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POSITIVE SOLUTIONS FOR A FOURTH ORDER BOUNDARY VALUE PROBLEM

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Abstract. We consider a boundary value problem for the beam equation, in which the boundary conditions mean that the beam is embedded at one end and free at the other end. Some new estimates to the positive solutions to the boundary value problem are obtained. Some sufficient conditions for the existence of at least one positive solution for the boundary value problem are established. An example is given at the end of the paper to illustrate the main results.

1. Introduction

In this paper, we consider the fourth order ordinary differential equation

$$u''''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.1}$$

together with boundary conditions

$$u(0) = u'(0) = u''(1) = u'''(1) = 0. \tag{1.2}$$

Throughout the paper, we assume that

(H1) $f : [0, \infty) \to [0, \infty)$ is continuous; and

(H2) $g : [0, 1] \to [0, \infty)$ is a continuous function such that $\int_0^1 g(t)\, dt > 0$.

Eq.(1.1) and the boundary conditions (1.2) have definite physical meanings. Eq.(1.1) is often referred to as the beam equation. It describes the deflection of a beam under a certain force. The boundary conditions (1.2) mean that the beam is embedded at the end $t = 0$, and free at the end $t = 1$. Eq.(1.1) has been studied by many authors under various boundary conditions. For example,

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Ma [23] studied the existence of positive solutions for the fourth order boundary value problem

$$u''''(x) = \lambda f(x, u(x), u'(x)),$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0$$

under some superlinear semipositone condition. For some other results on boundary value problems of the beam equation, we refer the reader to the papers of Bai and Wang [7], Davis and Henderson [8], Elgindi and Guan [10], Eloe, Henderson, and Kosmatov [11], Graef and Yang [13], Gupta [15], Kosmatov [21], Liu and Ge [22], Ma [24], Ma and Wang [25], Yang [27], and Yao [28].

The boundary conditions (1.2) are a special case of the $(p, n-p)$ right focal boundary conditions, in which $n = p = 2$. Extensive research has been done on focal boundary value problems. The reader is referred to the monograph [1] by Agarwal for a systematic survey of this area. For some recent results on focal boundary value problems, we refer the reader to the papers by Agarwal and O'Regan [2], Agarwal, O'Regan, and Lakshmikantham [4], Anderson [5], Anderson and Davis [6], Davis, Henderson, Prasad, and Yin [9], Harris, Henderson, Lanz, and Yin [16], Henderson and Kaufmann[17], and Henderson and Yin [19].

In this paper, we will study the existence and nonexistence of positive solutions of the problem (1.1)-(1.2). Note that if $f(0) = 0$, then $u(t) \equiv 0$ is a solution to the problem (1.1)-(1.2). But in this paper, we are interested only in positive solutions. By positive solution, we mean a solution $u(t)$ such that $u(t) > 0$ for $t \in (0, 1)$. In this paper, we'll concentrate on the existence of at least one positive solution for the problem (1.1)-(1.2).

The Green's function $G : [0, 1] \times [0, 1] \to [0, \infty)$ for the problem (1.1)-(1.2) is given by

$$G(t, s) = \begin{cases} \frac{1}{6} t^2 (3s - t), & \text{if} \quad 0 \le t \le s \le 1, \\ \frac{1}{6} s^2 (3t - s), & \text{if} \quad 0 \le s \le t \le 1. \end{cases}$$

And the problem (1.1)-(1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \le t \le 1. \tag{1.3}$$

To prove our results, we will use the following fixed point theorem, which is due to Krasnosel'skii [20].

Theorem K. Let $(X, \|\cdot\|)$ be a Banach space over the reals, and let $P \subset X$ be a cone in X. Let H_1 and H_2 be real numbers such that $H_2 > H_1 > 0$, and let

$$\Omega_i = \{v \in X \mid \|v\| < H_i\}, \quad i = 1, 2.$$

If

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

is a completely continuous operator such that, either

(K1) $\|Lv\| \le \|v\|$ if $v \in P \cap \partial\Omega_1$, and $\|Lv\| \ge \|v\|$ if $v \in P \cap \partial\Omega_2$,

or

(K2) $\|Lv\| \ge \|v\|$ if $v \in P \cap \partial\Omega_1$, and $\|Lv\| \le \|v\|$ if $v \in P \cap \partial\Omega_2$.

Then L has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

One of the purposes of this paper is to establish some new estimates to the positive solutions of the problem (1.1)-(1.2). These estimates are essential to the main results of this paper. It is based on these estimates that we can define an appropriate cone, and apply the Krasnosel'skii's fixed point theorem to prove the existence results.

This paper is organized as follows. In Section 2, we obtain some new estimates to the positive solutions to the problem (1.1)-(1.2). In Sections 3 and 4, we establish some existence and nonexistence results for positive solutions to the problem (1.1)-(1.2). In Section 5, we give an example to illustrate the main results of the paper.

2. Estimates to Positive Solutions

In this section, we shall give some nice estimates to the positive solutions to the problem (1.1)-(1.2). To this purpose, we define the functions $a : [0,1] \to [0,+\infty)$ and $b_1 : [0,1] \to [0,+\infty)$ by

$$a(t) = \frac{3}{2}t^2 - \frac{1}{2}t^3, \quad 0 \le t \le 1,$$

$$b_1(t) = 2t^2 - \frac{4t^3}{3} + \frac{t^4}{3}, \quad 0 \le t \le 1.$$

The functions $a(t)$ and $b_1(t)$ will be used to estimate the positive solutions of the problem (1.1)-(1.2). It's easy to see that $t \ge b_1(t) \ge a(t) \ge t^2$ for $t \in [0,1]$.

Lemma 2.1. If $u \in C^4[0,1]$ satisfies the boundary conditions (1.2), and such that

$$u''''(t) \ge 0 \quad \text{for} \quad 0 \le t \le 1, \tag{2.1}$$

then

$$u'''(t) \le 0, \quad u''(t) \ge 0, \quad u'(t) \ge 0, \quad u(t) \ge 0 \quad \text{for} \quad t \in [0,1]. \tag{2.2}$$

The proof of Lemma 2.1 is very straightforward and therefore omitted.

Lemma 2.2. If $u \in C^4[0,1]$ satisfies (1.2) and (2.1), then

$$a(t)u(1) \le u(t) \le tu(1) \quad \text{for} \quad t \in [0,1]. \tag{2.3}$$

Proof. If $u(1) = 0$, then because $u'(t) \ge 0$ for $t \in [0,1]$ and $u(0) = 0$, we have $u(t) \equiv 0$. And it is easy to see that (2.3) is true in this case.

Now let us prove (2.3) when $u(1) > 0$. Without loss of generality, we assume that $u(1) = 1$. If we define

$$h(t) = u(t) - a(t)u(1) = u(t) - \frac{3}{2}t^2 + \frac{t^3}{2}, \quad 0 \le t \le 1,$$

then we have

$$h'(t) = u'(t) - 3t + \frac{3}{2}t^2, \quad h''(t) = u''(t) - 3 + 3t,$$

$$h'''(t) = u'''(t) + 3,$$

$$h''''(t) = u''''(t) \geq 0 \quad \text{for} \quad t \in [0,1]. \tag{2.4}$$

It's easy to see, from the above equations and (1.2), that

$$h(0) = h(1) = h'(0) = h''(1) = 0.$$

By mean value theorem, because $h(0) = h(1) = 0$, there exists $r_1 \in (0,1)$ such that $h'(r_1) = 0$. Since $h'(0) = h'(r_1) = 0$, there exists $r_2 \in (0, r_1)$ such that $h''(r_2) = 0$. From (2.4) we see that $h''(t)$ is concave upward on the interval $(0,1)$. Since $h''(r_2) = h''(1) = 0$, we have

$$h''(t) \geq 0 \ \text{ for } \ t \in (0, r_2) \quad \text{and} \quad h''(t) \leq 0 \ \text{ for } \ t \in (r_2, 1).$$

So h' is nondecreasing on the interval $(0, r_2)$, and is nonincreasing on the intervals (r_2, r_1) and $(r_1, 1)$. Because $h'(0) = h'(r_1) = 0$, we have

$$h'(t) \geq 0 \ \text{ for } \ t \in (0, r_1) \quad \text{and} \quad h'(t) \leq 0 \ \text{ for } \ t \in (r_1, 1).$$

These, together with the fact that $h(0) = h(1) = 0$, imply that

$$h(t) \geq 0 \text{ on } (0,1).$$

Thus we proved the left half of (2.3) when $u(1) > 0$.

To prove the right half of (2.3) when $u(1) > 0$, we assume $u(1) = 1$, and define

$$y(t) = t - u(t), \quad 0 \leq t \leq 1,$$

then

$$y'(t) = 1 - u'(t), \quad y''(t) = -u''(t) \leq 0 \ \text{ for } \ t \in (0,1). \tag{2.5}$$

It's easy to see that

$$y(0) = y(1) = 0. \tag{2.6}$$

From (2.5) we see that $y(t)$ is concave downward. So (2.6) implies that $y(t) \geq 0$ for $t \in [0,1]$. The proof is complete.

Lemma 2.3. If $u \in C^4[0,1]$ satisfies (1.2) and (2.1), and $u''''(t)$ is nondecreasing on $[0,1]$, then

$$a(t)u(1) \leq u(t) \leq b_1(t)u(1) \quad \text{for} \quad t \in [0,1]. \tag{2.7}$$

Proof. The left half of (2.7) is already proved in Lemma 2.1. Therefore we need only to prove the right half of (2.7). Without loss of generality, we assume that $u(1) = 1$. If we define

$$y(t) = b_1(t) - u(t) = 2t^2 - \frac{4t^3}{3} + \frac{t^4}{3} - u(t), \quad 0 \le t \le 1,$$

then we have

$$y'(t) = 4t - 4t^2 + \frac{4t^3}{3} - u'(t), \quad y''(t) = 4 - 8t + 4t^2 - u''(t),$$
$$y'''(t) = -8 + 8t - u'''(t),$$
$$y''''(t) = 8 - u''''(t), \quad \text{for} \quad t \in [0, 1]. \tag{2.8}$$

It's easy to see, from the above equations and (1.2), that

$$y(0) = y(1) = y'(0) = y''(1) = 0.$$

By mean value theorem, because $y(0) = y(1) = 0$, there exists $r_1 \in (0, 1)$ such that $y'(r_1) = 0$. Since $y'(0) = y'(r_1) = 0$, there exists $r_2 \in (0, r_1)$ such that $y''(r_2) = 0$. Since $y''(r_2) = y''(1) = 0$, there exists $r_3 \in (r_2, 1)$ such that $y'''(r_3) = 0$. It is easy to verify that $y'''(1) = 0$. Because $y'''(r_3) = y'''(1) = 0$, there exists $r_4 \in (r_3, 1)$ such that

$$y''''(r_4) = 0. \tag{2.9}$$

Note that $u''''(t)$ is nondecreasing on $[0, 1]$. From (2.8), we see that $y''''(t)$ is nonincreasing on $[0, 1]$. This, together with (2.9), imply that

$$y''''(t) \ge 0 \ \text{ for } \ t \in (0, r_4) \quad \text{and} \quad y''''(t) \le 0 \ \text{ for } \ t \in (r_4, 1).$$

Because $y'''(r_3) = y'''(1) = 0$, we have

$$y'''(t) \le 0 \ \text{ for } \ t \in (0, r_3) \quad \text{and} \quad y'''(t) \ge 0 \ \text{ for } \ t \in (r_3, 1).$$

Because $y''(r_2) = y''(1) = 0$, we have

$$y''(t) \ge 0 \ \text{ for } \ t \in (0, r_2) \quad \text{and} \quad y''(t) \le 0 \ \text{ for } \ t \in (r_2, 1).$$

Because $y'(0) = y'(r_1) = 0$, we have

$$y'(t) \geq 0 \quad \text{for} \quad t \in (0, r_1) \quad \text{and} \quad y'(t) \leq 0 \quad \text{for} \quad t \in (r_1, 1).$$

These, together with the fact that $y(0) = y(1) = 0$ imply that

$$y(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq 1.$$

The proof is complete.

Lemma 2.4. Suppose that (H1) and (H2) hold. If $u(t)$ is a nonnegative solution to the problem (1.1)-(1.2), then $u(t)$ satisfies (2.2) and (2.3).

Proof. If $u(t)$ is a nonnegative solution to the problem (1.1)-(1.2), then $u(t)$ satisfies the boundary conditions (1.2), and

$$u''''(t) = g(t)f(u(t)) \geq 0, \quad 0 \leq t \leq 1.$$

Now Lemma 2.4 follows directly from Lemmas 2.1 and 2.2. The proof is complete.

Lemma 2.5. Suppose that (H1), (H2), and the following condition hold.

(H3) Both f and g are nondecreasing functions.

If $u(t)$ is a nonnegative solution to the problem (1.1)-(1.2), then $u(t)$ satisfies (2.2) and (2.7).

Proof. From Lemma 2.1 we see that $u(t)$ satisfies (2.2), therefore $u(t)$ is nondecreasing. From (H3), we see that $u''''(t) = g(t)f(u(t)) \geq 0$, and $u''''(t)$ is nondecreasing on the interval $[0, 1]$. Now from Lemma 2.3 we see that $u(t)$ satisfies (2.7). The proof is complete.

3. Main Results

First, we define some important constants:

$$A = \int_0^1 G(1, s)g(s)a(s)\, ds, \quad B = \int_0^1 G(1, s)g(s)s\, ds,$$

$$F_0 = \limsup_{x \to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x \to 0^+} \frac{f(x)}{x},$$

$$F_\infty = \limsup_{x\to+\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x\to+\infty} \frac{f(x)}{x}.$$

Throughout the rest of the paper, we let $X = C[0,1]$ be with norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \ v \in X,$$

and let

$$P = \{v \in X \mid v(1) \geq 0, \ a(t)v(1) \leq v(t) \leq tv(1) \ \text{ for } \ t \in [0,1]\}.$$

Clearly X is a Banach space, and P is a positive cone in X. We can restate Lemma 2.4 as follows.

Lemma 3.1. If $u(t)$ is a nonnegative solution to the problem (1.1)-(1.2), then $u \in P$.

The next lemma shows that if $u \in P$, then $u(t)$ achieves its maximum at $t = 1$.

Lemma 3.2. If $u \in P$, then $u(1) = \|u\|$.

Proof. If $u \in P$, then we have

$$0 \leq u(t) \leq tu(1) \leq u(1), \quad \text{for each} \quad t \in [0,1].$$

The proof is complete.

Define the operator $T : P \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \leq t \leq 1. \tag{3.1}$$

It is well known that $T : P \to X$ is a completely continuous operator.

Lemma 3.3. $T(P) \subset P$.

Proof. If $u \in P$, then (3.1) implies that $Tu(t)$ satisfies the boundary conditions (1.2), and

$$(Tu)''''(t) = g(t)f(u(t)) \geq 0, \quad 0 \leq t \leq 1.$$

Now the lemma follows immediately from Lemma 2.2. The proof is complete.

It is clear that the integral equation (1.3) is equivalent to the equality

$$Tu = u, \quad u \in P.$$

In order to find a positive solution to the problem (1.1)-(1.2), we need only to find a fixed point u of T such that $u \in P$ and $u(1) = \|u\| > 0$.

Now we are ready to prove the main results on the existence of at least one positive solution to the problem (1.1)-(1.2).

Theorem 3.4. Suppose that (H1) and (H2) hold. If $BF_0 < 1 < Af_\infty$, then the problem (1.1)-(1.2) has at least one positive solution.

Proof. First, we choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \leq 1$. From the definition of F_0 we see that there exists $H_1 > 0$ such that

$$f(x) \leq (F_0 + \varepsilon)x \ \text{ for } \ 0 < x \leq H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\
&\leq \int_0^1 G(1,s)g(s)(F_0 + \varepsilon)u(s)\,ds \\
&\leq (F_0 + \varepsilon)\|u\| \int_0^1 G(1,s)g(s)s\,ds \\
&= (F_0 + \varepsilon)\|u\|B \\
&\leq \|u\|,
\end{aligned}$$

which means $\|Tu\| \leq \|u\|$. Thus, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$\|Tu\| \leq \|u\| \ \text{ for } \ u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\delta > 0$ and $c \in (0, 1/4)$ such that

$$\int_c^1 G(1,s)g(s)a(s)\,ds \cdot (f_\infty - \delta) \geq 1.$$

There exists $H_3 > 0$ such that

$$f(x) \geq (f_\infty - \delta)x \text{ for } x \geq H_3.$$

Let $H_2 = \max\{H_3 c^{-2}, 2H_1\}$. If $u \in P$ such that $\|u\| = H_2$, then for each $t \in [c, 1]$, we have

$$u(t) \geq H_2 a(t) \geq H_2 t^2 \geq H_2 c^2 \geq H_3.$$

Therefore, for each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned} (Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\geq \int_c^1 G(1,s)g(s)f(u(s))\,ds \\ &\geq \int_c^1 G(1,s)g(s)(f_\infty - \delta)u(s)\,ds \\ &\geq \int_c^1 G(1,s)g(s)a(s)\,ds \cdot (f_\infty - \delta)\|u\| \\ &\geq \|u\|, \end{aligned}$$

which means $\|Tu\| \geq \|u\|$. Thus, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$\|Tu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_2.$$

Now that the condition (K1) of Theorem K is satisfied, there exists a fixed point of T in $P \cap (\overline{\Omega_2} - \Omega_1)$. The proof is now complete.

Theorem 3.5. Suppose that (H1) and (H2) hold. If $BF_\infty < 1 < Af_0$, then the problem (1.1)-(1.2) has at least one positive solution.

Proof. First, we choose $\varepsilon > 0$ such that $(f_0 - \varepsilon)A \geq 1$. There exists $H_1 > 0$ such that

$$f(x) \geq (f_0 - \varepsilon)x \quad \text{for} \quad 0 < x \leq H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\
&\geq \int_0^1 G(1,s)g(s)u(s)\,ds \cdot (f_0 - \varepsilon) \\
&\geq \int_0^1 G(1,s)g(s)a(s)\,ds \cdot (f_0 - \varepsilon)\|u\| \\
&= A(f_0 - \varepsilon)\|u\| \\
&\geq \|u\|,
\end{aligned}$$

which means $\|Tu\| \geq \|u\|$. Thus, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$\|Tu\| \geq \|u\| \quad \text{for} \quad u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\delta \in (0,1)$ such that $(F_\infty + \delta)B < 1$. There exists $H_3 > 0$ such that

$$f(x) \leq (F_\infty + \delta)x \quad \text{for} \quad x \geq H_3.$$

If we let $M = \max_{0 \leq x \leq H_3} f(x)$, then

$$f(x) \leq M + (F_\infty + \delta)x \quad \text{for} \quad x \geq 0.$$

Let

$$K = M \int_0^1 G(1,s)g(s)\,ds,$$

and let

$$H_2 = \max\{2H_1, K(1 - (F_\infty + \delta)B)^{-1}\}. \tag{3.2}$$

Note that (3.2) implies that

$$K + (F_\infty + \delta)BH_2 \leq H_2.$$

For each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\
&\leq \int_0^1 G(1,s)g(s)(M + (F_\infty + \delta)u(s))\,ds \\
&\leq K + (F_\infty + \delta)\int_0^1 G(1,s)g(s)u(s)\,ds \\
&\leq K + (F_\infty + \delta)H_2\int_0^1 G(1,s)g(s)s\,ds \\
&= K + (F_\infty + \delta)H_2 B \\
&\leq H_2,
\end{aligned}$$

which means $\|Tu\| \leq \|u\|$. Thus, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$\|Tu\| \leq \|u\| \;\text{ for }\; u \in P \cap \partial\Omega_2.$$

Hence, the condition (K2) of Theorem K is satisfied, so T has at least one fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$, which implies that the problem (1.1)-(1.2) has at least one positive solution. The proof is complete.

The next two lemmas provide sufficient conditions for the nonexsitence of positive solutions to the problem (1.1)-(1.2).

Theorem 3.6. If (H1), (H2), and the following condition hold.

(H4) $Bf(x) < x$ for all $x \in (0, +\infty)$.

Then the problem (1.1)-(1.2) has no positive solutions.

Proof. Assume the contrary that $u(t)$ is a positive solution of the problem (1.1)-(1.2). Then $u \in P$, $u(t) > 0$ for $0 < t \leq 1$, and

$$\begin{aligned} u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\leq B^{-1}\int_0^1 G(1,s)g(s)s\,ds \cdot u(1) \\ &= B^{-1}Bu(1) \\ &= u(1), \end{aligned}$$

which is a contradiction. The proof is complete.

Theorem 3.7. If (H1), (H2), and the following condition hold.

(H5) $Af(x) > x$ for all $x \in (0, +\infty)$.

Then the problem (1.1)-(1.2) has no positive solutions.

The proof of Theorem 3.7 is quite similar to that of Theorem 3.6 and therefore omitted.

4. More Results

Throughout this section, we assume that (H1), (H2), and (H3) hold, and we define the Banach space X, the constants A, F_0, F_∞, f_0 and f_∞, and the operator T the same way as in Section 3. In this section, we define a new constant B_1 by

$$B_1 = \int_0^1 G(1,s)g(s)b_1(s)\,ds,$$

and the positive cone P_1 of X by

$$P_1 = \left\{ v \in X \;\middle|\; \begin{array}{c} v(1) \geq 0,\ v(t) \text{ is nondecreasing on } [0,1], \\ a(t)v(1) \leq v(t) \leq b_1(t)v(1) \text{ on } [0,1]. \end{array} \right\}.$$

We can restate Lemma 2.5 as follows.

Lemma 4.1. If $u(t)$ is a nonnegative solution to the problem (1.1)-(1.2), then $u \in P_1$.

Lemma 4.2. If $u \in P_1$, then $u(1) = \|u\|$.

The proof is similar to that of Lemma 3.2 and therefore omitted.

Lemma 4.3. If (H1), (H2), and (H3) hold, then $T(P_1) \subset P_1$.

Proof. If $u \in P_1$, then $u(t)$ is nondecreasing. Now (3.1) implies that $Tu(t)$ satisfies the boundary conditions (1.2), and

$$(Tu)''''(t) = g(t)f(u(t)) \geq 0, \quad 0 \leq t \leq 1.$$

From (H3) we see that $(Tu)''''(t)$ is nondecreasing. Now the lemma follows immediately from Lemma 2.3.

Theorem 4.4. Suppose that (H1), (H2), and (H3) hold. If $B_1F_0 < 1 < Af_\infty$, then the problem (1.1)-(1.2) has at least one positive solution.

Theorem 4.5. Suppose that (H1), (H2), and (H3) hold. If $B_1F_\infty < 1 < Af_0$, then the problem (1.1)-(1.2) has at least one positive solution.

The proofs of Theorems 4.4 and 4.5 are very similar to those of Theorems 3.4 and 3.5. The only difference is that we use the positive cone P_1, instead of P, in the proofs of Theorems 4.4 and 4.5. The proofs of Theorems 4.4 and 4.5 are omitted.

Theorem 4.6. If (H1), (H2), (H3) and the following condition hold.

(H6) $B_1f(x) < x$ for all $x \in (0, +\infty)$.

Then the problem (1.1)-(1.2) has no positive solutions.

The proof of Theorem 4.6 is quite similar to that of Theorem 3.6 and therefore omitted.

5. Example

Example 5.1. Consider the problem

$$u''''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{5.1}$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{5.2}$$

where

$$g(t) = 1 + 3t, \quad 0 \le t \le 1, \tag{5.3}$$

$$f(u) = \frac{\lambda u(1+2u)}{1+u}, \quad u \ge 0, \tag{5.4}$$

and $\lambda > 0$ is a parameter. It is easy to see that $f_0 = F_0 = \lambda$, $f_\infty = F_\infty = 2\lambda$, and

$$\lambda u < f(u) < 2\lambda u, \quad \text{for} \quad u > 0.$$

It is also easy to verify, by direct calculation, that

$$A = \frac{303}{1120} \quad \text{and} \quad B = \frac{37}{120}.$$

From Theorem 3.4 we see that if

$$1.848 \approx \frac{1}{2A} < \lambda < \frac{1}{B} \approx 3.243,$$

then the problem (5.1)-(5.2) has at least one positive solution. From Theorem 3.6 we see that if

$$\lambda \le \frac{1}{2B} \approx 1.622,$$

then the problem (5.1)-(5.2) has no positive solutions. From Theorem 3.7 we see that if

$$\lambda \ge \frac{1}{A} \approx 3.696,$$

then the problem (5.1)-(5.2) has no positive solutions.

Note that the functions g and f given by (5.3) and (5.4) are increasing functions. Therefore Theorems 4.4 and 4.6 apply. It is easy to verify that

$$B_1 = \frac{1061}{3780}.$$

From Theorem 4.4 we see that if

$$1.848 \approx \frac{1}{2A} < \lambda < \frac{1}{B_1} \approx 3.5627,$$

then the problem (5.1)-(5.2) has at least one positive solution. From Theorem 4.6 we see that if

$$\lambda \leq \frac{1}{2B_1} \approx 1.7813,$$

then the problem (5.1)-(5.2) has no positive solutions.

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