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Instability of traveling waves for a generalized diffusion model in population problems

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Abstract. In this paper, we study the instability of the traveling waves of a generalized diffusion model in population problems. We prove that some traveling wave solutions are nonlinear unstable under H^2 perturbations. These traveling wave solutions converge to a constant as $x \to \infty$.
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1 Introduction

In this paper we consider the following equation

$$\frac{\partial u}{\partial t} + D^4 u + D^2 u = D^2 u^3 + g(u). \tag{1.1}$$

The equation (1.1) arises naturally as a continuum model for growth and dispersal in a population, see [1]. Here $u(x,t)$ denotes the concentration of population, the term $g(u)$ is nonlinear function, denotes reaction term or power with typical example as $g(u) = a(1-u^2), a > 0$. During the past years, many authors have paid much attention to the equation (1.1), see [2, 3, 4]. Liu and Pao [2] based on the fixed point principle, proved the existence of classical solutions for periodic boundary problem. Chen and Lü [3] proved the existence, asymptotic behavior and blow-up of classical solutions for initial boundary value problem. Chen [4] proved existence of solutions for Cauchy problem.

In this paper we study instability of the traveling waves of the equation (1.1) for $g(u) = a(1-u^2), a > 0$. The stability and instability of special solutions for the equation (1.1) are very important in the applied fields.

E. A. Carlen, M. C. Carvalho and E. Orlandi [8] proved the nonlinear stability of fronts for the equation (1.1) with $g(u) = 0$, under L^1 perturbations. We prove that it is nonlinearly unstable under H^2 perturbations, for some traveling wave solution that is asymptotic to a constant as $x \to \infty$. Our proof is based on the principle of linearization. We invoke a general theorem that asserts that linearized instability implies nonlinear instability.

Our main result is as follows

Theorem 1.1 *All the traveling waves $\varphi(x-ct)$ of the equation (1.1) satisfying $\varphi \in L^\infty(R)$, $\varphi^{(n)} \in L^\infty(R) \cap L^2(R)$ $(n = 1, 2, 3, 4)$ are nonlinearly unstable in the space $H^2(R)$. Where $\varphi^{(n)}$ denotes nth derivative of φ.*

This paper is organized as follows. We first find a exact traveling wave solution for the equation(1.1) in Section 2, and then give the proof of our main theorem in Section 3.

2 Exact Traveling Wave Solution

In this section, we construct an exact traveling wave which satisfies all conditions of theorem 1.1.

If $\varphi(x-ct)=\varphi(z)$ is a traveling wave solution of (1.1), then φ satisfies the ordinary differential equation

$$-c\varphi' + \varphi'''' = (3\varphi^2-1)\varphi'' + 6\varphi\varphi'^2 + a(1-\varphi^2). \tag{2.1}$$

Let $\varphi' = \dfrac{\partial\varphi}{\partial z} = k(1-\varphi^2)$. Then

$$\varphi'' = \frac{\partial}{\partial z}(k(1-\varphi^2)) = -2k^2\varphi(1-\varphi^2),$$

$$\varphi''' = \frac{\partial}{\partial z}(-2k^2\varphi(1-\varphi^2)) = 2k^3(-1+3\varphi^2)(1-\varphi^2),$$

$$\varphi'''' = \frac{\partial}{\partial z}(2k^3(-1+3\varphi^2)(1-\varphi^2)) = 2k^4(8\varphi-12\varphi^3)(1-\varphi^2).$$

Substituting above equation into (2.1), we have

$$\begin{aligned} & -ck(1-\varphi^2) + 2k^4(8\varphi-12\varphi^3)(1-\varphi^2) \\ = \quad & (3\varphi^2-1)\varphi'' + 6\varphi k^2(1-\varphi^2)^2 + a(1-\varphi^2). \end{aligned}$$

Then comparing the order of φ, we get

$$-ck = a,$$
$$16k^4 - 8k^2 = 0,$$
$$-24k^4 + 12k^2 = 0.$$

A simple calculation shows that $k=\frac{1}{\sqrt{2}}$, $c=-\sqrt{2}a$. Hence, we obtain

$$\varphi' = \frac{1}{\sqrt{2}}(1-\varphi^2),$$

that is

$$\frac{1}{2}\ln\frac{1+\varphi}{1-\varphi} = \frac{1}{\sqrt{2}}z,$$

i.e.

$$\varphi(z) = \frac{e^{\frac{1}{\sqrt{2}}z} - e^{-\frac{1}{\sqrt{2}}z}}{e^{\frac{1}{\sqrt{2}}z} + e^{-\frac{1}{\sqrt{2}}z}} = \tanh\frac{1}{\sqrt{2}}z.$$

We easily proved that

$$\lim_{z\to+\infty}\varphi(z) = 1, \qquad \lim_{z\to-\infty}\varphi(z) = -1$$

and $\varphi(z)$ satisfies the conditions of the theorem.

3 Proof of the Result

To prove the theorem 1.1, we first consider an evolution equation

$$\frac{\partial u}{\partial t} = Lu + F(u), \tag{3.1}$$

where L is a linear operator that generates a strongly continuous semigroup e^{tL} on a Banach space X, and F is a strongly continuous operator such that $F(0) = 0$. In [9] the authors considered the whole problem only on space X, that is to say, the nonlinear operator maps X into X. However, many equations possess nonlinear terms that include derivatives and therefore F maps into a large Banach space Z. Hence, they again got the following lemma.

Lemma 3.1 [5] *Assume the following*
(i) X, Z are two Banach spaces with $X \subset Z$ and $\|u\|_Z \leq C_1\|u\|_X$ for $u \in X$.
(ii) L generates a strongly continuous semigroup e^{tL} on the space Z, and the semigroup e^{tL} maps Z into X for $t > 0$, and $\int_0^1 \|e^{tL}\|_{Z\to X}dt = C_4 < \infty$.
(iii) The spectrum of L on X meets the right half-plane, $\{Re\lambda > 0\}$.
(iv) $F : X \to Z$ is continuous and $\exists\ \rho_0 > 0$, $C_3 > 0$, $\alpha > 1$ such that $\|F(u)\|_Z < C_3\|u\|_X^\alpha$, for $\|u\|_X < \rho_0$.
Then the zero solution of (3.1) is nonlinearly unstable in the space X.

In this paper, we are going to use Lemma 3.1 for proof of the theorem.

Definition 3.1 *A traveling wave solution $\varphi(x-ct)$ of the equation (1.1) is said to be nonlinearly unstable in the space X, if there exist positive ε_0 and C_0, a sequence $\{u_n\}$ of solutions of the equation (1.1), and a sequence of time $t_n > 0$ such that $\|u_n(0) - \varphi(x)\|_X \to 0$ but $\|u_n(t_n) - \varphi(\cdot - ct_n)\|_X \geq \varepsilon_0$.*

If $\varphi(x - ct) \in H^2(R)$ is a traveling wave solution of the equation (1.1), then letting $w(x,t) = u(x,t) - \varphi(x - ct)$, we have

$$\begin{aligned}(w+\varphi)_t + \partial_x^4(w+\varphi) &= \partial_x^2[(w+\varphi)^3 - (w+\varphi)] + a(1-(w+\varphi)^2)\\ &= \partial_x^2[w^3 + 3w\varphi^2 + 3w^2\varphi + \varphi^3 - w - \varphi]\\ &\quad +a(1 - w^2 - \varphi^2 - 2w\varphi),\end{aligned}$$

that is

$$w_t + \partial_x^4 w = \partial_x^2[w^3 + 3w\varphi^2 + 3w^2\varphi - w] + a(-w^2 - 2w\varphi),$$

i.e.

$$w_t + \partial_x^4 w - (3\varphi^2 - 1)\partial_x^2 w - 12\varphi\varphi'\partial_x w - (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)w = F(w) \tag{3.2}$$

where

$$F(w) = (3\varphi'' - a)w^2 + 12\varphi' w\partial_x w + 6\varphi w\partial_x^2 w + 6\varphi(\partial_x w)^2 + 6w(\partial_x w)^2 + 3w^2\partial_x^2 w,$$

with initial value

$$w(x,0) = w_0(x) \equiv u_0(x) - \varphi(x). \tag{3.3}$$

So the stability of traveling wave solutions of (1.1) is translated into the stability of the zero solution of (3.2). In order to prove Theorem, taking $Z = L^2(R)$, $X = H^2(R)$, we need to prove that the four conditions of Lemma 3.1 are satisfied by the associated equation (3.2). The condition (i) is satisfied, by our choice of Z and X.

Denote the linear partial differential operator in (3.2) by $L = -(\partial_x^4 + \partial_x^2) + [3\varphi^2\partial_x^2 + 12\varphi\varphi'\partial_x + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)] = L_0 + [3\varphi^2\partial_x^2 + 12\varphi\varphi'\partial_x + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)]$ with $L_0 = -(\partial_x^4 + \partial_x^2)$. Then (3.2) may be rewritten in the form (3.1)

$$w_t = Lw + F(w),$$

Note that F maps $H^2(R)$ into $L^2(R)$, using Sobolev embedding theorem, we have

$$\|F(w)\|_{L^2} \le C\|w\|_{H^2}^2, \quad C > 0, \quad \text{for } \|w\|_{H^2} < 1. \tag{3.4}$$

So, the condition (iv) is satisfied.

To prove condition (ii) in Lemma 3.1, we need the following two lemmas.

Lemma 3.2 *Let $L_0 = -(\partial_x^4 + \partial_x^2)$. Then*

$$\|e^{tL_0}\|_{H^m \to H^m} \le e^{t/4} \quad \textit{for } m \in R^+,\ 0 \le t < \infty, \tag{3.5}$$

$$\|e^{tL_0}\|_{L^2 \to H^2} \le a(t) \equiv 5t^{-1/4} \quad \textit{for } 0 < t \le 1. \tag{3.6}$$

Proof. We write $u(x,t) = e^{tL_0}u_0(x)$. By Fourier transformation,

$$\hat{u}(\xi,t) = e^{-t(\xi^4 - \xi^2)}\hat{u}_0(\xi).$$

$$\begin{aligned}
\|u\|_{H^m}^2 &\equiv \int_{-\infty}^{\infty} (1+\xi^2)^m |\hat{u}(\xi,t)|^2 d\xi \\
&= \int_{-\infty}^{\infty} (1+\xi^2)^m e^{-2t(\xi^4-\xi^2)} |\hat{u}_0(\xi)|^2 d\xi \\
&\le \sup_{\xi\in R} e^{-2t(\xi^4-\xi^2)} \int_{-\infty}^{\infty} (1+\xi^2)^m |\hat{u}_0(\xi)|^2 d\xi \\
&= e^{t/2}\|u_0\|_{H^m}^2.
\end{aligned}$$

Hence

$$\|e^{tL_0}\|_{H^m \to H^m} \le e^{t/4}.$$

On the other hand, letting $s = \xi^2$, we have

$$\|u\|_{H^2}^2 \le \sup_{s\in R^+} f(s) \int_{-\infty}^{\infty} |\hat{u}_0(\xi)|^2 d\xi$$

with $f(s) = (1+s)^2 e^{-2t(s^2-s)}$, $t > 0$. Elementary computation shows that

$$\sup_{s>0} f(s) \le (\frac{3}{2} + \frac{1}{\sqrt{2}} t^{-1/2}) e^{t/2}.$$

Thus

$$\|u(x,t)\|_{H^2} \le (\frac{3}{2} + \frac{1}{\sqrt{2}}t^{-1/2})^{1/2} e^{t/4} \|u_0\|_{L^2}$$

and

$$\|e^{tL_0}\|_{L^2\to H^2} \le (\frac{3}{2} + \frac{1}{\sqrt{2}}t^{-1/2})^{1/2} e^{t/4} \le 5t^{-1/4} \text{ for } 0 < t \le 1,$$

since $e^{t/4} \le e^{1/4} < 2$. Thus Lemma 3.2 has been proved.

Lemma 3.3 *Let* $L = -(\partial_x^4 + \partial_x^2) + [3\varphi^2\partial_x^2 + 12\varphi\varphi'\partial_x + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)] = L_0 + [3\varphi^2\partial_x^2 + 12\varphi\varphi'\partial_x + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)]$ *with* $\varphi \in L^\infty(R)$, $\varphi' \in L^\infty(R)$, $\varphi'' \in L^\infty(R)$. *Then*

$$\|e^{tL}\|_{L^2\to H^2} \le C_1 t^{-1/4} \quad \textit{for } \ 0 < t \le 1, \tag{3.7}$$

$$\|e^{tL}\|_{H^2\to H^2} \le C_2 < \infty \quad \textit{for } \ 0 < t \le 1. \tag{3.8}$$

Proof. Consider the initial value problem

$$u_t = Lu = L_0 u + 3\varphi^2\partial_x^2 u + 12\varphi\varphi'\partial_x u + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)u,$$

$$u(x,0) = u_0(x).$$

Then $u(x,t) = e^{tL}u_0(x)$, $t \ge 0$, $x \in R$. Thus

$$u(x,t) = e^{tL_0}u_0 + \int_0^t e^{(t-\tau)L_0}[3\varphi^2\partial_x^2 u + 12\varphi\varphi'\partial_x u + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)u]d\tau.$$

Denote $A = \|\varphi\|_{L^\infty}$, $B = \|\varphi'\|_{L^\infty}$, $C = \|\varphi''\|_{L^\infty}$ and $M = 3A^2 + 12AB + 6AC + 6B^2 + 2aA$.

$$\begin{aligned}
& \|u(t)\|_{H^2} \\
\le\ & \|e^{tL_0}\|_{L^2\to H^2}\|u_0\|_{L^2} + \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2} 3\|\varphi\|_{L^\infty}^2 \|\partial_x^2 u\|_{L^2} d\tau \\
& + \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2} 12\|\varphi\|_{L^\infty}\|\varphi'\|_{L^\infty}\|\partial_x u\|_{L^2} d\tau \\
& + \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2} 6\|\varphi\|_{L^\infty}\|\varphi''\|_{L^\infty}\|u\|_{L^2} d\tau \\
& + \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2} 6\|\varphi'\|_{L^\infty}^2\|u\|_{L^2} d\tau \\
& + \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2} 2a\|\varphi\|_{L^\infty}\|u\|_{L^2} d\tau \\
\le\ & a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau)\|u(\tau)\|_{H^2} d\tau,
\end{aligned} \tag{3.9}$$

where $a(t)$ is defined in Lemma 3.2 and we use $u(t)$ to denote $u(\cdot, t)$.

By iteration,

$$\begin{aligned}\|u(t)\|_{H^2} \le\ & a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau) \\ & [a(\tau)\|u_0\|_{L^2} + M\int_0^\tau a(\tau-s)\|u(s)\|_{H^2}ds]d\tau \\ =\ & a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}d\tau \\ & +M^2\int_0^t\int_0^\tau a(t-\tau)a(\tau-s)\|u(s)\|_{H^2}dsd\tau.\end{aligned} \tag{3.10}$$

The second term on the right of (3.10) is

$$\begin{aligned}& M\int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}d\tau \\ =\ & M\|u_0\|_{L^2}\int_0^t 5(t-\tau)^{-1/4}5\tau^{-1/4}d\tau \\ =\ & 25M\|u_0\|_{L^2}\int_0^t t^{-1/2}\left(1-\frac{\tau}{t}\right)^{-1/4}\left(\frac{\tau}{t}\right)^{-1/4}d\tau \\ =\ & 25MC_3t^{1/2}\|u_0\|_{L^2}, \quad \text{for } 0 < t \le 1,\end{aligned} \tag{3.11}$$

where $C_3 = \int_0^1 (1-r)^{-1/4}r^{-1/4}dr$. By exchanging the order of integration, we get from the third term on the right side of (3.10),

$$\int_0^t\int_0^\tau a(t-\tau)a(\tau-s)\|u(s)\|_{H^2}dsd\tau = \int_0^t\left[\int_s^t a(t-\tau)a(\tau-s)d\tau\right]\|u(s)\|_{H^2}ds.$$

Now

$$\begin{aligned}\int_s^t a(t-\tau)a(\tau-s)d\tau =\ & 25\int_s^t (t-\tau)^{-1/4}(\tau-s)^{-1/4}d\tau \\ =\ & 25C_3(t-s)^{1/2} \le 25C_3, \ \text{for } 0 < s \le t \le 1.\end{aligned} \tag{3.12}$$

Therefore (3.9)-(3.12) imply

$$\begin{aligned}\|u(t)\|_{H^2} \le\ & [a(t) + 25C_3M]\|u_0\|_{L^2} \\ & +25C_3M^2\int_0^t \|u(s)\|_{H^2}ds, \quad \text{for } 0 < t \le 1.\end{aligned} \tag{3.13}$$

Let $v(t) = \int_0^t \|u(s)\|_{H^2}ds$. Then

$$\frac{dv(t)}{dt} \le [a(t) + 25C_3M]\|u_0\|_{L^2} + 25C_3M^2v(t), \quad \text{for } 0 < t \le 1.$$

Multiplying both sides of the above inequality by $e^{-25C_3M^2t}$, we have

$$\frac{d(e^{-25C_3M^2t}v(t))}{dt} \le e^{-25C_3M^2t}[a(t) + 25C_3M]\|u_0\|_{L^2}, \quad \text{for } 0 < t \le 1.$$

Integrating the above inequality with respect to t over $(0, t)$, we obtain

$$e^{-25C_3M^2t}v(t) \le \int_0^t e^{-25C_3M^2s}[a(s) + 25C_3M]ds\|u_0\|_{L^2},$$

that is

$$v(t) \le e^{25C_3M^2t}\int_0^t e^{-25C_3M^2s}[a(s) + 25C_3M]ds\|u_0\|_{L^2}.$$

Observing that $v(t) = \displaystyle\int_0^t \|u(s)\|_{H^2}ds$, and substituting above inequality into (3.13), we get

$$\|u(t)\|_{H^2} \le C_1t^{-1/4}\|u_0\|_{L^2} \ \text{ for } 0 < t \le 1,\ C_1 > 0. \tag{3.14}$$

Thus (3.7) has been proven. To prove (3.8), replacing the first term on the right side of (3.9) by $\|e^{tL_0}\|_{H^2\to H^2}\|u_0\|_{H^2}$ and using (3.5), we have

$$\|u(t)\|_{H^2} \le e^{t/4}\|u_0\|_{H^2} + M\int_0^t a(t-\tau)\|u(\tau)\|_{H^2}d\tau \ \text{ for } 0 < t \le 1. \tag{3.15}$$

Similarly iterating and computing as above, we obtain

$$\|u(t)\|_{H^2} \le [2 + 25C_3M]\exp[25C_3M^2]\|u_0\|_{H^2} \equiv C_2\|u_0\|_{H^2}. \tag{3.16}$$

Hence (3.8) is proven and proof of Lemma 3.3 is finished. By Lemma 3.3 condition (ii) is proved.

We now proceed to verify condition (iii) of Lemma 3.1. Observing that if $u(x, t)$ satisfies

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + 3\varphi^2\frac{\partial^2 u}{\partial x^2} + 12\varphi\varphi'\frac{\partial u}{\partial x} + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)u$$

then $u(x, s+t)$ also satisfies the above equation. By uniqueness of solution, we know that L generates a strongly continuous semigroup on the Banach space $H^2(R)$ (see [6] p.344). By Fourier transformation, the essential spectrum of L_0 on $H^2(R)$ is

$$\sigma(L_0) \supset \{-\xi^4 + \xi^2|\ \ \xi \in R\}.$$

The curve $\lambda = -\xi^4 + \xi^2$ meets the vertical lines $Re\lambda = \alpha$ for $-\infty < \alpha \le 1/4$ because $-\infty < -\xi^4 + \xi^2 \le 1/4$.

We now prove that the same curve belongs to the essential spectrum of L.

Lemma 3.4 *The essential spectrum of L on $H^2(R)$ contains that of L_0.*

Proof. Let $\xi \in R$ and let $\lambda = P(\xi) = -\xi^4 + \xi^2$. Following Schechter [7], $\lambda \in \sigma(L)$ if there exists a sequence $\{\xi_n\} \subset H^2(R)$ with

$$\|\xi_n\|_{H^2} = 1, \qquad \|(L-\lambda)\xi_n\|_{H^2} \to 0,$$

and $\{\xi_n\}$ does not have a strongly convergent subsequence in $H^2(R)$. (here we use the definition: $\lambda \notin \sigma(L)$ if and only if $L-\lambda$ is Fredholm with index zero.) Now let $\xi_0 \not\equiv 0$ be a C^∞ function with compact support in $(0,\infty)$. Define

$$\xi_n(x) = c_n e^{i\xi x}\xi_0(x/n)/\sqrt{n}, \quad n = 1, 2, \cdots,$$

where c_n is chosen so that $\|\xi_n\|_{H^2} = 1$. In fact,

$$\|\xi_n\|_{L^2} = c_n\|\xi_0\|_{L^2} \quad \text{and} \quad 1 = \|\xi_n\|_{H^2} \le kc_n$$

for some positive constant k. Hence $c_n \ge 1/k > 0$. Since $\|\xi_n\|_{L^\infty} \to 0$ but $\|\xi_n\|_{L^2}$ is bounded away from zero, $\{\xi_n\}$ can have no convergent subsequence in $L^2(R)$.

It remains to show that $\|(L-\lambda)\xi_n\|_{H^2} \to 0$. We write

$$L - \lambda = L_0 - \lambda + 3\varphi^2\partial_x^2 + 12\varphi\varphi'\partial_x + (6\varphi\varphi'' + 6\varphi'^2 - 2a\varphi)$$

A simple calculation shows that

$$(L_0-\lambda)\xi_n(x) = e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\xi_0^{(s)}(\frac{x}{n})/(s!n^{1/2+s}),$$

$$\partial(L_0-\lambda)\xi_n(x) = i\xi(L_0-\lambda)\xi_n(x) + e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\xi_0^{(s+1)}(\frac{x}{n})/(s!n^{3/2+s}),$$

and

$$\begin{aligned}
&\partial^2(L_0-\lambda)\xi_n(x)\\
=\;& -\xi^2(L_0-\lambda)\xi_n(x) + 2i\xi e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\xi_0^{(s+1)}(\frac{x}{n})/(s!n^{3/2+s})\\
&+e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\xi_0^{(s+2)}(\frac{x}{n})/(s!n^{5/2+s}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\|(L_0-\lambda)\xi_n(x)\|_{H^2}\\
\le\;& (1+|\xi|^2)\sum_{1\le s\le 4}|P^{(s)}(\xi)|c_n\|\xi_0^{(s)}(\frac{x}{n})\|_{L^2}/(s!n^{1/2+s})\\
&+2|\xi|\sum_{1\le s\le 4}|P^{(s)}(\xi)|c_n\|\xi_0^{(s+1)}(\frac{x}{n})\|_{L^2}/(s!n^{3/2+s})\\
&+\sum_{1\le s\le 4}|P^{(s)}(\xi)|c_n\|\xi_0^{(s+2)}(\frac{x}{n})\|_{L^2}/(s!n^{5/2+s}) \to 0, \text{ as } n\to\infty.
\end{aligned}$$

Moreover, for any positive integer m, $\|\partial_x^m\xi_n\|_{L^\infty} \to 0$ as $n\to\infty$, so we have

$$\|3\varphi^2\partial_x^2\xi_n\|_{L^2}^2 \le \|\partial_x^2\xi_n\|_{L^\infty}^2\|3\varphi^2\|_{L^\infty}^2 \to 0,$$

$$\|\partial_x[3\varphi^2\partial_x^2\xi_n]\|_{L^2}^2 \le \|\partial_x^3\xi_n\|_{L^\infty}^2\|3\varphi^2\|_{L^2}^2 + \|\partial_x^2\xi_n\|_{L^\infty}^2\|6\varphi\varphi'\|_{L^\infty}^2 \to 0,$$

and

$$\begin{aligned}
&\|\partial_x^2[3\varphi^2\partial_x^2\xi_n]\|_{L^2}^2\\
\leq\ &\|\partial_x^3\xi_n\|_{L^\infty}^2\|12\varphi\varphi'\|_{L^2}^2+\|\partial_x^2\xi_n\|_{L^\infty}^2\|6\varphi\varphi''+6\varphi'^2\|_{L^2}^2+\|\partial_x^4\xi_n\|_{L^\infty}^2\|3\varphi^2\|_{L^\infty}^2\to 0.
\end{aligned}$$

From the assumptions on φ, we obtain

$$\|12\varphi\varphi'\partial_x\xi_n\|_{L^2}^2\leq\|\partial_x\xi_n\|_{L^\infty}^2\|12\varphi\varphi'\|_{L^2}^2\to 0,$$

$$\|\partial_x[12\varphi\varphi'\partial_x\xi_n]\|_{L^2}^2\leq\|\partial_x^2\xi_n\|_{L^\infty}^2\|12\varphi\varphi'\|_{L^2}^2+\|\partial_x\xi_n\|_{L^\infty}^2\|12\varphi\varphi''+12\varphi'^2\|_{L^2}^2\to 0,$$

and

$$\begin{aligned}
\|\partial_x^2[12\varphi\varphi'\partial_x\xi_n]\|_{L^2}^2\leq\ &\|\partial_x^3\xi_n\|_{L^\infty}^2\|12\varphi\varphi'\|_{L^2}^2+\|\partial_x^2\xi_n\|_{L^\infty}^2\|24\varphi\varphi''+24\varphi'^2\|_{L^2}^2\\
&+\|\partial_x\xi_n\|_{L^\infty}^2\|36\varphi'\varphi''+12\varphi\varphi'''\|_{L^2}^2\to 0.
\end{aligned}$$

In addition,

$$\|(6\varphi\varphi''+6\varphi'^2-2a\varphi)\xi_n\|_{L^2}^2\leq\|\xi_n\|_{L^\infty}^2\|6\varphi\varphi''+6\varphi'^2-2a\varphi\|_{L^2}^2\to 0,$$

$$\begin{aligned}
&\|\partial_x[(6\varphi\varphi''+6\varphi'^2-2a\varphi)\xi_n]\|_{L^2}^2\\
\leq\ &\|\partial_x\xi_n\|_{L^\infty}^2\|6\varphi\varphi''+6\varphi'^2-2a\varphi\|_{L^2}^2+\|\xi_n\|_{L^\infty}^2\|18\varphi'\varphi''+6\varphi\varphi'''-2a\varphi'\|_{L^2}^2\to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_x^2[(6\varphi\varphi''+6\varphi'^2-2a\varphi)\xi_n]\|_{L^2}^2\\
\leq\ &\|\partial_x^2\xi_n\|_{L^\infty}^2\|6\varphi\varphi''+6\varphi'^2-2a\varphi\|_{L^2}^2+2\|\partial_x\xi_n\|_{L^\infty}^2\|18\varphi'\varphi''+6\varphi\varphi'''-2a\varphi'\|_{L^2}^2\\
&+\|\xi_n\|_{L^\infty}^2\|24\varphi'\varphi'''+18\varphi''^2+6\varphi\varphi''''-2a\varphi''\|_{L^2}^2\to 0.
\end{aligned}$$

Thus

$$\|3\varphi^2\partial_x^2\xi_n+12\varphi\varphi'\partial_x\xi_n+(6\varphi\varphi''+6\varphi'^2-2a\varphi)\xi_n\|_{H^2}\to 0 \text{ as } n\to\infty.$$

So from the estimates above,

$$\|(L-\lambda)\xi_n\|_{H^2}\to 0,\quad \text{as } n\to\infty.$$

The proof of Lemma 3.4 is completed.

Therefore all the four conditions of Lemma 3.1 are satisfied by the linearized equation (3.2) and theorem 1.1 has been proved.

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