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Asymptotic behavior for minimizers of a p-energy functional associated with p-harmonic maps

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Abstract The author studies the asymptotic behavior of minimizers u_ε of a p-energy functional with penalization as $\varepsilon \to 0$. Several kinds of convergence for the minimizer to the p-harmonic map are presented under different assumptions.
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1 Introduction

Let $G \subset R^2$ be a bounded and simply connected domain with smooth boundary ∂G, and $B_1 = \{x \in R^2$ or the complex plane $C; x_1^2 + x_2^2 < 1\}$. Denote $S^1 = \{x \in R^3; x_1^2 + x_2^2 = 1, x_3 = 0\}$ and $S^2 = \{x \in R^3; x_1^2 + x_2^2 + x_3^2 = 1\}$. The vector value function can be denoted as $u = (u_1, u_2, u_3) = (u', u_3)$. Let $g = (g', 0)$ be a smooth map from ∂G into S^1. Recall that the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 dx + \frac{1}{2\varepsilon^2}\int_G u_3^2 dx$$

with a small parameter $\varepsilon > 0$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planner ferromagnets and antiferromagnets (see [9] and [12]). The asymptotic behavior of minimizers of $E_\varepsilon(u)$ had been studied by Fengbo Hang and Fanghua Lin in [7]. When the term $\frac{u_3^2}{2\varepsilon^2}$ replaced by $\frac{(1-|u|^2)^2}{4\varepsilon^2}$ and S^2 replaced by R^2, the problem becomes the simplified model of the Ginzburg-Landau theory for superconductors and was well studied in many papers such as [1][2] and [13]. These works show that the properties of harmonic map with S^1-value can be studied via researching the minimizers of the functional with some penalization terms. Indeed, Y.Chen and M.Struwe used the penalty method to establish the global existence of partial regular weak solutions of the harmonic map flow (see [4] and [6]). M.Misawa studied the p-harmonic maps by using the same idea of the penalty method in

[11]. Now, the functional

$$E_\varepsilon(u,G)=\frac{1}{p}\int_G|\nabla u|^p dx+\frac{1}{2\varepsilon^p}\int_G u_3^2dx,\quad p>2,$$

which equipped with the penalization $\frac{1}{2\varepsilon^p}\int_G u_3^2dx$, will be considered in this paper. From the direct method in the calculus of variations, it is easy to see that the functional achieves its minimum in the function class $W_g^{1,p}(G,S^2)$. Without loss of generality, we assume $u_3\geq 0$, otherwise we may consider $|u_3|$ in view of the expression of the functional. We will research the asymptotic properties of minimizers of this p-energy functional on $W_g^{1,p}(G,S^2)$ as $\varepsilon\to 0$, and shall prove the limit of the minimizers is the p-harmonic map.

Theorem 1.1 *Let u_ε be a minimizer of $E_\varepsilon(u,G)$ on $W_g^{1,p}(G,S^2)$. Assume* $\deg(g',\partial G)=0$. *Then*

$$\lim_{\varepsilon\to 0}u_\varepsilon=(u_p,0),\quad in\quad W^{1,p}(G,S^2),$$

where u_p is the minimizer of $\int_G|\nabla u|^pdx$ in $W_g^{1,p}(G,\partial B_1)$.

Remark. When $p=2$, [7] shows that if $\deg(g',\partial G)=0$, the minimizer of $E_\varepsilon(u)$ in $H_g^1(G,S^2)$ is just $(u_2,0)$, where u_2 is the energy minimizer, i.e., it is the minimizer of $\int_G|\nabla u|^2dx$ in $H_g^1(G,\partial B_1)$. When $p>2$, there may be several minimizers of $E_\varepsilon(u,G)$ in $W_g^{1,p}(G,S^2)$. The author proved that there exists a minimizer, which is called the regularized minimizer, is just $(u_p,0)$, where u_p is the minimizer of $\int_G|\nabla u|^pdx$ in $W_g^{1,p}(G,\partial B_1)$. For the other minimizers, we only deduced the result as Theorem 1.1.

Comparing with the assumption of Theorem 1.1, we will consider the problem under some weaker conditions. Then we have

Theorem 1.2 *Assume u_ε is a critical point of $E_\varepsilon(u,G)$ on $W_g^{1,p}(G,S^2)$. If*

$$E_\varepsilon(u_\varepsilon,K)\leq C \tag{1.1}$$

for some subdomain $K\subseteq G$. Then there exists a subsequence u_{ε_k} of u_ε such that as $k\to\infty$,

$$u_{\varepsilon_k}\to(u_p,0),\quad weakly\quad in\quad W^{1,p}(K,R^3), \tag{1.2}$$

where u_p is a critical point of $\int_K|\nabla u|^pdx$ in $W^{1,p}(K,\partial B_1)$, which is named p-harmonic map on K. Moreover, for any $\zeta\in C_0^\infty(K)$, when $\varepsilon\to 0$,

$$\int_K|\nabla u_{\varepsilon_k}|^p\zeta dx\to\int_K|\nabla u_p|^p\zeta dx, \tag{1.3}$$

$$\frac{1}{\varepsilon_k^p}\int_K u_{\varepsilon_k 3}\zeta dx \to 0. \tag{1.4}$$

The convergent rate of $|u'_\varepsilon| \to 1$ and $u_3 \to 0$ will be concerned with as $\varepsilon \to 0$.

Theorem 1.3 *Let u_ε be a minimizer of $E_\varepsilon(u,G)$ on $W_g^{1,p}(G,S^2)$. If (1.1) holds, then there exists a positive constant C, such that as $\varepsilon \to 0$,*

$$\int_K |\nabla|u'_\varepsilon||^p dx + \int_K |\nabla u_{\varepsilon 3}|^p dx + \frac{1}{\varepsilon^p}\int_K u_{\varepsilon 3}^2 dx \le C\varepsilon^\beta,$$

where $\beta = 1-\frac{2}{p}$ when $p \in (2,p_0]$; $\beta = \frac{2p}{p^2-2}$ when $p > p_0$. Here $p_0 \in (4,5)$ is a constant satisfying $p^3-4p^2-2p+4=0$.

2 Proof of Theorem 1.1

In this section, we always assume $deg(g',\partial G)=0$. By the argument of the weak low semi-continuity, it is easy to deduce the strong convergence in $W^{1,p}$ sense for some subsequence of the minimizer u_ε. To improve the conclusion of the convergence for all u_ε, we need to research the limit function: p-harmonic map.

From $deg(g',\partial G)=0$ and the smoothness of ∂G and g, we see that there is a smooth function $\phi_0 : \partial G \to R$ such that

$$g = e^{i\phi_0}, \quad on \quad \partial G. \tag{2.1}$$

Consider the Dirichlet problem

$$-div(|\nabla\Phi|^{p-2}\nabla\Phi) = 0, \quad in \quad G, \tag{2.2}$$

$$\Phi|_{\partial G} = \phi_0. \tag{2.3}$$

Proposition 2.1 *There exists the unique weak solution Φ of (2.2) and (2.3) in $W^{1,p}(G,R)$. Namely, for any $\phi \in W_0^{1,p}(G,R)$, there is the unique Φ satisfies*

$$\int_G |\nabla\Phi|^{p-2}\nabla\Phi\nabla\phi dx = 0 \tag{2.4}$$

Proof. By using the method in the calculus of variations, we can see the existence for the weak solution of (2.2) and (2.3).

If both Φ_1 and Φ_2 are weak solutions of (2.2) and (2.3), then, by taking the test function $\phi = \Phi_1 - \Phi_2$ in (2.4), there holds

$$\int_G (|\nabla\Phi_1|^{p-2}\nabla\Phi_1 - |\nabla\Phi_2|^{p-2}\nabla\Phi_2)\nabla(\Phi_1-\Phi_2)dx = 0.$$

In view of Lemma 1.2 in [5] we have

$$\int_G |\nabla(\Phi_1 - \Phi_2)|^p dx \le 0.$$

Hence, $\Phi_1 - \Phi_2 = Const.$ on $\overline{G}$. Noting the boundary condition, we see $\Phi_1 - \Phi_2 = 0$ on $\overline{G}$. Proposition is proved.

Recall that $u \in W_g^{1,p}(G, \partial B_1)$ is named p-harmonic map, if it is the critical point of $\int_G |\nabla u|^p dx$. Namely, it is the weak solution of

$$-div(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p \tag{2.5}$$

on G, or for any $\phi \in C_0^\infty(G, R^2 \ or \ C)$, it satisfies

$$\int_G |\nabla u|^{p-2}\nabla u \nabla \phi dx = \int_G u|\nabla u|^p \phi dx. \tag{2.6}$$

Assume Φ is the unique weak solution of (2.2) and (2.3). Set

$$u_p = e^{i\Phi}, \quad on \quad \overline{G}. \tag{2.7}$$

Proposition 2.2 *u_p defined in (2.7) is a p-harmonic map on G.*

Proof. Obviously, $u_p \in W_g^{1,p}(G, \partial B_1)$ since $\Phi \in W_{\phi_0}^{1,p}(G, R)$. We only need to prove that u_p satisfies (2.6) for any $\phi \in C_0^\infty(G, C)$. In fact,

$$\begin{aligned} &\textstyle\int_G (|\nabla u_p|^{p-2}\nabla u_p \nabla \phi - u_p \phi |\nabla u_p|^p) dx \\ =& \textstyle i\int_G |\nabla \Phi|^{p-2}\nabla \Phi(e^{i\Phi}\nabla\phi + ie^{i\Phi}\nabla\Phi\phi)dx = i\int_G |\nabla\Phi|^{p-2}\nabla\Phi\nabla(e^{i\Phi}\phi)dx \end{aligned}$$

for any $\phi \in C_0^\infty(G, C)$. Noting $e^{i\Phi}\phi \in W_0^{1,p}(G, C)$ and Φ is the weak solution of (2.2) and (2.3), we obtain

$$\int_G |\nabla u_p|^{p-2}\nabla u_p \nabla\phi dx - \int_G u_p \phi |\nabla u_p|^p dx = 0$$

for any $\phi \in C_0^\infty(G, C)$. Proposition is proved.

Since $W_g^{1,p}(G, \partial B_1) \ne \emptyset$ when $\deg(g', \partial G) = 0$, we may consider the minimization problem

$$Min\{\int_G |\nabla u|^p dx; u \in W_g^{1,p}(G, \partial B_1)\} \tag{2.8}$$

The solution is called p-energy minimizer.

Proposition 2.3 *The solution of (2.8) exists.*

Proof. The weakly low semi-continuity of $\int_G |\nabla u|^p dx$ is well-known. On the other hand, if taking a minimizing sequence u_k of $\int_G |\nabla u|^p dx$ in $W_g^{1,p}(G, \partial B_1)$, then there is a subsequence of u_k, which is still denoted u_k itself, such that as $k \to \infty$, u_k converges to u_0 weakly in $W^{1,p}(G, C)$. Noting that $W_g^{1,p}(G, \partial B_1)$ is the weakly closed subset of $W^{1,p}(G, C)$ since it is the convex closed subset, we see that $u_0 \in W_g^{1,p}(G, \partial B_1)$. Thus, if denote

$$\alpha = Inf\{\int_G |\nabla u|^p dx; u \in W_g^{1,p}(G, \partial B_1)\},$$

then

$$\alpha \le \int_G |\nabla u_0|^p dx \le \underline{\lim}_{k\to\infty} \int_G |\nabla u_k|^p dx \le \alpha.$$

This means u_0 is the solution of (2.8).

Obviously, the p-energy minimizer is the p-harmonic map.

Proposition 2.4 *The p-harmonic map is unique in* $W_g^{1,p}(G, \partial B_1)$.

Proof. It follows that $u_p = e^{i\Phi}$ is a p-harmonic map from Proposition 2.2. If u is also a p-harmonic map in $W_g^{1,p}(G, \partial B_1)$, then from $\deg(g', \partial G) = 0$ and using the results in [3], we know that there is $\Phi_0 \in W^{1,p}(G, R)$ such that

$$u = e^{i\Phi_0}, \quad on \quad \overline{G},$$

$$\Phi_0 = \phi_0, \quad on \quad \partial G.$$

Substituting these into (2.6), we see that Φ_0 is a weak solution of (2.2) and (2.3). Proposition 2.1 leads to $\Phi_0 = \Phi$, which implies $u = u_p$.

Now, we conclude that u_0 in Proposition 2.3 is just the p-harmonic map u_p. Furthermore, the p-energy minimizer is also unique in $W_g^{1,p}(G, \partial B_1)$.

Proof of Theorem 1.1. Noticing that u_ε is the minimizer, we have

$$E_\varepsilon(u_\varepsilon, G) \le E_\varepsilon((u_p, 0), G) \le C \tag{2.9}$$

with $C > 0$ independent of ε. This means

$$\int_G |\nabla u_\varepsilon|^p dx \le C, \tag{2.10}$$

$$\int_G u_{\varepsilon 3}^2 dx \le C\varepsilon^p. \tag{2.11}$$

Using (2.10), $|u_\varepsilon| = 1$ and the embedding theorem, we see that there exists a subsequence u_{ε_k} of u_ε and $u_* \in W^{1,p}(G, R^3)$, such that as $\varepsilon_k \to 0$,

$$u_{\varepsilon_k} \to u_*, \quad weakly \quad in \quad W^{1,p}(G, S^2), \tag{2.12}$$

$$u_{\varepsilon_k} \to u_*, \quad in \quad C^\alpha(\overline{G}, S^2), \quad \alpha \in (0, 1 - 2/p). \tag{2.13}$$

Obviously, (2.11) and (2.13) lead to $u_* \in W_g^{1,p}(G, S^1)$.

Applying (2.12) and the weak low semi-continuity of $\int_G |\nabla u|^p dx$, we have

$$\int_G |\nabla u_*|^p dx \leq \underline{\lim}_{\varepsilon_k \to 0} \int_G |\nabla u_{\varepsilon_k}|^p dx.$$

On the other hand, (2.9) implies

$$\int_G |\nabla u_{\varepsilon_k}|^p dx \leq \int_G |\nabla (u_p, 0)|^p dx,$$

hence,

$$\int_G |\nabla u'_*|^p dx \leq \int_G |\nabla u_p|^p dx.$$

This means that u'_* is also a p-energy minimizer. Noting the uniqueness we see $u_* = u_p$. Thus

$$\int_G |\nabla u_p|^p dx \leq \underline{\lim}_{\varepsilon_k \to 0} \int_G |\nabla u_{\varepsilon_k}|^p dx \leq \overline{\lim}_{\varepsilon_k \to 0} \int_G |\nabla u_{\varepsilon_k}|^p dx \leq \int_G |\nabla u_p|^p dx.$$

When $\varepsilon_k \to 0$,

$$\int_G |\nabla u_{\varepsilon_k}|^p \to \int_G |\nabla u_p|^p.$$

Combining this with (2.12) yields

$$\lim_{k \to \infty} \nabla u_{\varepsilon_k} = \nabla (u_p, 0), \quad in \quad L^p(G, S^2).$$

In addition, (2.13) implies that as $\varepsilon \to 0$,

$$u_{\varepsilon_k} \to (u_p, 0), \quad in \quad L^p(G, S^2).$$

Then

$$\lim_{k \to \infty} u_{\varepsilon_k} = (u_p, 0), \quad in \quad W^{1,p}(G, S^2).$$

Noticing the uniqueness of $(u_p, 0)$, we see the convergence above also holds for all u_ε.

3 Proof of Theorem 1.2

In this section, we always assume that u_ε is the critical point of the functional, and $E_\varepsilon(u_\varepsilon, K) \le C$ for some subdomain $K \subseteq G$, where C is independent of ε. The assumption is weaker than that of Theorem 1.1. So, all the results in this section will be derived in the weak sense.

The method in the calculus of variations shows that the minimizer $u_\varepsilon \in W_g^{1,p}(G, S^2)$ is a weak solution of

$$-div(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p + \frac{1}{\varepsilon^p}(uu_3^2 - u_3 e_3), \quad on \quad G, \tag{3.1}$$

where $e_3 = (0,0,1)$. Namely, for any $\psi \in W_0^{1,p}(G, R^3)$, u_ε satisfies

$$\int_G |\nabla u|^{p-2}\nabla u \nabla \psi dx = \int_G u\psi|\nabla u|^p dx + \frac{1}{\varepsilon^p}\int_G \psi(uu_3^2 - u_3 e_3)dx. \tag{3.2}$$

Proof of (1.2). $E_\varepsilon(u_\varepsilon, K) \le C$ means

$$\int_K |\nabla u_\varepsilon|^p dx \le C, \tag{3.3}$$

$$\int_K u_{\varepsilon 3}^2 dx \le C\varepsilon^p, \tag{3.4}$$

where C is independent of ε. Combining the fact $|u_\varepsilon| = 1$ a.e. on $\overline{G}$ with (3.3) we know that there exist $u_p \in W^{1,p}(K, \partial B_1)$ and a subsequence u_{ε_k} of u_ε, such that as $\varepsilon_k \to 0$,

$$u_{\varepsilon_k} \to (u_p, 0), \quad weakly \quad in \quad W^{1,p}(K), \tag{3.5}$$

$$u_{\varepsilon_k} \to (u_p, 0), \quad in \quad C^\alpha(\overline{K}), \tag{3.6}$$

for some $\alpha \in (0, 1 - \frac{2}{p})$. In the following we will prove that u_p is a weak solution of (2.5).

Let $B = B(x, 3R) \subset\subset K$. $\phi \in C_0^\infty(B(x,3R); [0,1]), \phi = 1$ on $B(x,R), \phi = 0$ on $B \setminus B(x, 2R)$ and $|\nabla\phi| \le C$, where C is independent of ε. Denote $u = u_{\varepsilon_k}$ in (3.2) and take $\psi = (0,0,\phi)$. Thus

$$\int_B |\nabla u|^{p-2}\nabla u_3 \nabla\phi dx + \frac{1}{\varepsilon_k^p}\int_B |u'|^2 \phi u_3 dx = \int_B u_3\phi|\nabla u|^p dx.$$

Applying (3.3) we can derive that

$$\frac{1}{\varepsilon_k^p}\int_B |u'|^2\phi|u_3|dx \le \int_B |\nabla u|^p \phi dx + \int_B |\nabla u|^{p-1}|\nabla\phi|dx \le C. \tag{3.7}$$

From (3.6) it follows $|u'| \geq 1/2$ when ε_k is sufficiently small. Noting $\phi = 1$ on $B(x,R)$, we have

$$\frac{1}{\varepsilon_k^p}\int_{B(x,R)} |u_3| dx \leq C. \tag{3.8}$$

Taking $\frac{1}{k} = \varepsilon_k, F_k = \frac{1}{\varepsilon_k^p}(u_{\varepsilon_k} u_{\varepsilon_k 3}^2 - u_{\varepsilon_k 3} e_3)$ in Lemma 3.11 of [8], noting $|F_k| = \frac{1}{\varepsilon_k^p}|u_3||u'|$ and applying (3.5) and (3.8) we obtain that for any $q \in (1,p)$, as $\varepsilon_k \to 0$, $\nabla u_{\varepsilon_k} \to \nabla u_p$, in $L^q(B(x,R))$. Since $B(x,R)$ is an arbitrary disc in K, we can see that as $\varepsilon_k \to 0$, for any $\xi \in C_0^\infty(B,R^3)$ there holds

$$\int_B |\nabla u_{\varepsilon_k}|^{p-2} \nabla u_{\varepsilon_k} \nabla \xi dx \to \int_B |\nabla u_p|^{p-2} \nabla u_p \nabla \xi dx. \tag{3.9}$$

Now, denote $u' = u'_{\varepsilon_k} = (u_1, u_2)$. Taking $\psi = (u_2, 0, 0)\zeta$ and $\psi = (0, u_1, 0)\zeta$ in (3.2), respectively, where $\zeta \in C_0^\infty(B,R)$, we have that for $m, j \in \{1,2\}$, and $m \neq j$,

$$\begin{aligned} &\tfrac{1}{\varepsilon_k^p}\textstyle\int_B u_3^2 u_m u_j \zeta dx + \int_B u_m u_j \zeta |\nabla u|^p dx \\ &= \textstyle\int_B |\nabla u|^{p-2} \nabla u_m \nabla u_j \zeta dx + \int_B u_j |\nabla u|^{p-2} \nabla u_m \nabla \zeta dx. \end{aligned}$$

One equation subtracts the other one, then

$$0 = \int_B |\nabla u|^{p-2} (u \wedge \nabla u) \nabla \zeta dx, \tag{3.10}$$

where $u \wedge \nabla u = u_1 \nabla u_2 - u_2 \nabla u_1$. On the other hand, since

$$\begin{aligned} &\textstyle\int_B u_2 |\nabla u|^{p-2} \nabla u_1 \nabla \zeta dx - \int_B u_{p2} |\nabla u_p|^{p-2} \nabla u_{p1} \nabla \zeta dx \\ = &\textstyle\int_B (|\nabla u|^{p-2} \nabla u_1 - |\nabla u_p|^{p-2} \nabla u_{p1}) u_{p2} \nabla \zeta dx \\ &+ \textstyle\int_B |\nabla u|^{p-2} \nabla u_1 \nabla \zeta (u_2 - u_{p2}) dx, \end{aligned}$$

we obtain that as $\varepsilon_k \to 0$,

$$\int_B u_2 |\nabla u|^{p-2} \nabla u_1 \nabla \zeta dx \to \int_B u_{p2} |\nabla u_p|^{p-2} \nabla u_{p1} \nabla \zeta dx \tag{3.11}$$

by using (3.3)(3.6) and (3.9). Similarly, we may also get that

$$\lim_{\varepsilon \to 0} \int_B u_1 |\nabla u|^{p-2} \nabla u_2 \nabla \zeta dx = \int_B u_{p1} |\nabla u_p|^{p-2} \nabla u_{p2} \nabla \zeta dx. \tag{3.12}$$

(3.12) subtracts (3.11), then

$$\lim_{\varepsilon \to 0} \int_B |\nabla u|^{p-2} (u \wedge \nabla u) \nabla \zeta dx = \int_B |\nabla u_p|^{p-2} (u_p \wedge \nabla u_p) \nabla \zeta dx.$$

Combining this with (3.10), we have

$$\int_B |\nabla u_p|^{p-2}(u_p \wedge \nabla u_p)\nabla\zeta dx = 0. \tag{3.13}$$

Let $u_* = u_{p1} + iu_{p2} : B \to C$. Thus

$$|\nabla u_*|^2 = |\nabla u_p|^2. \tag{3.14}$$

It is easy to see that $\overline{u_*}\nabla u_* = \nabla(|u_*|^2) + (u_* \wedge \nabla u_*)i = 0 + (u_* \wedge \nabla u_*)i$ since $|u_*|^2 = |u_{p1}|^2 + |u_{p2}|^2 = 1$. Substituting this into (3.13) yields

$$-i\int_B |\nabla u_*|^{p-2}\overline{u_*}\nabla u_*\nabla\zeta dx = 0$$

for any $\zeta \in C_0^\infty(B, R)$. Taking $\zeta = Re(u_*\phi_j)$ and $\zeta = Im(u_*\phi_j)$ $(j = 1, 2)$, respectively, where $\phi = (\phi_1, \phi_2) \in C_0^\infty(B, R^2)$, we can see that

$$\int_B |\nabla u_*|^{p-2}\overline{u_*}\nabla u_*\nabla Re(u_*\phi)dx + i\int_B |\nabla u_*|^{p-2}\overline{u_*}\nabla u_*\nabla Im(u_*\phi)dx = 0.$$

Namely

$$0 = \int_G |\nabla u_*|^{p-2}\overline{u_*}\nabla u_*\nabla(u_*\phi)dx.$$

Noting $\overline{u_*}\nabla u_* = -u_*\nabla\overline{u_*}$, we obtain

$$\begin{aligned} 0 &= \textstyle\int_B |\nabla u_*|^{p-2}\nabla u_*\nabla\phi dx - \int_B |\nabla u_*|^{p-2}u_*\nabla\overline{u_*}\nabla u_*\phi dx \\ &= \textstyle\int_B |\nabla u_*|^{p-2}\nabla u_*\nabla\phi dx - \int_B |\nabla u_*|^p u_*\phi dx := J \end{aligned}$$

By using (3.14) and $Re(J) = 0$, $Im(J) = 0$, we have

$$\int_B |\nabla u_p|^{p-2}\nabla u_{p1}\nabla\phi dx = \int_B |\nabla u_p|^p u_{p1}\phi dx \tag{3.15}$$

and

$$\int_B |\nabla u_p|^{p-2}\nabla u_{p2}\nabla\phi dx = \int_B |\nabla u_p|^p u_{p2}\phi dx.$$

Combining this with (3.15) yields that for any $\phi \in C_0^\infty(B, R^3)$,

$$\int_B |\nabla u_p|^{p-2}\nabla u_p\nabla\phi dx = \int_B |\nabla u_p|^p u_p\phi dx.$$

It shows that u_p is a weak solution of (2.5). (1.2) is completed.

Proof of (1.3). For simplification, denote $\varepsilon_k = \varepsilon$. From (3.3) and (3.6) it is deduced that as $\varepsilon \to 0$,

$$|\int_K u_3^2 \zeta |\nabla u|^p dx| \le \sup_K (1 - |u'|^2) \cdot \int_K |\nabla u|^p dx \to 0, \tag{3.16}$$

$$\begin{aligned} &|\textstyle\int_K u' u_p \zeta |\nabla u|^p dx - \int_K \zeta |\nabla u|^p dx| = |\int_K (u' u_p - u_p u_p)\zeta |\nabla u|^p dx| \\ &\le \sup_K |u' - u_p| \cdot |\textstyle\int_K u_p |\nabla u|^p dx| \to 0, \end{aligned} \tag{3.17}$$

and

$$\int_K (u - (u_p, 0))\zeta |\nabla u|^p dx \le \sup_K |u - (u_p, 0)| \cdot |\int_K u_p |\nabla u|^p dx| \to 0. \tag{3.18}$$

Similarly, (3.4) and (3.6) imply that as $\varepsilon \to 0$,

$$|\frac{1}{\varepsilon^p} \int_K u_3^2 \zeta dx - \frac{1}{\varepsilon^p} \int_K u_3^2 \zeta (1 - u_3^2) dx| \le \sup_K |1 - |u'|^2| \cdot \frac{1}{\varepsilon^p} |\int_K u_3^2 dx| \to 0 \tag{3.19}$$

and

$$|\frac{1}{\varepsilon^p} \int_K u_p \zeta u' u_3^2 dx - \frac{1}{\varepsilon^p} \int_K \zeta u_3^2 dx| \le \sup_K |u' - u_p| \cdot \frac{1}{\varepsilon^p} |\int_K u_p u_3^2 dx| \to 0. \tag{3.20}$$

Letting $\varepsilon \to 0$ in (3.2) we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} [\textstyle\int_K u\psi |\nabla u|^p dx + \frac{1}{\varepsilon^p} \int_K \psi (u u_3^2 - u_3 e_3) dx] \\ &= \textstyle\int_K |\nabla u_p|^{p-2} \nabla (u_p, 0) \nabla \psi dx = \int_G (u_p, 0)\psi |\nabla u_p|^p dx. \end{aligned} \tag{3.21}$$

Take $\psi = (0, 0, u_3 \zeta)$ where $\zeta \in C_0^\infty(K)$ we have

$$\lim_{\varepsilon \to 0} [\int_K u_3^2 \zeta |\nabla u|^p dx + \frac{1}{\varepsilon^p} \int_K u_3^2 \zeta (u_3^2 - 1) dx] = 0.$$

Combining this with (3.16) we derive

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_K u_3^2 \zeta (u_3^2 - 1) dx = 0.$$

Substituting this into (3.19) yields

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_K u_3^2 \zeta dx = 0. \tag{3.22}$$

Hence, as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon^p} |\int_K u u_3^2 \zeta dx| \le \frac{1}{\varepsilon^p} \int_K u_3^2 \zeta dx \to 0.$$

Thus, for any $\psi \in W_0^{1,p}(K, R^3)$, there holds

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_K u u_3^2 \psi dx = 0. \tag{3.23}$$

In addition, substituting (3.22) into (3.20) leads to

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_K u_p \zeta u' u_3^2 dx = 0. \tag{3.24}$$

Take $\psi = (u_p \zeta, 0)$ in (3.21) we have

$$\lim_{\varepsilon \to 0} [\int_K u' u_p \zeta |\nabla u|^p dx + \frac{1}{\varepsilon^p} \int_K u_p \zeta u' u_3^2 dx] = \int_K |\nabla u_p|^p \zeta dx,$$

which, together with (3.24), implies

$$\lim_{\varepsilon \to 0} \int_K u' u_p \zeta |\nabla u|^p dx = \int_K |\nabla u_p|^p \zeta dx.$$

Combining this with (3.17) we can see (1.3) at last.

Proof of (1.4). Obviously, (3.18) and (1.3) show that as $\varepsilon \to 0$,

$$\begin{aligned} &| \textstyle\int_K u |\nabla u|^p \psi dx - \int_K (u_p, 0) |\nabla u_p|^p \psi dx | \\ \le\, &| \textstyle\int_K (u - (u_p, 0)) |\nabla u|^p \psi dx | + | \int_K (u_p, 0)(|\nabla u|^p - |\nabla u_p|^p) \psi dx | \to 0. \end{aligned}$$

Substituting this and (3.23) into (3.21), we see that the left hand side of (3.21) becomes

$$\begin{aligned} &\lim_{\varepsilon \to 0} [\textstyle\int_K u \psi |\nabla u|^p dx + \frac{1}{\varepsilon^p} \int_K \psi (u u_3^2 - u_3 e_3) dx] \\ =\, &\textstyle\int_K (u_p, 0) |\nabla u_p|^p \psi dx - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_K \psi u_3 e_3 dx. \end{aligned}$$

Comparing this with the right hand side of (3.21), we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_K \psi u_3 e_3 dx = 0.$$

This is (1.4). Theorem 1.2 is proved.

4 A Preliminary Proposition

To present the convergent rate of $|u'_\varepsilon| \to 1$ and $u_{\varepsilon 3} \to 0$ in $W^{1,p}$ sense when $\varepsilon \to 0$, we need the following

Proposition 4.1 *Assume u_ε is a minimizer of $E_\varepsilon(u, G)$ on W. If $E_\varepsilon(u_\varepsilon, K) \le C$ for some domain $K \subseteq G$. Then there exists a positive constant C which is independent of $\varepsilon \in (0, 1)$, such that*

$$\frac{1}{p} \int_K |\nabla u_\varepsilon|^p dx + \frac{1}{\varepsilon^p} \int_K u_{\varepsilon 3}^2 dx \le C \varepsilon^{2/p} + \frac{1}{p} \int_K |\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}|^p dx. \tag{4.1}$$

Proof. Denote $w = \frac{u'_\varepsilon}{|u'_\varepsilon|}$. Choose $R > 0$ sufficiently small such that $\overline{B(x,3R)} \subset K$. It follows from (3.6) that

$$|u'_\varepsilon| \geq 1/2 \tag{4.2}$$

on $B(x,3R)$ as ε sufficiently small. This and (3.3) imply

$$\int_{B(x,3R)} |\nabla w|^p dx \leq 2^p \int_{B(x,3R)} |u'_\varepsilon|^p |\nabla w|^p dx \leq C \int_{B(x,3R)} |\nabla u_\varepsilon|^p dx \leq C. \tag{4.3}$$

Applying (1.1) and the integral mean value theorem, we know that there is a constant $r \in (2R, 3R)$ such that

$$\frac{1}{p} \int_{\partial B(x,r)} |\nabla u_\varepsilon|^p dx + \frac{1}{2\varepsilon^p} \int_{\partial B(x,r)} u_{\varepsilon 3}^2 dx = C_0(r) E_\varepsilon(u_\varepsilon, B_{3R} \setminus B_{2R}) \leq C. \tag{4.4}$$

Consider the functional

$$E(\rho, B) = \frac{1}{p} \int_B (|\nabla \rho|^2 + 1)^{p/2} dx + \frac{1}{2\varepsilon^p} \int_B (1-\rho)^2 dx,$$

where $B = B(x,r)$. It is easy to prove that the minimizer ρ_1 of $E(\rho, B)$ on $W^{1,p}_{|u'_\varepsilon|}(B, R^+ \cup \{0\})$ exists and solves

$$-div(v^{(p-2)/2} \nabla \rho) = \frac{1}{\varepsilon^p}(1-\rho) \quad on \quad B, \tag{4.5}$$

$$\rho|_{\partial B} = |u'_\varepsilon|, \tag{4.6}$$

where $v = |\nabla \rho|^2 + 1$. Since $1/2 < |u'_\varepsilon| \leq 1$, it follows from the maximum principle that on $\overline{B}$,

$$\frac{1}{2} < \rho_1 \leq 1. \tag{4.7}$$

Clearly, $(1 - |u'|)^2 \leq (1 - |u'|^2)^2 = u_3^4 \leq u_3^2$. Thus, by noting that ρ_1 is a minimizer, and applying (1.1) we see easily that

$$E(\rho_1, B) \leq E(|u'_\varepsilon|, B) \leq C E_\varepsilon(u_\varepsilon, B) \leq C. \tag{4.8}$$

Multiplying (4.5) by $(\nu \cdot \nabla \rho)$, where ρ denotes ρ_1, and integrating over B, we have

$$\begin{aligned} -\textstyle\int_{\partial B} v^{(p-2)/2} (\nu \cdot \nabla \rho)^2 d\xi \quad & + \textstyle\int_B v^{(n-2)/2} \nabla \rho \cdot \nabla(\nu \cdot \nabla \rho) dx \\ & = \tfrac{1}{\varepsilon^p} \textstyle\int_B (1-\rho)(\nu \cdot \nabla \rho) dx, \end{aligned} \tag{4.9}$$

where ν denotes the unit vector on $\overline{B}$, and it equals to the unit outside norm vector on ∂B.

Using (4.8) we obtain

$$\begin{aligned}
&|\int_B v^{(p-2)/2}\nabla\rho\cdot\nabla(\nu\cdot\nabla\rho)dx|\\
&\le C\int_B v^{(n-2)/2}|\nabla\rho|^2dx+\tfrac{1}{2}|\int_B v^{(p-2)/2}(\nu\cdot\nabla v)dx|\\
&\le C+\tfrac{1}{p}|\int_B \nu\cdot\nabla(v^{n/2})dx|\le C+\tfrac{1}{p}\int_B|div(v^{p/2}\nu)-v^{p/2}div\nu|dx\\
&\le C+\tfrac{1}{p}\int_{\partial B}v^{p/2}d\xi.
\end{aligned}$$

Combining (4.6), (4.4) and (4.8) we also have

$$\begin{aligned}
&|\tfrac{1}{\varepsilon^p}\int_B(1-\rho)(\nu\cdot\nabla\rho)dx|\le\tfrac{1}{2\varepsilon^p}|\int_B(1-\rho)^2div\nu dx-\int_{\partial B}(1-\rho)^2d\xi|\\
&\le\tfrac{1}{2\varepsilon^p}\int_B(1-\rho)^2|div\nu|dx+\tfrac{1}{2\varepsilon^p}\int_{\partial B}(1-\rho)^2d\xi\le C.
\end{aligned}$$

Substituting these into (4.9) yields

$$|\int_{\partial B}v^{(p-2)/2}(\nu\cdot\nabla\rho)^2d\xi|\le C+\frac{1}{p}\int_{\partial B}v^{p/2}d\xi. \tag{4.10}$$

Applying (4.6), (4.4) and (4.10), we obtain for any $\delta\in(0,1)$,

$$\begin{aligned}
&\int_{\partial B}v^{p/2}d\xi=\int_{\partial B}v^{(p-2)/2}[1+(\tau\cdot\nabla\rho)^2+(\nu\cdot\nabla\rho)^2]d\xi\\
&\le\int_{\partial B}v^{(p-2)/2}d\xi+\int_{\partial B}v^{(p-2)/2}(\nu\cdot\nabla\rho)^2d\xi\\
&+(\int_{\partial B}v^{p/2}d\xi)^{(p-2)/p}(\int_{\partial B}(\tau\cdot\nabla|u_\varepsilon'|)^pd\xi)^{2/p}\\
&\le C(\delta)+(\tfrac{1}{p}+2\delta)\int_{\partial B}v^{p/2}d\xi,
\end{aligned}$$

where τ denotes the unit tangent vector on ∂B. Hence it follows by choosing $\delta>0$ so small that

$$\int_{\partial B}v^{p/2}d\xi\le C. \tag{4.11}$$

Now we multiply both sides of (4.5) by $(1-\rho)$ and integrate over B. Then

$$\int_B v^{(p-2)/2}|\nabla\rho|^2dx+\frac{1}{\varepsilon^p}\int_B(1-\rho)^2dx=-\int_{\partial B}v^{(p-2)/2}(\nu\cdot\nabla\rho)(1-\rho)d\xi.$$

From this, using (4.4), (4.6), (4.7) and (4.11) we obtain

$$\begin{aligned}
&E(\rho_1,B)\le C|\int_{\partial B}v^{(p-2)/2}(\nu\cdot\nabla\rho)(1-\rho)d\xi|\\
&\le C|\int_{\partial B}v^{p/2}d\xi|^{(p-1)/p}|\int_{\partial B}(1-\rho)^2d\xi|^{1/p}\\
&\le C|\int_{\partial B}(1-|u_\varepsilon'|)^2d\xi|^{1/p}\le C\varepsilon
\end{aligned} \tag{4.12}$$

Since u_ε is a minimizer of $E_\varepsilon(u,G)$, we have

$$E_\varepsilon(u_\varepsilon,G) \le E_\varepsilon(U,G),$$

where

$$U=(\rho_1 w, \sqrt{1-\rho_1^2}) \quad on \quad B; \qquad U=u_\varepsilon \quad on \quad G\setminus B.$$

Namely,

$$E_\varepsilon(u_\varepsilon,G) \le E_\varepsilon(\rho_1 w,B)+E_\varepsilon(u_\varepsilon,G\setminus B).$$

Hence

$$\begin{aligned} &E_\varepsilon(u_\varepsilon,B) \le E_\varepsilon(\rho_1 w,B)\\ &=\tfrac{1}{p}\textstyle\int_B(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{p/2}dx+\tfrac{1}{2\varepsilon^p}\textstyle\int_B(1-\rho_1^2)dx, \end{aligned} \tag{4.13}$$

where $w=\frac{u'_\varepsilon}{|u'_\varepsilon|}$. On one hand,

$$\begin{aligned} &\textstyle\int_B(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{p/2}dx-\textstyle\int_B(\rho_1^2|\nabla w|^2)^{p/2}dx\\ &=\tfrac{p}{2}\textstyle\int_B\int_0^1[(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{(p-2)/2}s\\ &\quad+(\rho_1^2|\nabla w|^2)^{(p-2)/2}(1-s)]ds|\nabla\rho_1|^2dx\\ &\le C\textstyle\int_B(|\nabla\rho_1|^p+|\nabla\rho_1|^2|\nabla w|^{p-2})dx. \end{aligned} \tag{4.14}$$

On the other hand, by using (4.12) and (4.3) we have

$$\int_B|\nabla\rho_1|^2|\nabla w|^{p-2}dx \le (\int_B|\nabla\rho_1|^p dx)^{2/p}\cdot(\int_B|\nabla w|^p dx)^{(p-2)/p} \le C\varepsilon^{2/p}. \tag{4.15}$$

Combining (4.13)-(4.15), we can derive

$$E_\varepsilon(u_\varepsilon,B) \le \frac{1}{p}\int_B\rho_1^p|\nabla w|^p dx+C\varepsilon^{2/p}.$$

Thus (4.1) can be seen by noticing (4.7).

5 Proof of Theorem 1.3

Assume that u_ε is a minimizer, and $B=B(x,r)$. By noting $p>2$ and using Jensen's inequality, we have

$$E_\varepsilon(u_\varepsilon,B) \ge \frac{1}{p}\int_B|\nabla h|^p dx+\frac{1}{p}\int_B h^p|\nabla w|^p dx+\frac{1}{p}\int_B|\nabla u_3|^p dx+\frac{1}{2\varepsilon^p}\int_B u_3^2 dx,$$

where $h = |u'_\varepsilon|$. Thus, from (4.1) it follows that,

$$\begin{aligned}&\tfrac{1}{p}\int_B(|\nabla h|^p + |\nabla u_3|^p)dx + \tfrac{1}{p}\int_B(h^p - 1)|\nabla w|^p dx + \tfrac{1}{2\varepsilon^p}\int_B u_3^2 dx \\ &\le E_\varepsilon(u_\varepsilon, B) - \tfrac{1}{p}\int_B |\nabla w|^p dx \le C\varepsilon^{2/p}.\end{aligned} \tag{5.1}$$

Since $|u_3| \le 1$ and (1.1), we have

$$|u_3(x) - u_3(y)| \le C\|u_3\|_{W^{1,p}(K)}|x-y|^{1-2/p} \le C|x-y|^{1-2/p}, \quad \forall x, y \in K.$$

Hence, $u_3^2(x) \ge (|u_3(y)| - C\varepsilon^{1-2/p})^2$ when $x \in B(y,\varepsilon)$. Substituting this into (3.4) we obtain

$$\pi(|u_3(y)| - C\varepsilon^{1-2/p})^2\varepsilon^2 \le \int_{B(y,\varepsilon)} u_3^2(x)dx \le C\varepsilon^p$$

for any $y \in K$. This implies

$$\sup_{y\in K}|u_3(y)| \le C\varepsilon^{1-2/p}.$$

Thus, by using (4.2) and (3.3), we have that for any constant $\delta \in (0,1)$,

$$\begin{aligned}\tfrac{1}{p}\int_B(1-h^p)|\nabla w|^p dx \quad &\le \tfrac{2^p}{p}\int_B(1-h^p)h^p|\nabla w|^p dx \\ &\le C\int_B u_3^2|\nabla u_\varepsilon|^p dx \le C\varepsilon^{1-2/p}.\end{aligned} \tag{5.2}$$

Substituting this into (5.1), we can derive

$$\int_B |\nabla h|^p dx + \int_B |\nabla u_3|^p dx + \frac{1}{\varepsilon^p}\int_B u_3^2 dx \le C(\varepsilon^{1-2/p} + \varepsilon^{2/p}). \tag{5.3}$$

If $p \le 4$, then we have finished. If $p > 4$, we will prove

Theorem 5.1 *Let $p_0 \in (4,5)$ satisfy $p^3 - 4p^2 - 2p + 4 = 0$. Then*

$$\int_B |\nabla h|^p dx + \int_B |\nabla u_3|^p dx + \frac{1}{\varepsilon^p}\int_B u_3^2 dx \le C\varepsilon^{1-2/p}, \quad \textit{when} \quad p \in (4, p_0];$$

$$\int_B |\nabla h|^p dx + \int_B |\nabla u_3|^p dx + \frac{1}{\varepsilon^p}\int_B u_3^2 dx \le C\varepsilon^{\frac{2p}{p^2-2}}, \quad \textit{when} \quad p > p_0.$$

Proof. Step 1. The idea of Proposition 4.1 is used. At first, from (5.3) it follows that

$$\int_B u_3^2 dx \le C\varepsilon^{\frac{2}{p}+p}.$$

Using this and the integral mean theorem, we see that there exists $r_2 \in (2R, r)$ such that

$$\int_{\partial B(x,r_2)} u_3^2 dx \leq C\varepsilon^{\frac{2}{p}+p}.$$

Next, consider the minimizer ρ_2 of the functional

$$E(\rho, B(x,r_2)) = \frac{1}{p}\int_{B(x,r_2)} (|\nabla\rho|^2+1)^{p/2}dx + \frac{1}{2\varepsilon^p}\int_{B(x,r_2)} (1-\rho)^2 dx,$$

in $W^{1,p}_{|u'_\varepsilon|}(B(x,r_2), R^+ \cup \{0\})$. By the same argument of (4.12) we also obtain

$$E(\rho_2, B(x,r_2)) \leq C\varepsilon^{\frac{1}{p}(\frac{2}{p}+p)}.$$

Then, similar to the derivation of (4.1) we can see that

$$E_\varepsilon(u_\varepsilon, B(x,r_2)) \leq \frac{1}{p}\int_B |\nabla w|^p dx + C\varepsilon^{\frac{2}{p^2}(\frac{2}{p}+p)}.$$

At last, by processing as the proof of (5.3) we have

$$\int_{B(x,r_2)} |\nabla h|^p dx + \int_{B(x,r_2)} |\nabla u_3|^p dx + \frac{1}{\varepsilon^p}\int_{B(x,r_2)} u_3^2 dx \leq C(\varepsilon^{1-2/p} + \varepsilon^{\frac{2}{p^2}(\frac{2}{p}+p)}).$$

Step 2. Replacing (5.3) by the inequality above, and via the similar argument of Step 1, we also deduce that there exist $r_j \in (2R, r_{j-1})$ such that for any $j = 1, 2, \cdots$,

$$\int_{B(x,r_j)} |\nabla h|^p dx + \int_{B(x,r_j)} |\nabla u_3|^p dx + \frac{1}{\varepsilon^p}\int_{B(x,r_j)} u_3^2 dx \leq C(\varepsilon^{1-2/p} + \varepsilon^{a_j}), \quad (5.4)$$

where $a_1 = \frac{2}{p}$ and $a_j = \frac{2}{p^2}(a_{j-1}+p)$ for $j = 2, 3, \cdots$. Obviously, $\{a_j\}$ is a increasing and bounded sequence. So we see easily that its limit is $\frac{2p}{p^2-2}$. Letting $j \to \infty$ in (5.4) we have proved Theorem 5.1.

Combining Theorem 5.1 and (5.3) yields that Theorem 1.3 is proved.

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