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Existence of global solution for a nonlocal parabolic problem

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Abstract

In this paper, we study a non-local initial boundary-value problem arising in Ohmic heating. By using a dynamical systems approach, some existence and uniqueness results are proved and the existence of a compact attractor is shown.

Mathematics Subject Classifications: 35K20, 35K35, 35K45, 35K60.
Key words: thermistor, a nonlocal, existence, attractor global.

1 Introduction

In this paper, we shall deal with the following nonlocal parabolic problem

$$\frac{\partial u}{\partial t} - \triangle u = \lambda \frac{f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \text{ in } \Omega\times]0;T[, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega\times]0;T[, \tag{1.2}$$

$$u/_{t=0} = u_0 \quad \text{in } \Omega, \tag{1.3}$$

where $T > 0, \Omega$ is a regular open bounded subset of $\mathbb{R}^N, N \geq 1$, λ is a positive parameter and f a function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the hypotheses $(H_1)-(H_2)$ below. Problem $(1.1) - (1.3)$ represents, for example, static materiel such as thermistors [3, 6, 14, 15] and arises by reducing the system of two equations

$$u_t = \nabla.(k(u)\nabla u) + \sigma(u)|\nabla\varphi|^2, \tag{1.4}$$

$$\nabla(\sigma(u)\nabla\varphi) = 0, \tag{1.5}$$

to a simple but realistic equation (see [8]). More precisely, u represents the temperature produced by an electric current flowing through a conductor, φ the electric potential, $\sigma(u)$ is the electrical conductivity and $k(u)$ is the thermal conductivity. Taking the latter to be constant, problem $(1.4) - (1.5)$ can then be reduced to the single nonlocal equation(1.1), where $f(u) = \sigma(u)$ and $\lambda = \frac{I^2}{|\Omega|^2} \geq 0, I$ is the electric current which is supposed to be constant and $|\Omega|$ is the measure of Ω.

Our goal here concerns the existence and uniqueness of weak solutions to $(1.1)-(1.3)$. We shall also show existence of global attractor.
Let us first recall that problem $(1.1)-(1.3)$ has been the subject of variety of investigation in the past decade. Particularly, some results have been obtained by many authors in the case where $N=1$ and f taking particular forms: Montesinos and Gallego [11] proved the existence of weak solution under

$$0<\sigma_1\le\sigma(s)\le\sigma_2, \forall s\in\mathbb{R}. \tag{1.6}$$

Antontsev and Chipot [1] obtained also an existence and uniqueness results for $(1.4)-(1.5)$ supposing that $\sigma\in C^0(\Omega)$ and (1.6); furthermore, a study of smoothness of solutions was treated in that paper under some assumptions on the conductivity and initial data.
In [8, 9, 13], major emphasis is placed on cases where the spatial dimension N is 1 or 2 and f is of the form $f(u)=\exp(u) or \exp(-u)$. In these works, additional regularity assumptions are made on u_0 and a combination of usual Lyapounov functional and a comparison method is the main ingredient. Our purpose is to extend some of the results therein to problem $(1.1)-(1.3)$, where here, the condition (1.6) is weakened to (H_2) below. Following the frame work of Fioas and Temam [12], we shall also deal with the asymptotic behaviour of the solutions of problem $(1.1)-(1.3)$ via a dynamical systems approach. We start by proving the existence of absorbing sets in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$, which in turn paves the way for the existence of the global attractor. Cimatti [4] obtained similar results for particular cases, when $N=1$, by constructing a Lyapounov functional. As a concluding result, we show that the attractor is bounded subset of $H^2(\Omega)$ under restrictive assumptions on data.

2 Existence and regularity of global attractor.

a) Existence and uniqueness.
We assume the following

(H1) $f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitzian function .

(H2) There exist positive constants c_1, c_2 and α such that, for all $\xi\in\mathbb{R}$

$$\sigma\le f(\xi)\le c_1|\xi|^{\alpha+1}+c_2.$$

Let us denote by $\|.\|_k$ the norm in $L^k(\Omega)$.
We adopt the following weak formulation for $(1.1)-(1.3)$:

u is a solution of (1.1) – (1.3) if and only if

$$u \in L^{\infty}(\tau, +\infty, H_0^1(\Omega) \cap L^{\infty}(\Omega)) \text{ with} \frac{\partial u}{\partial t} \in L^2(\tau, +\infty, L^2(\Omega))$$
$$\text{for any } \tau > 0, \text{ and satisfying}$$
$$\int_0^T \int_\Omega u \frac{\partial}{\partial t}\phi - \nabla u \nabla \phi \, dx dt = \int_0^T \Big(\frac{\lambda}{\big(\int_\Omega f(u)\, dx\big)^2} \int_\Omega f(u)\phi dx \Big) dt,$$
$$\text{for any } \phi \in C^{\infty}((0,\infty), \Omega).$$

Now, we state our main result.

Theorem 2.1. *Let hypotheses* $(H_1) - (H_2)$ *be satisfied. Assume that* $u_0 \in L^{k_0+2}(\Omega)$ *with* k_0 *such that*

$$k_0 \geq \max\big(0, \frac{\alpha N}{2} - 2\big). \tag{2.1}$$

Then, there exists $d_0 > 0$ *such that if* $\|u_0\|_{k_0+2} < d_0$*, the problem* (1.1) *admits a solution* u *verifying for all* $\tau > 0$

$$u \in L^{\infty}(\tau, +\infty, L^{k_0+2}(\Omega)), \qquad |u|^{\gamma} u \in L^{\infty}(\tau, +\infty, H_0^1(\Omega)), \ \textit{with } \gamma = \frac{k_0}{2}.$$

Moreover, if $u_0 \in L^{\infty}(\Omega)$*, then* $u \in L^{\infty}(\tau, +\infty, L^{\infty}(\Omega))$ *and is unique.*

Remark. The value of d_0 will be given in the course of the proof.

Proof. We use a Faedo-Galerkin method see [10]. Let $u_m \subseteq D(\Omega)$ be such that $u_{0m} \rightarrow u_0$ in $H_0^1(\Omega)$ and let $(w_j)_j \subseteq H_0^1(\Omega)$ a special basis. We seek u to be the limit of a sequence $(u_m)_m$ such that

$$u_m(t) = \Sigma_{j=1}^m g_{jm}(t) w_j,$$

where g_{jm} is the solution of the following ordinary differential system

$$\begin{cases} \langle u_m', w_j \rangle + (u_m, w_j) = \frac{\lambda}{\left(\int_\Omega f(u_m)\, dx\right)^2} \langle f(u_m), w_j \rangle, \ 1 \leq j \leq m, \\ u_m(0) = u_{om}. \end{cases} \tag{2.2}$$

It is easy to see that (2.2) has a unique solution u_m according to hypotheses $(H_1) - (H_2)$ and Cartan's existence theorem concerning ordinary differential equations (see [5]). This solution is shown to exist on a maximal interval $[0; t_m[$. The following estimates enable us to assert that it can be continued on the whole interval $[0; T]$. We shall denote by C_i different positive constants, depending on data, but not on m.

Lemma 2.2. *For any $\tau > 0$, there exists a constant $c_3(\tau), c_4(\tau)$ such that*

$$\|u_m(t)\|_{k_0+2} \leq c_3(\tau), \forall t \geq \tau, \tag{2.3}$$

$$\|u_m(t)\|_\infty \leq c_4(\tau), \forall t \geq \tau. \tag{2.4}$$

Proof. (i) Multiplying the first equation of (2.2) by $|u_m|^k g_{jm}$, integrating on Ω, adding from $j = 1$ to m and using $(H_1) - (H_2)$, yields

$$\frac{1}{k+2}\frac{d}{dt}\|u_m\|_{k+2}^{k+2} + \frac{4}{(k+2)^2}\|\nabla|u_m|^{\frac{k}{2}}u_m\|_2^2 \leq c_5\|u_m\|_{k+\alpha+2}^{k+\alpha+2} + c_6. \tag{2.5}$$

By using well-known Sobolev and Gagliardo-Nirenberg's inequalities, we have

$$\|u_m\|_{k_0+\alpha+2}^{k_0+\alpha+2} \leq c_7\|u_m\|_{k_0+2}^{\alpha}\|\nabla|u_m|^\gamma u_m\|_2^2, \tag{2.6}$$

Thus, from (2.5) and (2.6), we obtain

$$\frac{1}{k_0+2}\frac{d}{dt}\|u_m\|_{k_0+2}^{k_0+2} \leq (c_8\|u_m\|_{k_0+2}^{\alpha} - \frac{4}{(k_0+2)^2})\|\nabla|u_m|^\gamma u_m\|_2^2 + c_6. \tag{2.7}$$

We shall make the following compatibility condition on u_0

$$\|u_0\|_{k_0+2} < \Big(\frac{4}{c_8(k_0+2)^2}\Big)^{\frac{1}{\alpha}} = d_0. \tag{2.8}$$

Then, there exists a small $\tau > 0$ such that

$$\|u_m(t)\|_{k_0+2} < d_0 \text{ for } t \in]0, \tau[. \tag{2.9}$$

Hence

$$\frac{1}{k_0+2}\frac{d}{dt}\|u_m\|_{k_0+2}^{k_0+2} + c_9\|\nabla|u_m|^\gamma u_m\|_2^2 \leq c_6 \quad \forall \quad 0 < t < \tau. \tag{2.10}$$

By Poincaré's inequality and after integrating, it follows that

$$\|u_m(t)\|_{k_0+2} \leq c_{10}, \quad \forall \quad 0 < t < \tau,$$

Therefore, relation (2.3) is achieved by iterating successively the same process on intervals of periode τ such as $[0, \tau], [\tau, t+\tau], ...$
(ii) By using Hôlder's inequality, we get

$$\|u_m\|_{k+\alpha+2}^{k+\alpha+2} \leq c_{11}\|u_m\|_{k+2}^{\theta_1}\|u_m\|_{k_0+2}^{\theta_2}\|u_m\|_q^{\theta_3}, \tag{2.11}$$

with θ_1, θ_2 and θ_3 satisfying

$$\frac{\theta_1}{k+2} + \frac{\theta_2}{k_0+2} + \frac{\theta_3}{q} = 1 \quad \text{ and } \quad \theta_1 + \theta_2 + \theta_3 = k + \alpha + 2.$$

We require moreover

$$\frac{\theta_1}{k+2} + \frac{\theta_3}{2(\gamma+1)} = 1.$$

Using the boundedness of $\|u_m\|_{k_0+2}$, the choice of q, Sobolev and Young's inequalities and relation (2.11), we derive that

$$\begin{aligned} c_5\|u_m\|_{k+\alpha+2}^{k+\alpha+2} &\le c_{12}\|u_m\|_{k+2}^{\theta_1}\|\nabla|u_m|^\gamma u_m\|_2^{\frac{\theta_3}{\gamma+1}} \\ &\le c_{13}(k+2)^{\theta_4}\|u_m\|_{k+2}^{k+2} + \frac{2}{(k+2)^2}\|\nabla|u_m|^\gamma u_m\|_2^2, \end{aligned}$$

where θ_4 is some positive constant. Hence (2.5) becomes

$$\frac{1}{k+2}\frac{d}{dt}\|u_m\|_{k+2}^{k+2} + \frac{c_{14}}{(k+2)^2}\|\nabla|u_m|^\gamma u_m\|_2^2 \le c_{15}(k+2)^{\theta_4}\|u_m\|_{k+2}^{k+2} + c_5.$$

Therefore, by applying lemma 4 ([7]) we conclude to (2.4).

Passage to the limit in (2.2) **as** $m \to \infty$**.** Multiplying the jth equation of system (2.2) by $g_{jm}(t)$, adding these equations for $j = 1, ..., m$ and integrating with respect to the time variable, we deduce the existence of a subsequence of u_m such that

$$\begin{aligned} u_m &\to u \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \\ u_m &\to u \text{ weak in } L^2(0,T;H_0^1(\Omega)), \\ u_{mt} &\to u_t \text{ weak in } L^2(0,T;H^{-1}(\Omega)), \\ u_m &\to u \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e in } Q_T. \end{aligned}$$

Straightforward standard compactness arguments allow us to assert that u is a solution of problem (1.1)

Uniqueness. Consider u_1 and u_2 two weak solutions of the problem (1.1) and define $w = u_1 - u_2$. Substracting the equations verified by u_1 and u_2, we obtain

$$\begin{aligned} \frac{dw}{dt} - \triangle w = {} & \frac{\lambda}{\big(\int_\Omega f(u_1)\,dx\big)^2}\Big(f(u_1) - f(u_2)\Big) \\ & + \lambda\frac{\Big(\int_\Omega f(u_2) - f(u_1)\,dx\Big)\Big(\int_\Omega f(u_2) + f(u_1)\,dx\Big)}{\big(\int_\Omega f(u_1)\,dx\big)^2\big(\int_\Omega f(u_2)\,dx\big)^2}f(u_2). \quad (2.12) \end{aligned}$$

Taking the inner product of (2.12) by w and using (H_1) and (2.4), we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 \le c_{16}\|w(t)\|_2^2,$$

which implies that $w = 0$. Hence the solution is unique. $\square$

b) We denote by $\{T(t), t \geq 0\}$ the continuous semi-group generated by (1.1) and defined by

$$\begin{array}{rcl} T(t) : L^\infty(\Omega) & \to & L^\infty(\Omega) \\ u_0 & \to & T(t)u_0 = u(t,.). \end{array}$$

In this part, we refer to [12] for used concepts.

Theorem 2.3. *Assume that $(H_1)-(H_2)$ are satisfied, Then $T(t)$ possesses a maximal attractor which is bounded in $H_0^1(\Omega)\bigcap L^\infty(\Omega)$, compact and connected in $L^\infty(\Omega)$.*

Proof.

(i) (2.4) imply that there exists an absorbing set in $L^k(\Omega)$, $\quad 1 \leq k \leq \infty$.

(ii) To obtain existence of absorbing sets in $H_0^1(\Omega)$ and the uniform compactness of $T(t)$, multiply (2.2) by $g'_{jm}(t)$, add from $j = 1$ to m and integrate on Ω by using Young's inequality, it follows therefore that, for any $t \geq \tau > 0$

$$\int_\Omega (\frac{\partial u_m}{dt})^2 dx + \frac{d}{dt}\|\nabla u_m\|_2^2 \leq c_{17}(\tau), \tag{2.13}$$

which gives

$$\frac{d}{dt}\|\nabla u_m\|_2^2 \leq c_{17}(\tau), \forall t \geq \tau > 0. \tag{2.14}$$

On the other hand, multiplying (2.2) by g_{jm}, adding and integrating on $\Omega \times [t, t+\tau]$ we get

$$\int_t^{t+\tau} \|\nabla u_m(s)\|_2^2 ds \leq c_{18}(\tau), \forall t \geq \tau > 0. \tag{2.15}$$

Then, by the uniform Gronwall's lemma (see [12], p.89) and the lower semi-continuity of the norm, we have

$$\|\nabla u(t)\|_2^2 \leq c_{19}(\tau), \forall t \geq \tau. \tag{2.16}$$

Therefore, the open ball $B(0, c_{19}(\tau))$ is an absorbing set in $H_0^1(\Omega)$.
Hence, by theorem (1.1)([12], p.23), we conclude to the results of theorem (2.3).

Theorem 2.4. *We suppose $(H_1) - (H_2)$ and*

(H3) $f \in C^1(\mathbb{R})$.

Then, we have

$$y(t) \equiv \|u_t\|^2 \le c_{20}(\tau), \quad \textit{for any } t \ge \tau > 0.$$

Proof. Differentiating equation (1.1) with respect to time(the justification of the formal derivatives can be done as in [5]), we get

$$u_{tt} - \triangle u_t = \frac{\lambda f'(u) u_t}{\left(\int_\Omega f(u)\,dx\right)^2} - 2\lambda f(u) \frac{\int_\Omega f'(u) u_t\,dx}{\left(\int_\Omega f(u)\,dx\right)^3}. \tag{2.17}$$

Multiplying (2.17) by u_t, integrating over Ω and using the L^∞ estimate of u and Hôlder's inequality, yields

$$\frac{1}{2} y'(t) \le c_{21}(\tau) y(t). \tag{2.18}$$

On the other hand, taking the scalar product of (1.1) with u_t, using Young's inequality, integrating on $[t, t+\tau]$ and using estimate (2.16), then gives

$$\int_t^{t+\tau} y(s) ds \le c_{23}(\tau), \text{for any } t \ge \tau. \tag{2.19}$$

From (2.18) and the uniform Gronwall's lemma, we have

$$y(t) \le c_{23}(\tau), \text{ for any } \ t \ge \tau.$$

Therefore,

$$u_t \in L^\infty(\tau, \infty, L^2(\Omega)).$$

By (1.1), we then get

$$-\triangle u \in L^\infty(\tau, \infty, L^2(\Omega)),$$

that is,

$$u(t) \quad \text{is in a bounded subset of } H^2(\Omega).$$

Hence the existence of an absorbing set in $H^2(\Omega)$ is shown.

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