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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOUR FOR A DEGENERATE DIFFUSIVE SEIR MODEL

TARIK ALIZIANE AND MICHEL LANGLAIS

Abstract. In this paper we analyze the global existence and asymptotic behavior of a reaction diffusion system with degenerate diffusion arising in modeling the spatial spread of an epidemic disease.

1. Introduction

In this paper we shall be concerned with a degenerate parabolic system of the form

$$\left\{\begin{array}{lll} \partial_t U_1 - \Delta U_1^{m_1} = -\gamma(U_1,U_2,U_3,U_4) - \nu U_1 & = f_1(U_1,U_2,U_3,U_4), \\ \partial_t U_2 - \Delta U_2^{m_2} = \gamma(U_1,U_2,U_3,U_4) - (\lambda+\mu)U_2 & = f_2(U_1,U_2,U_3,U_4), \\ \partial_t U_3 - \Delta U_3^{m_3} = \lambda\pi U_2 - (\alpha+m+\mu)U_3 & = f_3(U_1,U_2,U_3,U_4), \\ \partial_t U_4 - \Delta U_4^{m_4} = (1-\pi)\lambda U_2 + \alpha U_3 + \nu U_1 & = f_4(U_1,U_2,U_3,U_4). \end{array}\right. \tag{1.1}$$

in $\Omega \times (0,+\infty)$, subject to the initial conditions

$$U_i(x,0) = U_{i,0}(x) \geq 0, \quad x \in \Omega; \qquad i = 1..4. \tag{1.2}$$

and to the Neumann boundary conditions

$$\frac{\partial U_i^{m_i}}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \qquad i = 1..4. \tag{1.3}$$

Herein, Ω is an open, bounded and connected domain in $I\!R^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$; Δ is the Laplace operator in $I\!R^N$. Powers m_i verify $m_i > 1$, $i = 1..4$.

In the spatially homogeneous case and for $\nu = \mu = \alpha = m = 0$ and $\pi = 1$ this problem reduces to one of the models of propagation of an epidemic disease devised in Kermack and McKendricks [21], namely

$$\left\{\begin{array}{ll} S' & = -\gamma SI, \\ I' & = +\gamma SI - \lambda I, \\ R' & = +\lambda I. \end{array}\right.$$

In that setting it is known, *loc. cit.*, that $I(t) \to 0$ as $t \to +\infty$, while the large time behavior of $S(t)$ and $R(t)$ depends on the initial state (S_0, I_0, R_0); note that for $t > 0, S(t) + I(t) + R(t) = S_0 + I_0 + R_0$.

This basic model served as a starting point for many further developments, both from epidemiological or mathematical point of vue : see the books of Busenberg and Cooke [7] or Capasso [8] and their references. These lead to so-called $(S-E-I-R)$ models : S is the distribution of susceptible individuals in a given population, $\gamma(S,E,I,R)$ is the incidence

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term or number of susceptible individuals infected by contact with an infective individual I per time unit and becoming exposed E, while R is the density of removed or resistant (immune) individuals. Then λ (resp. α) is the inverse of the duration of the exposed stage (resp. infective stage) or rate at which exposed individuals enter the infective class (resp. infective individuals who do not die from the disease recover), m is the death-rate induced by the disease. The last two parameters are control parameters : first ν is a vaccination rate; next, for a population of animals, as it is considered here as in Anderson *et al* [5], Fromont *et al* [17], Courchamp *et al* [10] or Langlais and Suppo [23], μ is an elimination rate of exposed and infective individuals. Lastly, as it is suggested by the FeLV, a retrovirus of domestic cats (*Felis catus*) see [17], one also introduces a parameter π measuring the proportion of exposed individuals which actually develop the disease after the exposed stage, the remaining proportion $1-\pi$ becoming resistant.

The nonlinear incidence term γ takes various forms as it can be found from the literature; at least two of them are widely used in applications

$$\gamma(S,E,I,R) = \begin{cases} \gamma SI, & [5,\ 8,\ 21], \quad \begin{array}{l}\text{mass action in } [7,\ 8]\ ,\\ \text{or pseudo-mass action in } [20,\ 12]\ .\end{array} \\ \gamma \dfrac{SI}{S+E+I+R}, & [10,\ 17,\ 23], \quad \begin{array}{l}\text{proportionate mixing in } [7]\\ \text{or true mass action in } [20,\ 12]\ .\end{array} \end{cases}$$

We refer to De Jong *et al*, [20] and Diekmann *et al* [12] for a discussion supporting the second one in populations of varying size and Fromont *et al* [18] for a specific discussion in the case of a cat population. See Capasso and Serio [9] and Capasso [8] for more general incidence terms. Note that no demographical effect is considered in our model.

A mathematical analysis of the model of Kermack and McKendricks for spatially structured populations with linear diffusion, i.e. $m_i = 1,\ i = 1..4$, is performed in Webb [27]. Nonlinear but nondegenerate diffusion terms are introduced in Fitzgibbon *et al* [16]. Global existence and large time behavior results are derived therein. Homogeneous Neumann boundary conditions correspond to isolated populations.

A comprehensive analysis of generic $(S-E-I-R)$ models with linear diffusion is initiated in Fitzgibbon and Langlais [14] and Fitzgibbon *et al* [15]. These models include a logistic effect on the demography, yielding $L^1(\Omega)$ a priori estimates on solutions independent of the initial data for large time; this allows to use a bootstrapping argument to show global existence and exhibit a global attractor in $(C(\overline{\Omega}))^4$.

For degenerate reaction-diffusion equations, a similar approach is followed in Le Dung [13]. In our case, $L^1(\Omega)$ a priori estimates can be established for nonegative solutions upon integrating over $\Omega \times (0,t)$

$$\sum_{i=1}^{4} \int_\Omega U_i(x,t)dx \le \sum_{i=1}^{4} \int_\Omega U_{i,0}(x)dx \quad \text{for all } t>0,$$

but they cannot be found to be independent of the initial data. Moreover, generally speaking, the large time behavior of solutions depends on these initial datas, as it can be already seen for spatially homogeneous problems see §§5.3. This can also be checked on the disease free model: assuming $U_{i,0}(x) \equiv 0$ in Ω $i = 2..4$, the uniqueness result given in Theorem 1 implies $U_i(x,t) \equiv 0$ in $\Omega \times (0,+\infty)$ $i = 2..4$. Then, it should be clear that $\gamma(U_1, 0, 0, 0) = 0$ for any reasonable incidence term so that the equation for U_1 reads

$$\partial_t U_1 - \Delta U_1^{m_1} + \nu U_1 = 0 \ \text{ in } \ \Omega \times (0,+\infty); \tag{1.4}$$

this is the so-called porous medium equation. Now U_1 verifies homogeneous Neumann boundary conditions and it is well-known (see Alikakos [1]) that as $t \longrightarrow +\infty$

$$\begin{cases} U_1(.,t) \longrightarrow 0 & \text{if } \ \nu > 0, \\ U_1(.,t) \longrightarrow \dfrac{1}{mes(\Omega)} \displaystyle\int_\Omega U_{1,0}(x)dx & \text{if } \ \nu = 0. \end{cases}$$

The case of mass action incidence was studied by Aliziane and Moulay [4] and they established the long time behavior of the solution of the SIS model, Aliziane and Langlais [3] study the case of models include a logistic effect on the demography and they established global existence result of the solution and existence of periodic solution. We also obtain the existence of the global attractor. Finally Hadjadj *et al* [19] study the case where the source term depends on gradient of solution, they study existence of globally bounded weak solutions or blow-up, depending on the relations between the parameters that appear in the problem.

2. Main results

2.1. **Basic assumptions and notations.** Herein, Ω is an open, bounded and connected domain of the N-dimensional Euclidian space $I\!R^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, a $(N-1)$-dimensional manifold so that locally Ω lies on one side of $\partial\Omega$; $x = (x_1, ..., x_N)$ is the generic element of $I\!R^N$. Next we shall denote the gradient with respect to x by ∇ and the Laplace operator in $I\!R^N$ by Δ.
Then we set $\Omega \times (0,T) = Q_T$ and for $0 \leq \tau < T$, $\Omega \times (\tau, T) = Q_{\tau,T}$. The norm in $L^p(\Omega)$ is $\| \ \|_{p,\Omega}$ and the norm in $L^p(Q_{\tau,T})$ is $\| \ \|_{p,Q_{\tau,T}}$ for $1 \leq p \leq +\infty$.

Next we shall assume throughout this paper

- (H0) Powers m_i verify $m_i > 1$, $i = 1..4$.
- (H1) $\mu, \alpha, \nu, m, \lambda, \pi$ are nonnegative constants, $\lambda + \mu > 0$, $\alpha + m + \mu > 0$ and $0 \leq \pi \leq 1$.
- (H2) $U_{i,0} \in C(\bar{\Omega})$, $\quad U_{i,0}(x) \geq 0$, $x \in \Omega$, $\qquad i = 1..4$.
- (H3) $\gamma : I\!R_+^4 \longrightarrow I\!R_+$ is a locally lipschitz continuous function with polynomial growth and $\gamma(0, U_2, U_3, U_4) = 0$ on $I\!R_+^3$.
- (H4) There exists nonnegative constants C_1, C_2 and r such that $\gamma(U_1, U_2, U_3, U_4) \leq C_1 + C_2 U_1^r$ on $I\!R_+^4$.

Remark 1. *In the limiting case $\lambda + \mu = 0$ the equations for U_3 and U_4 do not depend on U_2, the equation for U_3 being a porous medium type equation as in* (1.4). *This condition also implies $\lambda = 0$ which is not relevant if one goes back to our motivating problem.*

In the limiting case $\alpha + m + \mu = 0$ one could not get $L^\infty(Q_{0,\infty})$ bounds for U_3, but one still has global existence.

The assumption $\gamma(0, U_2, U_3, U_4) = 0$ is required to make sure that the nonnegative orthant $\mathbb{R}^4_+$ is forward invariant by (1.1); this is a natural assumption for our motivating problem : no new exposed individuals when there is no susceptible ones.
$(H4)$ removes mass action incidence terms; in that case one can also get global existence results, but no $L^\infty(Q_{0,\infty})$ bounds for U_2 and U_3.

2.2. **Main results.** System (1.1) is degenerate : when $U_i = 0$ the equation for U_i degenerates into first order equation. Hence classical solutions cannot be expected for Problem $(1.1) - (1.3)$. A suitable notion of generalized solutions is required : we adopt the notion of weak solution introduced in Oleinik *et al* [25].

Definition 1. *A quadruple (U_1, U_2, U_3, U_4) of nonnegative and continuous functions $U_i : \Omega \times [0, +\infty) \to [0, +\infty)$, $i = 1..4$, is a weak solution of Problem $(1.1) - (1.3)$ in Q_T, $T > 0$ if for each $i = 1..4$ and for each $\varphi_i \in C^1(\bar{Q}_T)$, such that $\dfrac{\partial \varphi_i}{\partial \eta} = 0$ on $\partial\Omega \times (0, T)$.*

(i) *∇U^{m_i} exists in the sense of distribution and $\nabla U_i^{m_i} \in L^2(Q_T)$;*
(ii) *U_i verifies the identity*

$$\begin{aligned} &\int_\Omega U_i(x,T)\varphi_i(x,T)dx + \int_{Q_T} \nabla U_i^{m_i} \nabla \varphi_i(x,t)dxdt \\ &= \int_{Q_T} (\partial_t \varphi_i U_i - f_i \varphi_i)(x,t)dxdt + \int_\Omega U_{i,0}(x)\varphi_i(x,0)dx. \end{aligned} \tag{2.1}$$

We are now ready to state our first result.

Theorem 1. *For each quadruple of continuous nonnegative initial functions $(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0})$ there exists a unique weak solution (U_1, U_2, U_3, U_4) of Problem $(1.1) - (1.3)$ on Q_∞; furthermore*

(i) *For all $i = 1..3$, $U_i \in L^1 \cap L^\infty(Q_\infty)$ and $\nabla U_i^{m_i}$, $\partial_t U_i^{m_i} \in L^2(Q_{\tau,\infty})$, $\tau > 0$;*
(ii) *$U_4 \in L^1 \cap L^\infty(Q_T)$ and $\nabla U_4^{m_4}$, $\partial_t U_4^{m_4} \in L^2(Q_{\tau,T})$, $\tau > 0$.*

The proof is found in Section §4.
Now we look at the large time behavior of weak solutions.

Theorem 2. *There exist nonnegative constants U_1^*, U_4^* such that*

$$\begin{aligned} U_2(.,t), U_3(.,t) &\longrightarrow 0 \ , \ U_1(.,t) \longrightarrow U_1^* \ \text{ in } C(\bar{\Omega}) \text{ as } t \longrightarrow +\infty \\ \text{and } \overline{U_4}(t) &\longrightarrow U_4^* \ \text{ in } L^p(\Omega) \text{ forall } p \geq 1 \text{ as } t \longrightarrow +\infty; . \end{aligned}$$

moreover, if $\nu > 0$ then $U_1^* = 0$.

The proof is found in Section §5.

Remark 2. *In the non degenerate case $m_4 = 1$ one has that $U_4(.,t) \longrightarrow U_4^*$ in $C(\bar{\Omega})$. More generally, this still holds provided that U_4 lies in $L^\infty(Q_\infty)$, the proof being similar to the one for U_1 when $\nu = 0$, see subsection §§5.2.*

3. Auxiliary problem and a priori estimates

In this section we consider an auxiliary problem depending on a small parameter ε, with $0 < \varepsilon \le 1$. Namely let us introduce in $\Omega \times (0, +\infty)$ the quasilinear nondegenerate initial and boundary value problem

$$
\left\{
\begin{array}{l}
\partial_t U_1 - \Delta d_1(U_1) = -\gamma((U_1 - \varepsilon)^+, U_2, U_3, U_4) - \nu(U_1 - \varepsilon), \\
\partial_t U_2 - \Delta d_2(U_2) = \gamma((U_1 - \varepsilon)^+, U_2, U_3, U_4) - (\lambda + \mu)(U_2 - \varepsilon), \\
\partial_t U_3 - \Delta d_3(U_3) = \lambda\pi(U_2 - \varepsilon) - (\alpha + m + \mu)(U_3 - \varepsilon), \\
\partial_t U_4 - \Delta d_3(U_4) = (1 - \pi)\lambda(U_2 - \varepsilon) + \alpha(U_3 - \varepsilon) + \nu(U_1 - \varepsilon).
\end{array}
\right.
\tag{3.1}
$$

$$
\left\{
\begin{array}{l}
U_{i,\varepsilon}(x, 0) = U_{i,0,\varepsilon}(x) \ge 0, \quad x \in \Omega; \\
\dfrac{\partial d_i(U_{i,\varepsilon})}{\partial \eta}(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0,
\end{array}
\right.
\qquad i = 1..4.
\tag{3.2}
$$

Herein $(r)^+$ is the nonnegative part of the real number r; for each $i = 1..4$ $d_i : I\!R \longrightarrow (\frac{\varepsilon}{2}, +\infty)$ is a smooth and increasing functions with

$$
d_i(u) = u^{m_i}, \quad \varepsilon \le u;
\tag{3.3}
$$

$(U_{i,0,\varepsilon})_{i=1..4}$ is a quadruple of smooth functions over $\bar{\Omega}$ such that

$$
\left\{
\begin{array}{l}
U_{i,0,\varepsilon}(x) \ge \varepsilon, \quad x \in \Omega, \quad 0 < \varepsilon \le 1; \\
\displaystyle\int_\Omega (U_{i,0,\varepsilon}(x) - \varepsilon)dx = \int_\Omega U_{i,0}(x)dx \\
U_{i,0,\varepsilon} \longrightarrow U_{i,0} \ \text{ in } C(\bar{\Omega}), \ \text{ as } \ \varepsilon \longrightarrow 0;
\end{array}
\right.
\qquad i = 1..4;
\tag{3.4}
$$

we refer to [2, 19] for a construction of such a set of initial data. From standard results, i.e. [22] or [26], local existence and uniqueness of a quadruple $(U_{1,\varepsilon}, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})$, a classical solution of $(3.1) - (3.2)$ in some maximal interval $[0, T_{max,\varepsilon})$ is granted.

Looking at the equation for $U_{i,\varepsilon}$ it is checked that ε is a subsolution, thus $0 < \varepsilon \le U_{i,\varepsilon}(x, t)$, $x \in \Omega$, $0 < t < T_{max,\varepsilon}$. Next, from the maximum principle and the nonnegativity of γ, ν and $U_{1,\varepsilon} - \varepsilon$, it follows $U_{1,\varepsilon}(x, t) \le \|U_{1,\varepsilon,0}\|_{\infty,\Omega}$, $x \in \Omega$, $0 < t < T_{max,\varepsilon}$. As a consequence one has

$$
\left\{
\begin{array}{l}
0 < \varepsilon \le U_{1,\varepsilon}(x, t) \le \|U_{1,\varepsilon,0}\|_{\infty,\Omega}, \quad x \in \Omega, \quad t < T_{max,\varepsilon} \\
0 < \varepsilon \le U_{i,\varepsilon}(x, t), \quad x \in \Omega, \quad t < T_{max,\varepsilon}, \quad i = 2..4
\end{array}
\right.
\tag{3.5}
$$

Then one can apply results in [16] to show global existence, i.e. $T_{max,\varepsilon} = +\infty$, of a classical solution for $(3.1) - (3.2)$. Using (3.3) and (3.5) this yields global existence for the initial and boundary value problem

$$(3.6)\qquad \begin{cases} \partial_t U_1 - \Delta U_1^{m_1} = -\gamma(U_1-\varepsilon, U_2, U_3, U_4) - \nu(U_1-\varepsilon), \\ \partial_t U_2 - \Delta U_2^{m_2} = \gamma(U_1-\varepsilon, U_2, U_3, U_4) - (\lambda+\mu)(U_2-\varepsilon), \\ \partial_t U_3 - \Delta U_3^{m_3} = \lambda\pi(U_2-\varepsilon) - (\alpha+m+\mu)(U_3-\varepsilon), \\ \partial_t U_4 - \Delta U_4^{m_4} = (1-\pi)\lambda(U_2-\varepsilon) + \alpha(U_3-\varepsilon) + \nu(U_1-\varepsilon). \end{cases}$$

in $\Omega \times (0,+\infty)$, together with (3.2).

We derive a priori estimates. First, using the L^1 property of $U_{1,0,\varepsilon}$ in (3.4) and the nonnegativity of $U_{1,\varepsilon}-\varepsilon$, a straightforward integration of the equation for $U_{1,\varepsilon}$ over $\Omega \times (0,+\infty)$ gives:

$$(3.7)\qquad \int_{Q_T} (\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}) + \nu(U_{1,\varepsilon}-\varepsilon))(x,t)dxdt \le \int_\Omega U_{1,0}(x)dx.$$

In what follows T is a positive number, $M_1,..,M_n$ are positive constants independent of T and $\varepsilon, 0<\varepsilon\le 1$, and $F_1,..,F_n$ are non decreasing functions of T independent of $\varepsilon, 0<\varepsilon\le 1$.

Lemma 1. *There exists a constant M_1 and nondecreasing function F_1, independent of $\varepsilon, 0 < \varepsilon \le 1$ such that*

$$(3.8)\qquad 0<\varepsilon\le U_{i,\varepsilon}(x,t)\le M_1,\ x\in\Omega, t>0, i=1..3;$$

$$(3.9)\qquad \varepsilon \le U_{4,\varepsilon}(x,t) \le F_1(T), \quad x\in\Omega, \quad 0<t<T.$$

Proof. The estimate for $U_{1,\varepsilon}$ follows from (3.5) and the choice of $(U_{1,0,\varepsilon})_{\varepsilon>0}$.
Multiplying the equation for $U_{2,\varepsilon}$ by $p\,(U_{2,\varepsilon}-\varepsilon)^{p-1},\ p\ge 1$, and integrating over Ω one has

$$\frac{d}{dt}\|U_{2,\varepsilon}(.,t)-\varepsilon\|_{p,\Omega}^p + p(\lambda+\mu)\,\|U_{2,\varepsilon}(.,t)-\varepsilon\|_{p,\Omega}^p \le p\int_\Omega \gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})(U_{2,\varepsilon}-\varepsilon)^{p-1}(x,t)dx;$$

keeping in mind $\lambda+\mu>0$ from $(H1)$, one gets from Young's inequality

$$(3.10)\qquad \frac{d}{dt}\|U_{2,\varepsilon}(.,t)-\varepsilon\|_{p,\Omega}^p \le \left(\frac{1}{\lambda+\mu}\right)^{p-1}\int_\Omega [\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})]^p(x,t)dx.$$

A further integration over $(0,T)$ leads to

$$\|U_{2,\varepsilon}(.,T)-\varepsilon\|_{p,\Omega}^p \le \|U_{2,0,\varepsilon}-\varepsilon\|_{p,\Omega}^p + \left(\frac{1}{\lambda+\mu}\right)^{p-1}\int_{Q_T} [\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})]^p(x,t)dxdt.$$

Using the already known L^∞ estimate for $U_{1,\varepsilon}$, assumption $(H4)$ and inequality (3.7) one arrives at : for each $T>0$

$$(3.11)\qquad \|U_{2,\varepsilon}(.,T)-\varepsilon\|_{p,\Omega}^p \le \|U_{2,0,\varepsilon}-\varepsilon\|_{p,\Omega}^p + \left(\frac{1}{\lambda+\mu}\right)^{p-1}(C_1+C_2M_1^r)^{p-1}\|U_{1,0}\|_{1,\Omega}.$$

To conclude, one observes that $U_{2,\varepsilon} - \varepsilon$ being continuous on $\bar{\Omega} \times [0, +\infty)$ it follows

$$\lim_{p \to +\infty} \|U_{2,\varepsilon}(.,t) - \varepsilon\|_{p,\Omega} = \|U_{2,\varepsilon}(.,t) - \varepsilon\|_{\infty,\Omega}.$$

Hence for some constant M_2 independent of $\varepsilon, 0 < \varepsilon \leq 1$, one gets

$$0 < \varepsilon \leq U_{2,\varepsilon}(x,t) \leq M_2, \quad x \in \Omega, \quad t > 0. \tag{3.12}$$

Now, integrating the equation for $U_{2,\varepsilon}$ over $\Omega \times [0,T)$, using the L^1 property of $U_{2,0,\varepsilon}$ in (3.4), the nonnegativity of $U_{2,\varepsilon} - \varepsilon$ and (3.7) one has for $0 < \varepsilon \leq 1$

$$(\lambda + \mu) \int_{Q_T} (U_{2,\varepsilon} - \varepsilon)(x,t) dx dt \leq \int_\Omega [U_{1,0,\varepsilon}(x) + U_{2,0,\varepsilon}(x)] dx. \tag{3.13}$$

The estimate for $U_{3,\varepsilon}$ follows from computations similar to the ones for $U_{2,\varepsilon}$ above, carried over the equation for $U_{3,\varepsilon}$ and getting help from (3.13) and from the positivity of $\alpha + m + \mu$.

Along the same lines, from the equation for $U_{3,\varepsilon}$ one gets for $0 < \varepsilon \leq 1$

$$(\alpha + m + \mu) \int_{Q_T} (U_{3,\varepsilon} - \varepsilon) dx dt \leq \int_\Omega [U_{1,0,\varepsilon}(x) + U_{2,0,\varepsilon}(x) + U_{3,0,\varepsilon}(x)] dx. \tag{3.14}$$

Hence, going back to the equation for $U_{4,\varepsilon}$ one can derive the a priori estimate upon multiplying it by $p\,(U_{2,\varepsilon} - \varepsilon)^{p-1}$, $p \geq 1$ and using (3.13) – (3.14).

$\square$

Lemma 2. *There exist constants $M_{i,3}$, $i = 1..3$ and a nondecreasing function F_2, independent of $\varepsilon, 0 < \varepsilon \leq 1$ such that*

$$\int_{Q_T} \|\nabla U_{i,\varepsilon}^{m_i}\|^2(x,t) dx dt \leq M_{i,3}, \quad T > 0, \quad i = 1..3; \tag{3.15}$$

$$\int_{Q_T} \|\nabla U_{4,\varepsilon}^{m_i}\|^2(x,t) dx dt \leq F_2(T), \quad T > 0. \tag{3.16}$$

Proof. The estimate for $\nabla U_{1,\varepsilon}^{m_1}$ is obtained upon multiplying the equation for $U_{1,\varepsilon}$ by $U_{1,\varepsilon}^{m_1}$, integrating over $\Omega \times (0,T)$ and using the nonnegativity of γ and $U_{1,\varepsilon} - \varepsilon$. One finds

$$M_{1,3} = \frac{1}{m_1 + 1} \int_\Omega U_{1,\varepsilon}^{m_1+1}(x,0) dx$$

Proceeding along the same lines for $U_{2,\varepsilon}$ one gets

$$\frac{1}{m_2+1} \int_\Omega U_{2,\varepsilon}^{m_2+1}(x,T) dx + \int_{Q_T} \|\nabla U_{2,\varepsilon}^{m_2}(x,t)\|^2 dx dt \leq$$
$$\frac{1}{m_2+1} \int_\Omega U_{2,\varepsilon}^{m_2+1}(x,0) dx + \int_{Q_T} \gamma(U_{1,\varepsilon} - \varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})[U_{2,\varepsilon}]^{m_2}(x,t) dx dt.$$

Using the properties of $U_{2,0,\varepsilon}$, the uniform estimate for $U_{2,\varepsilon}$ in Lemma 1 and the L^1 estimate for γ in (3.7) we obtain

$$\int_{Q_T} \|\nabla U_{2,\varepsilon}^{m_2}(x,t)\|^2 dxdt \le M_{2,3}, \quad T > 0.$$

A similar computation supplies the estimate for $U_{3,\varepsilon}$. The estimate for $U_{4,\varepsilon}$ then follows. $\square$

Lemma 3. *For all $t > 0$*

$$\begin{aligned}
&\|\nabla U_{1,\varepsilon}^{m_1}(.,t)\|_{2,\Omega}^2 \le \frac{2}{t(m_1+1)} \int_\Omega U_{1,0,\varepsilon}^{m_1+1}(x)dx \\
&+m_1^2\nu^2 \|U_{1,0}\|_{\infty,\Omega}^{m_1-1} \int_{Q_{\frac{t}{2},t}} (U_{1,\varepsilon}-\varepsilon)^2(x,s)dxds \\
&+m_1^2 \|U_{1,0}\|_{\infty,\Omega}^{m-1} (C_1 + C_2\|U_{1,0}\|_{\infty,\Omega}^r) \int_{Q_{\frac{t}{2},t}} \gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})(x,s)dxds
\end{aligned} \tag{3.17}$$

Proof. Let us multiply the equation for $U_{1,\varepsilon}$ by $\partial_t U_{1,\varepsilon}^{m_1}$ and integrate over $\Omega\times(\tau,t)$, $\frac{t}{2} \le \tau \le t$; then one finds
(3.18)

$$\begin{aligned}
&(\frac{2}{m_1+1})^2 \int_{Q_{\tau,t}} (\partial_t U_{1,\varepsilon}^{\frac{m_1+1}{2}})^2(x,s)dxds + \|\nabla U_{1,\varepsilon}^{m_1}(.,t)\|_{2,\Omega}^2 \\
&\quad \le \int_{Q_{\tau,t}} (-\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}) - \nu(U_{1,\varepsilon}-\varepsilon))\partial_t U_{1,\varepsilon}^{m_1}(x,s)dxds + \|\nabla U_{1,\varepsilon}^{m_1}(.,\tau)\|_{2,\Omega}^2.
\end{aligned}$$

Next, for any suitably smooth and nonnegative function U and any $m > 1$ one gets $\partial_t U^m = \dfrac{2m}{m+1} U^{\frac{m-1}{2}} \partial_t U^{\frac{m+1}{2}}$ so that

$$\begin{aligned}
&\int_{Q_{\tau,t}} (-\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}) - \nu(U_{1,\varepsilon}-\varepsilon))\partial_t U_{1,\varepsilon}^{m_1}(x,s)dxds \\
&\quad \le \frac{2}{(m_1+1)^2} \int_{Q_{\frac{t}{2},t}} (\partial_t U_{1,\varepsilon}^{(\frac{m_1+1}{2})})^2(x,s)dxds \\
&\quad +m_1^2\nu^2 \int_{Q_{\frac{t}{2},t}} [(U_{1,\varepsilon}-\varepsilon)U_{1,\varepsilon}^{\frac{m-1}{2}}]^2(x,s)dxds \\
&\quad +m_1^2 \int_{Q_{\frac{t}{2},t}} [\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})U_{1,epsilon}^{\frac{m-1}{2}}]^2(x,s)dxds.
\end{aligned} \tag{3.19}$$

The last term on the right hand side of this inequality is bounded from above by

$$m_1^2 \|U_{1,\varepsilon}\|_{\infty,Q_\infty}^{m-1} \|\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})\|_{\infty,Q_\infty} \int_{Q_{\frac{t}{2},t}} \gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})(x,s)dxds.$$

putting this estimate in (3.18) one obtains

$$\begin{aligned}(3.20)\qquad &\|\nabla U_{1,\varepsilon}^{m_1}(.,t)\|_{2,\Omega}^2 \le \|\nabla U_{1,\varepsilon}^{m_1}(.,\tau)\|_{2,\Omega}^2 + m_1^2\nu^2\|U_{1,0}\|_{\infty,\Omega}^{m_1-1}\int_{Q_{\frac{t}{2},t}}(U_{1,\varepsilon}-\varepsilon)^2(x,s)dxds \\ &+m_1^2\|U_{1,0}\|_{\infty,\Omega}^{m_1-1}(C_1+C_2\|U_{1,0}\|_{\infty,\Omega}^r)\int_{Q_{\frac{t}{2},t}}\gamma(U_{1,\varepsilon}-\varepsilon,U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})(x,t)dxdt.\end{aligned}$$

Integrating this inequality in τ over $(\frac{t}{2},t)$ and using the explicit value for $M_{1,3}$ found in the proof of Lemma 2 we deduce the desired result.

□

Lemma 4. *There exists a constant M_1 and non decreasing function F_1, independent of $\varepsilon, 0<\varepsilon\le 1$ such that*

$$(3.21)\qquad \int_{Q_T}|(U_{i,\varepsilon}^{m_i})_t|^2(x,t)dxdt \le M_2,\ \ T>0,\ \ i=1..3;$$

$$(3.22)\qquad \int_{Q_T}|(U_{4,\varepsilon}^{m_i})_t|^2(x,t)dxdt \le F_2(T),\ \ T>0.$$

Proof. The estimate for $U_{1,\varepsilon}$ is immediatly deduced from (3.18) keeping in mind that

$$|(U_{1,\varepsilon}^{m_1})_t|^2(x,t)\le \frac{{m_1}^2}{2}\|U_{1,\varepsilon}\|_{\infty,\Omega}^{m_1-1}(U_{1,\varepsilon}^{\frac{m_1+1}{2}})_t^2(x,t).$$

And one can establish such estimates for $U_{2,\varepsilon}, U_{3,\varepsilon}$ and $U_{4,\varepsilon}$ in the same way.

□

4. Existence and uniqueness: proofs

In this section we supply a quick proof of Theorem 1.

4.1. **Existence.** Let us fix $T>0$. From the estimates established in the previous section one has : for each $i=1..4$ $(U_{i,\varepsilon}-\varepsilon)_{0<\varepsilon\le1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\le1}$ are respectively bounded in $L^2(Q_T)$ and $(L^2(Q_T))^N$. Then there exists two sequences which one still denotes $(U_{i,\varepsilon}-\varepsilon)_{0<\varepsilon\le1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\le1}$ such that for $i=1..4$ as $\varepsilon\to0$: $(U_{i,\varepsilon}-\varepsilon)_{0<\varepsilon\le1}$ is weakly convergent to some U_i in $L^2(Q_T)$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\le1}$ is weakly convergent to some V_i in $(L^2(Q_T))^N$.
On the other hand $(U_{i,\varepsilon})_{0<\varepsilon\le1}$ is bounded in $L^\infty(Q_T)$; using a weak formulation of the equation for $U_{i,\varepsilon}$ one can invoke the results in Di Benedetto [11] to get : $(U_{i,\varepsilon})_{0<\varepsilon\le1}$ is a relatively compact subset of $C(\overline{\Omega}\times[0,T])$. It follows that actually $(U_{i,\varepsilon}-\varepsilon)_{0<\varepsilon\le1}$ is convergent to U_i in $C(\overline{\Omega}\times[0,T])$ and $(U_{i,\varepsilon}^{m_i})_{0<\varepsilon\le1}$ is convergent to $U_i^{m_i}$ in $C(\overline{\Omega}\times[0,T])$.
As a first consequence one has : $V_i=\nabla U_i^{m_i}$; next one also has :

$$\gamma(U_{1,\varepsilon}-\varepsilon,U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})\ \to\ \gamma(U_1,U_2,U_3,U_4)\ \text{ in }\ C(\overline{\Omega}\times[0,T])\ \text{ as }\ \varepsilon\to0.$$

From standard arguments one may conclude that the quadruple (U_1,U_2,U_3,U_4) is a desirable weak solution.

The regularity results for $\nabla U_i^{m_i}$ and $\partial_t U_i^{m_i}$ follow from the a priori estimates in Lemma 2 and Lemma 4.

4.2. **Uniqueness.** Assume there exist two quadruples $(U_{j,1}, U_{j,2}, U_{j,3}, U_{j,4})_{j=1,2}$, both weak solutions of Problem (1.1) – (1.3). They verify the integral identity, for $i = 1..4$

(4.1)
$$\int_\Omega (U_{1,i} - U_{2,i})(x,T)\varphi_i(x,T)dx + \int_{Q_T} \nabla(U_{1,i}^{m_i} - U_{2,i}^{m_i})\nabla\varphi_i(x,t)dxdt$$
$$= \int_{Q_T} [\partial_t\varphi_i(U_{1,i} - U_{2,i}) - (f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_i](x,t)dxdt$$

for every $\varphi_i \in C^1(\bar{Q}_T)$, such that $\dfrac{\partial\varphi_i}{\partial\eta} = 0$ on $\partial\Omega \times (0,T)$ and $\varphi_i > 0$.

We follow an idea of [24] and introduce a function ψ_i as follows

$$\psi_i(x,t) = \begin{cases} \dfrac{U_{1,i}^{m_i} - U_{2,i}^{m_i}}{U_{1,i} - U_{2,i}} & \text{if } U_{1,i} \neq U_{2,i}, \\ 0 & \text{otherwise.} \end{cases} \quad i = 1..4. \tag{4.2}$$

Let us consider a sequence of smooth functions $(\psi_{i,\varepsilon})_{\varepsilon \geq 0}$ such that $\psi_{i,\varepsilon} \geq \varepsilon$, $\psi_{i,\varepsilon}$ is uniformly bounded in $L^\infty(Q_T)$ and

$$\lim_{\varepsilon\to 0} \|(\psi_{i,\varepsilon} - \psi_i)/\sqrt{\psi_{i,\varepsilon}}\|_{L^2(Q_T)} = 0.$$

For any $0 < \varepsilon \leq 1, \sigma > 0$ let us introduce the adjoint nondegenerate boundary value problem

$$\begin{cases} \partial_t\varphi_i + \psi_{i,\varepsilon}\Delta\varphi_i = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial\varphi_i}{\partial\eta}(x,t) = 0 & \text{in } \partial\Omega \times (0,T) \quad i = 1..4. \\ \varphi_i(x,T) = \chi_i & \text{in } \Omega \end{cases} \tag{4.3}$$

For any smooth χ_i with $0 \leq \chi_i(x,t) \leq 1$, $i = 1..4$, any $0 < \varepsilon \leq 1$ and any $\sigma > 0$ this problem has a unique classical solution $\varphi_{i,\varepsilon}$ such that see [24]

$$0 \leq \varphi_{i,\varepsilon}(x,t) \leq 1 \tag{4.4}$$

$$\int_{Q_T} \psi_{i,\varepsilon}(\Delta\varphi_{i,\varepsilon})^2 dxdt \leq K_1, \tag{4.5}$$

If in (4.1) we replace φ_i by $\varphi_{i,\varepsilon}$, which is the solution of problem (4.3) with $\chi_i = sign((U_i - V_i)^+)$ we obtain.

$$\begin{aligned} &\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)\varphi_{i,\varepsilon}(x,T)dx + \int_{Q_T} (\psi_i - \psi_{i,\varepsilon})(U_{1,i} - U_{2,i})\Delta\varphi_{i,\varepsilon}dxdt \\ &= \int_{Q_T} (f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_{i,\varepsilon}dxdt \end{aligned} \tag{4.6}$$

Using the local lipschitz continuity of f_i and the properties of $\psi_{i,\varepsilon}$ and $\varphi_{i,\varepsilon}$ we deduce by letting $\varepsilon \to 0$

$$\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)dx \leq K \int_{Q_T} \sum_{i=1}^4 |U_{1,i} - U_{2,i}|(x,t)dxdt \tag{4.7}$$

In a similar fashion we establish an analogous inequality for $(U_i - V_i)^-$ and deduce

$$\int_\Omega \sum_{i=1}^{4} |U_{1,i} - U_{2,i}|(x,T)dx \leq K \int_{Q_T} \sum_{i=1}^{4} |U_{1,i} - U_{2,i}|(x,t)dxdt \tag{4.8}$$

Uniqueness follows from Gronwall's Lemma.

5. Large time behavior: proofs

The semi-orbit $\{(U_1(.,t), U_2(.,t), U_3(.,t)), t \geq 0\}$ is relatively compact in $(C(\bar{\Omega}))^3$: it is actually bounded in $(L^\infty(Q_\infty))^3$ by (3.15) and then one may use a result of [11].

5.1. **Case $\nu > 0$.** A convergence and continuity argument allows to deduce from (3.7)

$$\int_{Q_T} \gamma(U_1, U_2, U_3, U_4)(x,t)dxdt + \nu \int_{Q_T} U_1\ (x,t)dxdt\ \leq \|U_{1,0}\|_{1,\Omega},\ \ T > 0. \tag{5.1}$$

Hence $U_1 \in L^1(\Omega \times (0, +\infty))$ and there is a sequence $(\tau_j)_{j\geq 0}$ such that $\tau_j \longrightarrow +\infty$ as $j \longrightarrow +\infty$ and $\int_\Omega U_1(x, \tau_j)dx \longrightarrow 0$ as $j \longrightarrow +\infty$. Next, given any $t > \tau_j$, one has

$$0 \leq \int_\Omega U_1(x,t)dx \leq \int_\Omega U_1(x,\tau_j)dx; \tag{5.2}$$

actually such an identity holds for $U_{1,\varepsilon}$ from a straightforward integration over $\Omega \times (\tau_j, t)$ and is preserved upon letting $\varepsilon \longrightarrow 0$ because $U_{1,\varepsilon} \to U_1$ in $C^0(\overline{\Omega} \times (0, +\infty)$ as $\varepsilon \to 0$. This shows that $U_1(.,t) \longrightarrow 0$ in $L^1(\Omega)$ as $t \longrightarrow +\infty$ and also in $C(\bar{\Omega})$.

Then, along the same lines, from (3.13) and (3.14) one has for $T > 0$

$$\begin{aligned} &(\lambda + \mu) \int_{Q_T} U_2\ (x,t)dxdt + (\alpha + m + \mu) \int_{Q_T} U_3\ (x,t)dxdt \\ &\leq 2\|U_{1,0}\|_{1,\Omega} + 2\|U_{2,0}\|_{1,\Omega} + \|U_{3,0}\|_{1,\Omega}. \end{aligned} \tag{5.3}$$

Again, for some sequence $(\tau_j)_{j\geq 0}$ such that $\tau_j \longrightarrow +\infty$ one has $\int_\Omega U_2(x,\tau_j)dx \longrightarrow 0$ as $j \longrightarrow +\infty$. Integrating over $\Omega \times (\tau_j, t)$ the equation in (3.6) for $U_{2,\varepsilon}$ and letting $\varepsilon \longrightarrow 0$ one finds

$$0 \leq \int_\Omega U_2(x,t)dx\ \leq \int_{\tau_j}^{t} \int_\Omega \gamma(U_1, U_2, U_3, U_4)(x,\tau)dxd\tau + \int_\Omega U_2(x,\tau_j)dx; \tag{5.4}$$

thus again $U_2(.,t) \longrightarrow 0$ in $L^1(\Omega)$ and in $C(\bar{\Omega})$ because γ lies in $L^1(\Omega \times (0, +\infty)$.

The conclusion for $U_3(.,t)$ is derived in the same fashion, using the third equation in (3.6).

Now we will establish the long time behavior of U_4, to do this let us consider for any $\tau > 0$ the following problem

$$\begin{cases} \partial_t V - \Delta V^{m_4} = 0, & (x,t) \in \Omega \times (0, +\infty) \\ V(x,0) = U_4(x,\tau), & x \in \Omega; \\ \dfrac{\partial V^{m_4}}{\partial \eta}(x,t) = 0. & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{5.5}$$

It is well known see [1] that $\lim\limits_{t\to+\infty} V(.,t) = \overline{V}(0) = \overline{U_4}(\tau)$ in $L^p(\Omega)$, for all $p \geq 1$, and in another hand from [6] we have for all $p \geq 1$

$$\|U_4(x,\tau+h) - V(x,h)\|_{p,\Omega} \leq \int_h^{\tau+h} \|f_4(x,s)\|_{p,\Omega}\, ds, \tag{5.6}$$

with $f_4(x,t) = (1-\pi)\lambda U_2(x,t) + \alpha U_3(x,t) + \nu U_1(x,t)$. Set $\tau = h = \frac{t}{2}$, we can write

$$\begin{aligned} \|U_4(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega} &\leq \|U_4(x,t) - V(x,\frac{t}{2})\|_{p,\Omega} + \|V(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega}, \\ &\leq \int_{\frac{t}{2}}^{t} \|f_4(x,s)\|_{p,\Omega} ds + \|V(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega};\ p \geq 1, \end{aligned}$$

since $f_4 \in L^1(Q_\infty) \cap L^\infty(Q_\infty)$ we deduce that $\lim_{t\to+\infty} \|U_4(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega} = 0$, furthermore $f_4 \geq 0$ allow to show that $t \longrightarrow \overline{U_4}(t)$ is bounded and nondecreasing and then converges to some nonnegative constant U_4^* and this yields $\lim\limits_{t\to+\infty} U_4(.,t) = \lim\limits_{t\to+\infty} \overline{U_4}(t) = U_4^*$ in $L^p(\Omega)$ for all $p \geq 1$.

5.2. **Case** $\nu = 0$**.** The analysis of the behavior of $\{U_1(.,t), t > 0\}$ requires modifications because it is not known, and actually it is not true, that $U_1 \in L^1(\Omega \times (0,+\infty))$. Set

$$\bar{\phi}(t) = \frac{1}{mes(\Omega)} \int_\Omega \phi(x,t)dx;$$

then multiplying the equation for $U_{1,\varepsilon}$ in (3.7) by $\frac{1}{m_1} U_{1,\varepsilon}^{m_1-1}$ and integrating over $\Omega \times (\tau, \tau+t)$ yields

$$\overline{U_{1,\varepsilon}^{m_1}}(\tau) \geq \overline{U_{1,\varepsilon}^{m_1}}(\tau+t) \geq 0, \tau > 0, t > 0; \tag{5.7}$$

so that upon letting $\varepsilon \longrightarrow 0$, the average $\bar{U}_1^{m_1}$ is a nonincreasing function of time. From the inequality of Poincaré-Wirtinger one can conclude the existence of a constant $K(\Omega)$ such that for $t > 0$

$$\|U_1^{m_1}(.,t) \;-\; \overline{U_1^{m_1}}(t)\|_{2,\Omega} \;\leq\; K(\Omega)\, \|\nabla U_1^{m_1}(.,t)\|_{2,\Omega}. \tag{5.8}$$

Now, one gets from Lemma 3 with $\nu = 0$ that

$$\begin{aligned} &\|\nabla U_1^{m_1}(.,t)\|_{2,\Omega}^2 \leq \frac{2}{t(m_1+1)} \int_\Omega U_{1,0}^{m_1+1}(x)dx \\ &+ m_1^2 \|U_{1,0}\|_{\infty,\Omega}^{m-1} (C_1 + C_2 \|U_{1,0}\|_{\infty,\Omega}^{r}) \int_{Q_{\frac{t}{2},t}} \gamma(U_1,U_2,U_3,U_4)(x,s)dxds \end{aligned} \tag{5.9}$$

It follows that $\|\nabla U_1^{m_1}(.,t)\|_{2,\Omega} \longrightarrow 0$ as $t \longrightarrow +\infty$, so that $\|U_1^{m_1}(.,t) \;-\; \bar{U}_1^{m_1}(t)\|_{2,\Omega} \longrightarrow 0$ by (5.8). The monotonicity of $t \longrightarrow \overline{U_1^{m_1}}(t)$ yields $\lim\limits_{t\to+\infty} U_1^{m_1}(.,t) = \lim\limits_{t\to+\infty} \bar{U}_1^{m_1}(t) = U_1^*$ in $L^2(\Omega)$ and also in $C(\bar{\Omega})$.

5.3. **An elementary spatially homogeneous system.** Let us consider the system of ordinary differential equation

$$(5.10)\qquad \begin{cases} U_1' &= -\gamma(U_1,U_2,U_3,U_4) \\ U_2' &= \gamma(U_1,U_2,U_3,U_4) - \lambda U_2 - \mu U_2, \\ U_3' &= \lambda\pi U_2 - \alpha U_3 - \mu U_3 - mU_3, \\ U_4' &= (1-\pi)\lambda U_2 + \alpha U_3. \end{cases}$$

With $U_i(0) \geq 0,\ \ i = 1..4,\ \ U_1(0) > 0,\ \ U_3(0) \geq 0$ and,

$$\begin{cases} \lambda > 0,\ \ \alpha > 0,\ \ m \geq 0,\ \ \mu \geq 0, \\ \gamma(U_1,U_2,U_3,U_4) = \sigma(U_2,U_3,U_4)\,U_1, \end{cases}$$

γ having either a masse action or a proportionate mixing form : see the introduction.

Then $U_1(t) = U_1(0)exp\left(-\int_0^t \sigma(U_2,U_3,U_4)(\tau)d\tau\right)$ so that $U_1(t) \searrow U_1^* \geq 0$ as $t \longrightarrow +\infty$ and $U_1^* = 0$ if and only if $\displaystyle\int_0^{+\infty} \sigma(U_2,U_3,U_4)(\tau)d\tau = +\infty$.

Next $U_1 + U_2 = -(\lambda+\mu)U_2(t)$ and upon integrating over $(0,+\infty)$ one gets U_2 lies in $L^1(0,+\infty)$ so that $U_2(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ because U_2' is bounded.

A similar argument yields U_3 lies in $L^1(0,+\infty)$ and $U_3(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. Then one has $U_4(t) = U_4(0) + (1-\pi)\lambda\displaystyle\int_0^t U_2(\tau)d\tau + \alpha\int_0^t U_3(\tau)d\tau$. Here $U_4(t) \nearrow U_4^* > 0$ as $t \longrightarrow +\infty$.

To conclude that $U_1^* > 0$ note that

- When $\gamma(U_1,U_2,U_3,U_4) = \gamma U_1U_3$ then $\sigma(U_2,U_3,U_4) = \gamma U_3$ lies in $L^1(0,+\infty)$.
- When $\gamma(U_1,U_2,U_3,U_4) = \gamma\dfrac{U_1U_3}{U_1+U_2+U_3+U_4}$ then $\sigma(U_2,U_3,U_4) = \gamma\dfrac{U_3}{U_1+U_2+U_3+U_4}$

  Now $(U_1+U_2+U_3+U_4)(t) \longrightarrow U_1^*+U_4^*$ as $t \longrightarrow +\infty$ and $U_1^*+U_4^* > 0$, because $U_4^* > 0$ and $U_1^* \geq 0$; hence for $t \geq t_0$ one has

  $$\frac{1}{2}(U_1^*+U_4^*) \leq (U_1+U_2+U_3+U_4)(t) \leq (U_3+U_4)(0)$$

  which implies

  $$\frac{U_3(t)}{(U_3+U_4)(0)} \ \leq\ \sigma(U_2,U_3,U_4)(t) \ \leq\ 2\frac{U_3(t)}{U_1^*+U_4^*},\quad t \geq t_0.$$

  As a conclusion $\sigma(U_2,U_3,U_4)$ lies in $L^1(0,+\infty)$ and $U_1^* > 0$.

Last when $m = \mu = 0,\ \ U_1^*+U_4^* = (U_1+U_2+U_3+U_4)(0)$.

References

[1] N. Alikakos and R. Rostamian, *Large time behavior of solution neumann boundary value problem of the porous medium equation*, Indiana University Mathematics Journal **30** (1981), no. 5, 749–785.

[2] T. Aliziane, *Etude de la régularité pour un problème d'évolution dégénéré en dimension supérieure de l'espace*, USTHB, Algiers, 1993.

[3] T. Aliziane and M. Langlais, *Global existence and periodic solution for a system of degenerate evolution equations*, (preprint) Faculté de Mathématiques USTHB, 2003.

[4] T. Aliziane and M. S. Moulay, *Global existence and asymptotic behaviour for a system of degenerate evolution equations*, Maghreb Math Rev **9** (2000), no. 1 & 2, 9–22.
[5] R.M. Anderson, H.C. Jackson, R.M. May, and A.D.M. Smith, *Population dynamics of fox rabies in europe*, Nature **289** (1981), 765–770.
[6] P. Baras, *Compacité de l'opérateur $f \longmapsto u$ solution d'une équation non linéaire $(du/dt) + Au \ni f$*, C. R. Acad. Sc. (1978), 1113 – 1116.
[7] S. Busenberg and K.C. Cooke, *Vertically transmitted diseases*, Biomathematics, vol. 23, Springer Verlag, Berlin, 1993.
[8] V. Capasso, *Mathematical structures of epidemic systems*, Lecture Notes in Biomathematics (1993), no. 97.
[9] V. Capasso and G. Serio, *A generalization of the kermack-mckendrick deterministic epidemic model*, Math. Biosci. **42** (1978), 41–61.
[10] F. Courchamp, Ch. Suppo, E. Fromont, and C. Bouloux, *Dynamics of two feline retroviruses (fiv and felv) within one population of cats*, Proc. Roy. Soc. London. **B 264** (1997), 785–794.
[11] E. DiBenedetto, *Continuity of weak solutions to a general pourous medium equation*, Indiana University Mathematic Journal **32** (1983), no. 1, 83–118.
[12] O. Diekmann, M. C. De Jong, A. A. De Koeijer, and P. Reijders, *The force of infection in populations of varying size: A modeling problem*, Journal of Biological Systems **3** (1995), no. 2, 519–529.
[13] Le Dung, *Global attractors for a class of degenerate nonlinear parabolic systems*, J. Diff. Eqns., 147 (1998), pp. 1–29.
[14] W.E. Fitzgibbon and M. Langlais, *Diffusive seir models with logistic population control*, Communication on Applied Nonlinear Analysis **4** (1997), 1–16.
[15] W.E. Fitzgibbon, M. Langlais, and J.J. Morgan, *Eventually uniform bounds for a class of quasipositive reaction-diffusion systems*, Japan Journal of Industrial And Applied Mathematics **16** (1999), no. 2, 225–241.
[16] W.E. Fitzgibbon, J.J. Morgan, and S.J Waggoner, *A quasilinear system modeling the spread of infectious disease*, Rocky Mountain Journal of Mathematics **2** (1992), 579–592.
[17] E. Fromont, M. Artois, M. Langlais, F. Courchamp, and D. Pontier, *Modelling the feline leukemia virus (felv) in a natural population of cats (it Felis catus)*, Theoretical Population Biology **52** (1997), 60–70.
[18] E. Fromont, D. Pontier, and M. Langlais, *Dynamics of a feline retroviretrovirusrus (felv) in host populations with variable structures*, (1998).
[19] L. Hadjadj, T. Aliziane, and M. S. Moulay, *Non linear reaction diffusion system of degenerate parabolic type* (submitted).
[20] M. C. De Jong, O. Diekmann, and H. Heesterbeek, *How does transmission of infection depend on population size? epidemic models*, Publication of the Newton Institute (1995), no. 5, 84–94.
[21] W.O Kermack and A.G. McKendrick, *Contribution to the mathematical theory of epidemics. ii-the problem of endemicity*, Proc. Roy. Soc. Edin **A 138** (1932), 55–83.
[22] O.A. Ladyzenskaja, V.A. Solonikov, and N.N. Ural'ceva, *Linear and quasilinear equation of parabolic type*, Transl. Math. Monographs, vol. 23, American Math. Society, Providence, 1968.
[23] M. Langlais and Ch. Suppo, *A remark on a generic seirs model and application to cat retroviruses and fox rabies*, Mathematical and Computer Modelling **31** (2000), 117–124.
[24] L. Maddalena, *Existence, uniqueness and qualitative properties of the solution of a degenerate nonlinear parabolic system*, J. Math. Anal. Applications **127** (1987), 443–458.
[25] O. A. Oleinik, A. S. Kalashnikov, and C. Yui-Lin, *The cauhy problem and boundary value problem for the equation of insteady filtration type*, Izv. Akad. Nauk SSSR, Soc. Math. **22** (1958), 667–704.
[26] J. Smoller, *Shock waves and reaction-diffusion equation*, Springer, Berlin Heidelberg New York Tokyo, 1983.
[27] G.F. Webb, *Reaction diffusion model for deterministic diffusive epidemics*, J. Math. App **84** (1981), 150–161.

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University of Sciences and Technology Houari Boumediene, Faculty of Mathematics, Po.Box 32 El Alia, Algiers, Algeria.
E-mail address: `taliziane@usthb.dz`

UPRES A 5466, "Mathématiques Appliquées de Bordeaux" U.F.R M.I.2S, B.P 26, Université Victor Segalen, 33076 Bordeaux Cedex. FRANCE
E-mail address: `langlais@sm.u-bordeaux2.fr`