(\mathbf{v},\ q)}{\|\mathbf{v}\|_X\|q\|_M} \geq \beta. \tag{26}$$

By taking *inf-sup* of (26), we get the following inequality

$$\inf_{q\in M}\ \sup_{\mathbf{v}\in X}\ \frac{\mathbf{b}(\mathbf{v},\ q)}{\|\mathbf{v}\|_X\|q\|_M} \geq \beta,$$

which completes the proof.

□

3. **The iteration-by-subdomain procedure and its convergence.** The purpose of this section is to prove that the solution of VIC-ID problem can be obtained as a limit of solutions of two subproblems in the subdomains Ω_- and Ω_+, respectively, of Ω.

Let $\{\tilde{p}_+^m\}$ be a sequence of functions in $L^2(\Gamma)$ such that $\tilde{p}_+^m \longrightarrow p_+$, for some $p_+ \in L^2(\Gamma)$. We define the sequence of function pairs $(\mathbf{u}_-^m, p_-^m)_{m\geq 1}$ by solving for each m the following viscous interfacial data problem in Ω_-:

$$\begin{aligned} \alpha\mathbf{u}_-^m - \nu\Delta\mathbf{u}_-^m + \nabla p_-^m &= \mathbf{f}_-\ in\ \Omega_- \\ \nabla\cdot\mathbf{u}_-^m &= 0\ in\ \Omega_- \\ \mathbf{u}_-^m &= 0\ in\ \Gamma_- \\ \nu\frac{\partial\mathbf{u}_-^m}{\partial n_-} - p_-^m n_- &= \tilde{p}_+^m n_+\ \ on\ \Gamma. \end{aligned} \tag{27}$$

We have to find the variational problem corresponding to the viscous interfacial data problem using the saddle point theory. For that we consider the first and the second equation of (27) to get the pair of inner product equations as follows:

$$\begin{aligned} \alpha(\mathbf{u}_-^m,\ \mathbf{v}_-)_- - \nu(\Delta\mathbf{u}_-^m,\ \mathbf{v}_-)_- + (\nabla p_-^m,\ \mathbf{v}_-)_- &= (\mathbf{f}_-,\ \mathbf{v}_-)_-, \\ (\nabla\cdot\mathbf{u}_-^m,\ q_-) &= 0. \end{aligned} \tag{28}$$

Using the well known identities for vector functions,

$$(\Delta \mathbf{u}_-^m,\ \mathbf{v}_-)_- = (\nabla \mathbf{u}_-^m,\ \nabla \mathbf{v}_-)_- - \int_\Gamma \frac{\partial \mathbf{u}_-}{\partial n_-} \mathbf{v}_- d\sigma,$$

$$(\nabla p_-^m,\ \mathbf{v}_-)_- = - \int_\Gamma p_-^m \mathbf{v}_- n_- d\sigma - \int_\Omega \nabla \cdot \mathbf{v}_- p_-^m dx,$$

by combining the equations in (28), and making the appropriate substitutions we can state the following weak formulation of the viscous interfacial data problem: find $(\mathbf{u}_-^m,\ p_-^m) \in X_- \times M_-$ such that

$$\mathbf{A}_-[(\mathbf{u}_-^m,\ p_-^m),\ (\mathbf{v}_-,\ q_-)] = (\mathbf{f}_-,\ \mathbf{v}_-)_- + \langle \tilde{p}_+^m n_+,\ \mathbf{v}_- \rangle_\Gamma \ \forall\, (\mathbf{v}_-,\ q_-) \in X_- \times M_-, \tag{29}$$

where

$$X_- = \{\mathbf{v}_- \in \mathbf{H}^1(\Omega_-), \mathbf{v}_-|_{\Gamma_-} = 0\} \quad \text{and} \quad M_- = L(\Omega_-),$$

and $\mathbf{A}_-$ is defined by

$$\begin{aligned} \mathbf{A}_-[(\mathbf{u}_-^m,\ p_-^m),\ (\mathbf{v}_-,\ q_-)] =& \alpha(\mathbf{u}_-^m,\ \mathbf{v}_-)_- + \nu(\nabla \mathbf{u}_-^m,\ \nabla \mathbf{v}_-)_- \\ &- (\nabla \cdot \mathbf{v}_-,\ p_-^m)_- + (\nabla \cdot \mathbf{u}_-^m,\ \mathbf{q}_-)_-. \end{aligned}$$

The next step is to make use of the following theorem, from the Navier-Stokes equations literature(for more details see [2]) that guarantees the existence and uniqueness of the solution for the weak formulation of the viscous interfacial data problem.

Theorem 5. *For all* $\mathbf{f}_- \in \mathbf{L}^2(\Omega_-)$ *and* $\tilde{p}_+^m \in L^2(\Gamma)$, *the variational problem (29) admits one solution; furthermore, its solution* $(\mathbf{u}_-^m,\ p_-^m)$ *satisfies*

$$\|\mathbf{u}_-^m\|_{1,\ \Omega_-} + \|p_-^m\|_{0,\ \Omega_-} \le c_3(\|\mathbf{f}\|_{0,\ \Omega_-} + \|\tilde{p}_+^m\|_{0,\ \Gamma}), \tag{30}$$

where c_3 *depends on* α *and* ν.

By the theorem we can define a sequence of function pairs $(\mathbf{u}_-^m,\ p_-^m)$ to state the inviscid interfacial data problem in Ω_+ as follows:

$$\begin{aligned} \alpha \mathbf{u}_+^m + \nabla p_+^m &= \mathbf{f}_+ \ in\ \Omega_+ \\ \nabla \cdot \mathbf{u}_+^m &= 0 \ in\ \Omega_+ \\ \mathbf{u}_+^m n_+ &= 0 \ in\ \Gamma_+ \\ \mathbf{u}_+^m n_+ &= \varphi^m \ on\ \Gamma. \end{aligned} \tag{31}$$

The functions φ^m are defined by:

$$\varphi^m = e^{-t_m} \mathbf{u}_- n_-|_\Gamma + t_m \mathbf{u}_-^m n_-|_\Gamma,$$

where the terms e^{-t_m} , t_m are the non-uniform relaxation parameters, and these functions satisfy the following compatibility condition

$$\int_\Gamma \varphi^m d\sigma = 0 \ \forall m.$$

Where $\{t_m\}$ is a sequence of non-negative numbers such that $t_m \longrightarrow 0$ as $m \longrightarrow \infty$. As discussed in the viscous case, we need to find the corresponding variational problem for the inviscid interfacial data problem. From the first and the third equations of (31) we obtain the following pair of inner product equations

$$\begin{aligned} \alpha(\mathbf{u}_+^m,\ \mathbf{v}_+)_+ (\nabla p_-^m,\ \mathbf{v}_+)_+ &= (\mathbf{f}_+,\ \mathbf{v}_-)_+, \\ (\mathbf{u}_+^m n_+,\ q_+)_+ &= \langle \varphi^m,\ q_+\rangle_+, \end{aligned} \tag{32}$$

and using the identity

$$(\nabla q_+,\ \mathbf{u}_+^m)_+ = \int_\Gamma q_+ \mathbf{u}_+^m n_- d\sigma - \int_\Omega \nabla\cdot \mathbf{u}_+^m q_+ dx,$$

the variational problem for the inviscid interfacial data problem can be stated as: find a pair of functions $(\mathbf{u}_+^m,\ p_+^m) \in X_+ \times M_+$ such that

$$\mathbf{A}_+[(\mathbf{u}_+^m,\ p_+^m),\ (\mathbf{v}_+,\ q_+)] = (\mathbf{f}_+,\ \mathbf{v}_+)_+ - \langle \varphi^m,\ q_+\rangle_\Gamma \ \forall\ (\mathbf{v}_+,\ q_+) \in X_+ \times M_+, \tag{33}$$

where $\langle .\ ,\ .\rangle_\Gamma$ denotes the pairing between the space $H^{1/2}(\Gamma)$ and its dual space $H^{-1/2}(\Gamma)$. The form $\mathbf{A}_+$ is defined by

$$\mathbf{A}_+[(\mathbf{u}_+^m,\ p_+^m),\ (\mathbf{v}_+,\ q_+)] = \alpha(\mathbf{u}_+^m,\ \mathbf{v}_+)_+ + (\mathbf{v}_+,\ \nabla p_+^m)_+ - (\mathbf{u}_+^m,\ \nabla q_+)_+,$$

where the spaces X_+ and M_+ are given as:

$$X_+ = \mathbf{L}^2(\Omega_+),$$

$$M_+ = H^1(\Omega_+) \cap L_0^2(\Omega_+),$$

and these are equipped by the following norms:

$$\|\ .\ \|_{X_+} = \|\ .\ \|_{0,\ \Omega_+}, \quad \|\ .\ \|_{M_+} = |\ .\ |_{1,\ \Omega_+}.$$

The following theorem states the existence and uniqueness of the weak formulation for the inviscid interfacial data problem.

Theorem 6. *For all $\mathbf{f}_+ \in \mathbf{L}^2(\Omega_+)$and $\varphi^m \in H^{-1/2}(\Gamma)$, the variational problem (33) admits a unique solution; furthermore, its solution $(\mathbf{u}_+^m,\ p_+^m)$ satisfies the following condition*

$$\|\mathbf{u}_+^m\|_{0,\ \Omega_+} + |p_+^m|_{1,\ \Omega_+} \leq (\frac{1}{\alpha} + 2)\|\mathbf{f}_+\|_{0,\ \Omega_+} + 2(1+\alpha)\|\varphi^m\|_{-1/2,\ \Gamma}. \tag{34}$$

If $\mathbf{f}_+ = 0$ *then the following inequalities hold*

$$\|\mathbf{u}_+^m\|_{0,\ \Omega_+} \le 2\|\varphi^m\|_{-1/2,\ \Gamma}, \tag{35}$$

$$|p_+^m|_{1,\ \Omega_+} \le \alpha\|\mathbf{u}_+^m\|_{0,\ \Omega_+} \le 2\alpha\|\varphi^m\|_{-1/2,\ \Gamma}. \tag{36}$$

Proof. In order to prove the theorem Saddle point theory is used. The well posedness of equation (33) can be shown by solving its equivalent saddle problem and is given as follows

$$\begin{aligned} \mathbf{a}_+(\mathbf{u}_+^m,\ \mathbf{v}_+) + \mathbf{b}_+(\mathbf{v}_+,\ p_+^m) = (\mathbf{f}_+,\ \mathbf{v}_+) \quad \forall \mathbf{v}_+ \in X_+ \\ \mathbf{b}_+(\mathbf{u}_+^m,\ q_+) = \langle \varphi^m,\ q_+\rangle_\Gamma \quad \forall q_+ \in M_+. \end{aligned} \tag{37}$$

The above bilinear forms $\mathbf{a}_+$ and $\mathbf{b}_+$ are defined by

$$\mathbf{a}_+(\mathbf{u},\ \mathbf{v}) = \alpha(\mathbf{u},\ \mathbf{v})_+,\ \forall \mathbf{u},\mathbf{v} \in X_+$$

$$\mathbf{b}_+(\mathbf{u},\ q) = (\mathbf{v},\ \nabla q)_+,\ \forall \mathbf{v} \in X_+, \forall q \in M_+,$$

which has to satisfy the requirements discussed in the previous section, so that it guarantees the uniqueness of the solution of the weak problem and is given as

(i) The form $\mathbf{a}_+$ is continuous:

$$|\mathbf{a}_+(\mathbf{u},\ \mathbf{v})| = |\alpha(\mathbf{u},\ \mathbf{v})_+| \le \alpha\|\mathbf{u}\|_{0,\ \Omega_+}\|\mathbf{v}\|_{0,\ \Omega_+} \le \alpha\|\mathbf{u}\|_{X_+}\|\mathbf{v}\|_{X_+}.$$

(ii) the form $\mathbf{a}_+$ is coercive:

$$\mathbf{a}_+(\mathbf{u},\ \mathbf{u}) = \alpha(\mathbf{u},\ \mathbf{u})_+ = \alpha\|\mathbf{u}\|^2_{0,\ \Omega_+} = \alpha\|\mathbf{u}\|^2_{X_+}.$$

(iii) the form $\mathbf{b}_+$ is continuous:

$$|\mathbf{b}_+(\mathbf{u},\ q)| \le \|\mathbf{v}\|_{0,\ \Omega_+}\|\nabla q\|_{0,\ \Omega_+} = \|\mathbf{v}\|_{0,\ \Omega_+}|q|_{1,\ \Omega_+} = \|\mathbf{v}\|_{X_+}\|q\|_{M_+}.$$

(iv) the form $\mathbf{b}_+$ satisfies the *inf-sup* condition:

By replacing $\mathbf{v} = \nabla q$, in the bilinear form $\mathbf{b}_+$ and using the norms defined above, we obtain

$$\mathbf{b}_+(\mathbf{u},\ q) = (\mathbf{v},\ \nabla q)_+ = (\nabla q,\ \nabla q)_+ = \|q\|^2_{0,\ \Omega_+} = \|\nabla q\|_{0,\ \Omega_+}|q|_{1,\ \Omega_+},$$

then

$$\mathbf{b}_+(\mathbf{u},\ q) = \|\mathbf{v}\|_{X_+}\|q\|_{M_+},$$

which implies that the form $\mathbf{b}_+$ satisfies the *inf-sup* condition for the case $\beta = 1$.

Therefore, by the saddle point theory, the problem (33) has a unique solution $(\mathbf{u}_+^m,\ p_+^m)$. So the next objective is to prove the inequalities described in (34), (35), and (36). For every pair of functionals $\mathbf{f}_+,\ \varphi^m \in L^2(\Omega_+)\times H^{-1/2}(\Gamma)$, there exist exactly one pair of solution $(\mathbf{u}_+^m,\ p_+^m)$ corresponding to the saddle point problem which satisfies the following conditions:

$$\|\mathbf{u}_+^m\|_{X_+} \le \alpha^{-1}\|\mathbf{f}_+\|_{0,\ \Omega_+} + \beta^{-1}(1+\frac{c}{\alpha})\|\varphi^m\|_{-1/2,\ \Gamma},$$

$$\|p_+^m\|_{M_+} \le \beta^{-1}(1+\frac{c}{\alpha})\|\mathbf{f}_+\|_{0,\ \Omega_+} + \beta^{-1}(1+\frac{c}{\alpha})\frac{c}{\beta}\|\varphi^m\|_{-1/2,\ \Gamma}.$$

In the above inequalities, β the constant of *inf-sup* condition of $\mathbf{b}_+$ is 1, α and c are the constants for coerciveness and continuity of $\mathbf{a}_+$.

Let $\mathbf{V}$ and $\mathbf{V}(\varphi^m)$ be linear subspaces defined as follows:

$$\mathbf{V}(\varphi^m) = \{\mathbf{v}\ \in X_+ : \mathbf{b}_+(\mathbf{v},\ q) = \langle \varphi^m,\ q\rangle,\ \forall\ q\ \in\ M_+\}$$

$$\mathbf{V} = \{\mathbf{v}\ \in X_+ : \mathbf{b}_+(\mathbf{v},\ q) = 0,\ \ \forall\ q\ \in\ M_+\};$$

$\mathbf{V}$ is a closed subspace of X_+, since $\mathbf{b}_+$ is continuous. By the *inf-sup* condition there exists $\mathbf{u}_0^m\ \in\ \mathbf{V}^\perp$ with $\mathbf{B}\mathbf{u}_0^m = \varphi^m$, where $\mathbf{B} : X_+ \longrightarrow H^{-1/2}(\Gamma)$ is a mapping associated to $\mathbf{b}_+$, defined by

$$\langle \mathbf{B}\mathbf{u}^m,\ q_+\rangle = \mathbf{b}_+(\mathbf{u}^m,\ q_+),\ \forall\ q\ \in\ M_+,$$

moreover,

$$\|\mathbf{u}_0^m\|_{X_+} \le \beta^{-1}\|\varphi^m\|_{-1/2,\ \Gamma}. \tag{38}$$

By setting $\mathbf{w} = \mathbf{u}_+^m - \mathbf{u}_0^m$, the equivalent saddle point problem (37) can be rewritten as

$$\begin{aligned} \mathbf{a}_+(\mathbf{w},\ \mathbf{v}_+) + \mathbf{b}_+(\mathbf{v}_+,\ p_+^m) = (\mathbf{f}_+,\ \mathbf{v}_+) - \mathbf{a}_+(\mathbf{u}_0^m,\ \mathbf{v}_+) \quad \forall\ \mathbf{v}_+ \in X_+ \\ \mathbf{b}_+(\mathbf{w},\ q_+) = 0 \quad \forall q_+ \in M_+. \end{aligned} \tag{39}$$

Since $\mathbf{a}_+$ is coercive, the real valued function

$$\frac{1}{2}\ \mathbf{a}_+(\mathbf{v}_+,\ \mathbf{v}_+) - (\mathbf{f}_+,\ \mathbf{v}_+) + \mathbf{a}_+(\mathbf{u}_0^m,\ \mathbf{v}_+)$$

attains a minimum for some $\mathbf{w} \in \mathbf{V}$ having the following property

$$\|\mathbf{w}\|_{X_+} \le \alpha^{-1}(\|\mathbf{f}_+\|_{0,\ \Omega_+} + c\|\mathbf{u}_+^m\|_{0,\ \Omega_+}). \tag{40}$$

In particular , the characterization theorem [3] implies

$$\mathbf{a}_+(\mathbf{w},\ \mathbf{v}_+) = (\mathbf{f}_+,\ \mathbf{v}_+) - \mathbf{a}_+(\mathbf{u}_0^m,\ \mathbf{v}_+) \quad \forall\ q_+ \in M_+. \tag{41}$$

The equation (39) will be satisfied if we can find $p_+^m \in M_+$ such that

$$\mathbf{b}_+(\mathbf{v}_+,\ p_+^m) = (\mathbf{f}_+,\ \mathbf{v}_+) - \mathbf{a}_+(\mathbf{u}_0^m + \mathbf{w},\ \mathbf{v}_+) \quad \forall\ \mathbf{v}_+ \in X_+, \tag{42}$$

holds. The right-hand side of (42) defines a functional in the dual space X'_+, since $\mathbf{b}_+$ satisfies the *inf-sup* condition, it follows that the functional can be represented as $\mathbf{B}' p_+^m$ with $p_+^m \in M_+$, which satisfies the following condition

$$\|p_+^m\|_{M_+} \leq \beta^{-1}(\|\mathbf{f}_+\|_{0,\ \Omega_+} + c\|\mathbf{u}_+^m\|_{X_+}), \tag{43}$$

where β and c are the constants for the *inf-sup* and continuity conditions for the bilinear form $\mathbf{b}_+$. This establishes the solvability for equation (42). Since $\mathbf{w} = \mathbf{u}_+^m - \mathbf{u}_0^m$, it follows that $\mathbf{u}_+^m = \mathbf{w} + \mathbf{u}_0^m$, then by triangular inequality, we obtain

$$\|\mathbf{u}_+^m\|_{X_+} \leq \|\mathbf{w}\|_{X_+} + \|\mathbf{u}_0^m\|_{X_+}.$$

By using inequality (40), we have

$$\|\mathbf{u}_+^m\|_{X_+} \leq \alpha^{-1}(\|\mathbf{f}_+\|_{0,\ \Omega_+} + c\|\mathbf{u}_+^m\|_{0,\ \Omega_+}) + \|\mathbf{u}_0^m\|_{X_+},$$

and by (38) we get

$$\|\mathbf{u}_+^m\|_{X_+} \leq \alpha^{-1}(\|\mathbf{f}_+\|_{0,\ \Omega_+} + c\beta^{-1}\|\varphi^m\|_{-1/2,\ \Gamma}) + \beta^{-1}\|\varphi^m\|_{-1/2,\ \Gamma}.$$

By combining the above terms, we obtain the following inequality

$$\|\mathbf{u}_+^m\|_{X_+} \leq \alpha^{-1}\|\mathbf{f}_+\|_{0,\ \Omega_+} + (1 + \frac{c}{\alpha})\beta^{-1}\|\varphi^m\|_{-1/2,\ \Gamma}.$$

If we substitute $\beta = 1$, the *inf-sup* condition for $\mathbf{b}_+$ and $c = \alpha$, the continuity condition for $\mathbf{a}_+$, we have the following inequality

$$\|\mathbf{u}_+^m\|_{X_+} \leq \alpha^{-1}\|\mathbf{f}_+\|_{0,\ \Omega_+} + 2\|\varphi^m\|_{-1/2,\ \Gamma}. \tag{44}$$

From inequalities (43) and (44), we get

$$\|p_+^m\|_{M_+} \leq \beta^{-1}\|\mathbf{f}_+\|_{0,\ \Omega_+} + \beta^{-1}c(\alpha^{-1}\|\mathbf{f}_+\|_{0,\ \Omega_+} + 2\|\varphi^m\|_{-1/2,\ \Gamma}),$$

which is equivalent to

$$\|p_+^m\|_{M_+} \leq \beta^{-1}(1 + \frac{c}{\alpha})\|\mathbf{f}_+\|_{0,\ \Omega_+} + 2c\beta^{-1}\|\varphi^m\|_{-1/2,\ \Gamma},$$

again by substituting $\beta = 1$ and $c = 1$ the continuity condition for $\mathbf{b}_+$, we obtain

$$\|p_+^m\|_{M_+} \leq 2\|\mathbf{f}_+\|_{0,\ \Omega_+} + 2\alpha\|\varphi^m\|_{-1/2,\ \Gamma}. \tag{45}$$

By combining inequalities (44) and (45) we proved inequality (34) stated in theorem 6 asfollows:

$$\|\mathbf{u}_+^m\|_{0,\ \Omega_+} + |p_+^m|_{1,\ \Omega_+} \leq (\frac{1}{\alpha} + 2)\|\mathbf{f}_+\|_{0,\ \Omega_+} + 2(1+\alpha)\|\varphi^m\|_{-1/2,\ \Gamma}.$$

If $\mathbf{f}_+ = 0$, we get (35) and (36) from (44) and (45), respectively, i.e

$$\|\mathbf{u}_+^m\|_{0,\Omega_+} \leq 2\|\varphi^m\|_{-1/2,\Gamma},$$

and

$$|p_+^m|_{1,\Omega_+} \leq \alpha\|\mathbf{u}_+^m\|_{0,\Omega_+} \leq 2\alpha\|\varphi^m\|_{-1/2\Gamma}.$$

$\square$

3.1 **Convergence of the iteration-by-subdomain procedure.** To prove convergence we use the fact that the relaxation parameter in φ^m is non-uniform and the properties of the norms imposed on the interface Γ and the spaces X_+ and X_-. The next step is to prove the sequence $\{\varphi^m\}$ converges to $\mathbf{u}_- n_-$ with respect to the dual norm of $H^{\frac{1}{2}}(\Gamma)$. For that we have the following estimates:

$$\|\varphi^m - \mathbf{u}_- n_-\|_{-1/2,\Gamma} = \sup_{\substack{\mu \in H^{1/2}(\Gamma) \\ \mu \neq 0}} \frac{|\langle \varphi^m - \mathbf{u}_- n_-,\ \mu \rangle_\Gamma|}{\|\mu\|_{1/2,\Gamma}},$$

since $\varphi^m - \mathbf{u}_- n_- \in\ L^2(\Gamma)$, so we have,

$$\langle \varphi^m - \mathbf{u}_- n_-,\ \mu \rangle_\Gamma = \int_\Gamma (\varphi^m - \mathbf{u}_- n_- \mu) d\sigma = \int_\Gamma (e^{-t_m} - 1) \mathbf{u}_- n_- \mu d\sigma + t_m \int_\Gamma \mathbf{u}_-^m n_- \mu d\sigma.$$

Using Schwartz inequality, we have the following

$$|\langle \varphi^m - \mathbf{u}_- n_-,\ \mu \rangle_\Gamma| \leq |e^{-t_m} - 1| \int_\Gamma |\mathbf{u}_- n_- \mu| d\sigma + |t_m| \int_\Gamma |\mathbf{u}_-^m n_- \mu| d\sigma.$$

By taking *sup* and dividing by $\|\mu\|_{1/2,\Gamma}$, we obtain

$$\|\varphi^m - \mathbf{u}_- n_-\|_{-1/2,\Gamma} \leq |e^{-t_m} - 1|\ \|\mathbf{u}_- n_-\|_{-1/2,\ \Gamma} + |t_m|\ \|\mathbf{u}_-^m n_-\|_{-1/2,\ \Gamma}.$$

By using the following estimate (see [1] and [11] for further details),

$$\|\mathbf{u}_-^m n_-\|_{-1/2,\ \Gamma} \leq c_4 (\|\mathbf{u}_-^m\|_{0,\ \Omega_-} + |\|\nabla . \mathbf{u}_-^m\|_{0,\ \Omega_-}) \leq c_4 \|\mathbf{u}_-^m\|_{1,\ \Omega_-},$$

and equation (30) in theorem 5, , we obtain the following

$$\begin{aligned} \|\mathbf{u}_-^m n_-\|_{-1/2,\ \Gamma} &\leq c_4 (\|\mathbf{u}_-^m\|_{0,\ \Omega_-} + \|\nabla . \mathbf{u}_-^m\|_{0,\ \Omega_-}) \\ &\leq c_4 \|\mathbf{u}_-^m\|_{1,\ \Omega_-} \leq c_5 (\|\mathbf{f}_-\|_{0,\ \Omega_-} + \|\tilde{p}_+^m\|_{0,\ \Gamma}). \end{aligned}$$

Therefore,

$$\|\varphi^m - \mathbf{u}_- n_-\|_{-1/2,\Gamma} \leq |e^{-t_m} - 1|\ \|\mathbf{u}_- n_-\|_{-1/2,\ \Gamma} + c_5 |t_m|\ (\|\mathbf{f}_-\|_{0,\ \Omega_-} + \|\tilde{p}_+^m\|_{0,\ \Gamma}).$$

Since $t_m \longrightarrow 0$ as $m \longrightarrow \infty$, and $\tilde{p}_+^m \longrightarrow p_+$ as $m \longrightarrow \infty$ we obtain $\|\varphi^m - \mathbf{u}_- n_-\|_{-1/2,\Gamma} \longrightarrow 0$, therefore $\varphi^m \longrightarrow \mathbf{u}_- n_-$ as $m \longrightarrow \infty$. The next objective is to prove that $\mathbf{u}_+^m - \mathbf{u}_+ \longrightarrow 0$ in Ω_+. By taking $\mathbf{f}_+ = 0$, and using the variational form $\mathbf{A}_+$ as discussed in section 4.1 for the inviscid problem, we have

$$\mathbf{A}_+ [(\mathbf{u}_+^m - \mathbf{u}_+,\ p_+^m - p_+), (\mathbf{v}_+,\ q_+)] = \langle \mathbf{u}_- n_- - \varphi^m,\ q_+ \rangle_\Gamma,$$

and (34) from theorem 6, we get

$$\|\mathbf{u}_+^m - \mathbf{u}_+\|_{0,\Omega_+} + \|p_+^m - p_+\|_{M_+} \leq 2(1 + \alpha) \|\mathbf{u}_- n_- - \varphi^m\|_{-1/2,\ \Gamma}.$$

Since $\varphi^m - \mathbf{u}_- n_- \longrightarrow 0$ as $m \longrightarrow \infty$, it follows that $\mathbf{u}_+^m \longrightarrow \mathbf{u}_+$ in X_+, and $p_+^m \longrightarrow p_+$ in M_+ as $m \longrightarrow \infty$. By similar argument we show that $\mathbf{u}_-^m - \mathbf{u}_- \longrightarrow 0$ in Ω_-. By taking $\mathbf{f}_- = 0$, and using the fact $p_-^m - p_- \in L^2(\Gamma)$, the form $\mathbf{A}_-$ take the form

$$\mathbf{A}_-[(\mathbf{u}_-^m - \mathbf{u}_-,\ p_-^m - p_-), (\mathbf{v}_-,\ q_-)] = \langle (p_-^m - p_-)n_+,\ \mathbf{v}_-)\rangle_\Gamma,$$

and by (30) in theorem 5, we get

$$\|\mathbf{u}_-^m - \mathbf{u}_-\|_{1,\ \Omega_-} + \|p_-^m - p_-\|_{0,\ \Omega_-} \leq\ c_3\ \|\tilde{p}_+^m - p_+\|_{0,\ \Gamma}.$$

Using the hypothesis $\tilde{p}_+^m \longrightarrow p_+$
$textas\ m \longrightarrow \infty$, it follows that $\mathbf{u}_-^m \longrightarrow \mathbf{u}_-$ inX_- and $p_-^m \longrightarrow p_-$ as $m \longrightarrow \infty$ in M_-, which concludes the proof of convergence of the iteration-by-subdomain procedure.

4. **Exact solutions with weaker boundary conditions.** During the investigation of the viscous/inviscid coupled problem we found few exact solutions with weak boundary conditions. For computational purposes it is always advantageous to have as many test functions as possible, and the visualization for analyzing and characterizing the behavior of the problem under investigation, see [12]. Also with the purpose of designing newer algorithms and testing the ones suggested by others. We briefly describe two kinds of solutions with weak boundary conditions found in earlier literatures. The first kind of solutions $(\mathbf{u},\ p)$ are those that do not satisfy some of the boundary conditions in both subdomains, but satisfy the interface conditions. In such solutions the vector field component, $\mathbf{u}(x, y) = (\mathbf{u}_1(x, y), \mathbf{u}_2(x, y))$, has a particular form, in such a way that one of the component, either $\mathbf{u}_1(x, y) = 0$ or $\mathbf{u}_2(x, y) = 0$. In other words, the graph of the vector field $\mathbf{u}$ is embedded in $\mathbb{R}^3$. Almost all the solutions found in earlier articles are of this kind, see [18]).

The second kind of solutions $(\mathbf{u},\ p)$ are those that do not satisfy any of the boundary conditions but satisfy the interface conditions. In this case the graph of the vector field component is in $\mathbb{R}^4$. In fact only one of this kind of solution was found by Xu in his articles [16]. New examples of these solutions are found in Ramirez's dissertation [13].

Exact solutions with weaker boundary conditions are those that satisfy all the boundary conditions in at least one of the subdomains, and the interface conditions. In fact these solutions are an improvement over the exact solutions with weak boundary conditions found in earlier literatures. Here we present two types of solutions, which are exponential and polynomial in nature.

Given the domain Ω as follows:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -2\ \leq\ x\ \leq 2\ \&\ -1 \leq\ y\ \leq 1\},$$

which is decomposed into the two subdomains Ω_+ and Ω_- as

$$\Omega_+ = \{(x, y) \in \mathbb{R}^2 : 0\ \leq\ x\ \leq 2\ \&\ -1 \leq\ y\ \leq 1\},$$

and

$$\Omega_- = \{(x,y)\in\mathbb{R}^2 : -2 \ \le \ x \ \le 0 \ \& \ -1 \le \ y \ \le 1\}$$

Example 1: Exact solution with weaker boundary condition in Ω_+.
Consider the following data:
(i) the external force in the subdomain Ω_- is of the following form

$$\begin{aligned}
\mathbf{f}_-(x,y) =&(\alpha[4ye^{1-y^2}(e^{1-y^2}-1)(e^{8+x^3}-1)^2] - 4\nu ye^{1-y^2}(e^{1-y^2}-1)18x^4e^{2(x^3+8)}\\
&-4\nu ye^{1-y^2}(e^{1-y^2}-1)[6xe^{8+x^3}(e^{8+x^3}-1)(2+3x^3)]\\
&-\nu(e^{8+x^3}-1)^2[-16ye^{2(1-y^2)}[3-4y^2] - \nu(8e^{8+x^3}-1)^2ye^{1-y^2}[3-2y^2]]\\
&+\pi\cos(\pi x)\cos(\pi y)\ ,\ \alpha[6x^2(e^{1-y^2}-1)^2e^{8+x^3}(e^{8+x^3}-1)]\\
&-\nu(e^{1-y^2}-1)^2[3x^2(e^{2(8+x^3)})[12x+18x^4]\\
&+(e^{8+x^3}-1)e^{8+x^3}[12+108x^3+54x^6]+e^{2(8+x^3)}[72x^3+108x6]]\\
&+6x^2e^{x^3+8}(e^{x^3+8}-1)[8y^2e^{2(1-y^2)}\\
&+e^{1-y^2}(e^{1-y^2}-1)[-4+8y^2]] - \pi\sin(\pi y)\sin(\pi x)),
\end{aligned}$$

and
(ii) the external force in the subdomain Ω_+ is of the following form

$$\begin{aligned}
\mathbf{f}_+(x,y) =&(\alpha[4ye^{1-y^2}(e^{1-y^2}-1)(e^{8+x^3}-1)^2] + \pi\cos(\pi x)\cos(\pi y)\ ,\\
&\alpha[6x^2(e^{1-y^2}-1)^2e^{8+x^3}(e^{8+x^3}-1)] - \pi\sin(\pi y)\sin(\pi x)).
\end{aligned}$$

Then the vector field component $\mathbf{u}$ of the exact solution in the entire domain Ω is given by

$$\mathbf{u}(x,y) = (4ye^{1-y^2}(e^{1-y^2}-1)(e^{8+x^3}-1)^2\ ,\ 6x^2(e^{1-y^2}-1)^2e^{8+x^3}(e^{8+x^3}-1)),$$

and the pressure component of the solution in Ω is

$$p_+(x,y) = p_-(x,y) = \sin(\pi x)\cos(\pi y).$$

Figure 2 below shows the behavior of the velocity vectors close to the boundary in Ω and on the interface Γ. In other words, the graph below shows that the velocity vectors have an uniform behavior in the viscous part, contrary to the inviscid part where the behavior of the velocity vectors change dramatically close to the boundary, this happens because some of the boundary conditions are not satisfied in the inviscid part.

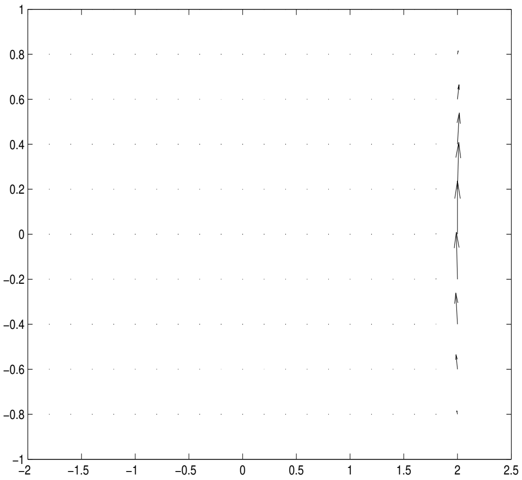

Figure 2: Velocity vectors of $\mathbf{u}$ in the domain Ω

Example 2: Exact Solution with weaker boundary condition in Ω_-.

The external forces in the subdomains Ω_- and Ω_+ are of the forms:

$$\begin{aligned}\mathbf{f}_-(x,y) =&(\alpha 4y(1-y^2)(x^3-8)^2 - 4\nu y(1-y^2)[30x^4+96x] - 24y(x^3-8)^2 - 3x \ , \\ &\alpha 6x^2(x^3-8)(1-y^2)^2 - \nu(1-y^2)^2[120x^3+96] + 6x^2(x^3-8)(-4+12y^2)+2y),\end{aligned}$$

and

$$\mathbf{f}_+(x,y) = (\alpha 4y(1-y^2)(x^3-8)^2 + 2x - 2\nu y \ , \ \alpha 6x^2(x^3-8)(1-y^2)^2 + 2y - 2\nu x).$$

Then the vector field component $\mathbf{u}$ of the exact solution in the domain Ω can be written as

$$\mathbf{u}(x,y) = (4y(1-y^2)(x^3-8)^2 \ , \ 6x^2(x^3-8)(1-y^2)^2).$$

For the completeness of the entire solution in the domain Ω, the pressures in both subdomain Ω_+ and Ω_- are given as follows:
pressure p_+ in the subdomain Ω_+ is

$$p_+(x,y) = x^2 + y^2 - yx \ ,$$

and pressure p_- in the subdomain Ω_- is

$$p_-(x,y) = \frac{-3}{2}x^2 + y^2.$$

Figure 3 below shows the behavior of the velocity vectors close to the boundary $\partial\Omega$ and on the interface Γ. From the graph we can observe the smooth behavior of the velocity vectors in the inviscid part without any hindrance.

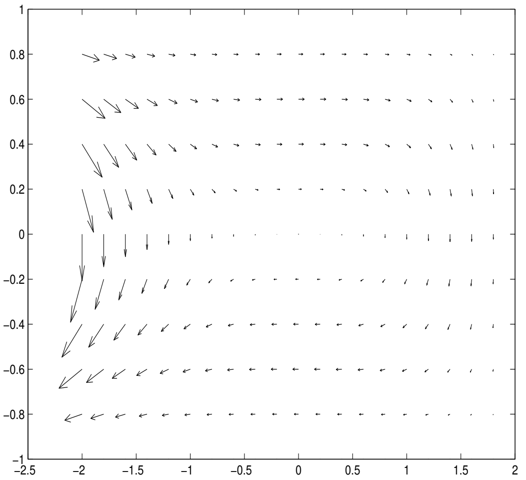

Figure 3: Velocity vectors of $\mathbf{u}$ in the domain Ω

For more information of these solutions see [13].

5. **Conclusions.** We have presented in this article (i) existence and uniqueness of the viscous-inviscid coupled problem with interfacial data, when suitable conditions are imposed on the interface (see theorem 4). (ii) convergence of the algorithm-subdomain procedure without using lifting operators (see section 3.2). (iii) exact solutions with weaker boundary condition in at least one of the subdomains for the viscous-inviscid coupled problem (see section 4), an improvement over those exact solutions with weak boundary conditions found in earlier literatures.

Further extensions of this work will be focused in several objectives: (i) Approximate the solutions of the viscous/inviscid coupled problem using finite element methods, with non-uniform relaxation parameters found along this work to improve convergence. (ii) Investigate the existence and uniqueness theorem for general case of the Navier-Stokes equations and apply to special applications in the field of fluid dynamics, see [6] and [9]. (iii) Extend the viscous-inviscid coupled problem with interfacial data to the unsteady Navier-Stokes equations by applying a similar methodology as Xu did in his work.

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