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On the local integrability and boundedness of solutions to quasilinear parabolic systems*

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Abstract

We introduce a structure condition of parabolic type, which allows for the generalization to quasilinear parabolic systems of the known results of integrability, and boundedness of local solutions to singular and degenerate quasilinear parabolic equations.

1 Introduction

In this note, we investigate under which conditions it is possible to extend to systems the results of local integrability and local boundedness known to hold for solutions to a general class of degenerate and singular quasilinear parabolic equations. In particular, we show that the results presented by DiBenedetto in [1, Chp. VIII] are true for a larger class of problems, by providing conditions under which one can recover for weak solutions of quasilinear parabolic systems the work contained in [5, 6]. Fundamental to our approach is a new condition for the parabolicity of systems, which can be viewed as the extension of an analogous notion for parabolic equations, introduced in [1, Lemma 1.1 pg 19].

Generalizations of the results in [1, Chp. VIII] to initial-boundary value problems for systems have been proven in [7].

We study systems of the general form:

$$\frac{\partial}{\partial t}u_i - \frac{\partial}{\partial x_j}A_{ij}(x,t,u,\nabla u) = B_i(x,t,u,\nabla u) \tag{1}$$

for $i = 1, 2, ..., n$, and $(x,t) \in \Omega_T \equiv \Omega \times (0,T)$ with $\Omega \subseteq \mathbf{R}^N$; where we assume A_{ij} and B_i to be measurable functions in $\Omega \times (0,T) \times \mathbf{R}^n \times \mathbf{R}^{Nn}$, here $i = 1, 2, ..., n$; $j = 1, 2, ..., N$.

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By a weak solution of (1), we mean a function $u = (u_1, u_2, \dots, u_n)$ with $u \in L_{\infty,loc}(0,T; L_{2,loc}(\Omega)) \cap L_{p,loc}(0,T; W^1_{p,loc}(\Omega))$ for some $p > 1$, which verifies

$$\iint_{\Omega_T} \left\{ -u_i \frac{\partial \phi_i}{\partial t} + A_{ij}(x,t,u,\nabla u) \frac{\partial \phi_i}{\partial x_j} \right\} dx\, dt = \iint_{\Omega_T} B_i(x,t,u,\nabla u) \phi_i \, dx\, dt \quad (2)$$

for all $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in C_0^\infty(\Omega_T; \mathbf{R}^n)$.

To the system (1), we add the following classical structure conditions (see [1, Chp. VIII]). For a.e. $(x,t) \in \Omega_T$, every $u \in \mathbf{R}^n$, and $v \in \mathbf{R}^{Nn}$, we assume that

(H1) $\displaystyle\sum_{j=1}^{N} \sum_{i=1}^{n} A_{ij}(x,t,u,v) v_{ij} \geq C_0 |v|^p - C_3 |u|^\delta - \phi_0(x,t)$;

(H2) $|A_{ij}(x,t,u,v)| \leq C_1 |v|^{p-1} + C_4 |u|^{\delta(1-\frac{1}{p})} + \phi_1(x,t)$;

(H3) $|B_i(x,t,u,v)| \leq C_2 |v|^{p(1-\frac{1}{\delta})} + C_5 |u|^{\delta - 1} + \phi_2(x,t)$,

for $C_0 > 0$, $C_1, C_2, \dots, C_5 \geq 0$, with δ s.t. $1 < p \leq \delta < \left(\frac{N+2}{N}\right) p \equiv m$, and where ϕ_0, ϕ_1, ϕ_2 are non-negative functions which satisfy

(H4) $\phi_0 \in L_{1,loc}(\Omega_T)$, $\phi_1 \in L_{\frac{p}{p-1},loc}(\Omega_T)$, and $\phi_2 \in L_{\frac{m}{m-1},loc}(\Omega_T)$.

Finally, we introduce and assume the parabolicity condition

(H5) $\displaystyle\sum_{i,k=1}^{n} \sum_{j=1}^{N} A_{ij}(x,t,u,v) u_i u_k v_{kj} \geq 0.$

The main result of our work is the complete recovery for systems of the form (1) of Theorem 1 in [5]:

Theorem 1 *Let u be a weak solution of (1), and suppose that the structure conditions (H1)-(H5) hold true, together with the following additional hypotheses:*

(H6) *$\phi_0 \in L_{\mu,loc}(\Omega_T)$, $\phi_1, \phi_2 \in L_{s,loc}(\Omega_T)$, where $\mu > 1$ and $s > \dfrac{(N+2)p}{(N+2)p - N}$;*

(H7) *$u \in L_{r,loc}(\Omega_T)$, with $r > 1$ and $N(p-2) + rp > 0$.*

If $s, \mu > \frac{(N+p)}{p}$, then $u \in L_{\infty,loc}(\Omega_T)$;

if $s = \mu = \frac{(N+p)}{p}$, then $u \in L_{q,loc}(\Omega_T)$ for any $q < \infty$;

if $s, \mu < \frac{(N+p)}{p}$, then $u \in L_{q,loc}(\Omega_T)$ for any $q < q^*$, where

$$q^* = \min \left\{ \frac{s(Np + p - N)}{sN - (s-1)(N+p)}, \frac{\mu(Np + 2p)}{\mu N - (\mu - 1)(N+p)} \right\}.$$

Remark We would like to point out that the parabolicity condition (H5) is a quite natural one to consider. In fact, for the case of a single equation it reduces to the condition

$$A_j(x,t,u,v)u^2v_j \geq 0,$$

which, for $u \neq 0$, is equivalent to the weak parabolicity condition presented in [1, Lemma 1.1, p.19].

Further, in the simple case where

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x_j}\left(a_{jm}(x,t,u,\nabla u)\frac{\partial u_i}{\partial x_m}\right) = B_i(x,t,u,\nabla u);$$

our requirement is satisfied if the matrix $a_{jm}(x,t,u,\nabla u)$ is for example positive definite. Indeed, since for the above system one has the identity

$$\sum_{i,k=1}^{n}\sum_{j=1}^{N} A_{ij}(x,t,u,v)u_iu_kv_{kj} = \sum_{i,k=1}^{n}\sum_{j,m=1}^{N} a_{jm}(x,t,u,v)\left(u_iv_{im}\right)\left(u_kv_{kj}\right),$$

(H5) can be rewritten as

$$\sum_{i,k=1}^{n}\sum_{j=1}^{N} A_{ij}(x,t,u,v)u_iu_kv_{kj} = \sum_{j,m=1}^{N} a_{jm}(x,t,u,v)w_mw_j \geq 0,$$

where we set $w_h = \sum_l u_lv_{lh}$.

Finally, we note that (H5) is not so restrictive that the equation must have one of these simple forms. For example, consider the perturbation

$$A_{ij}(x,t,u,v) = a_{jm}(x,t,u,v)v_{im} + \alpha_{ij}(x,t,u,v)$$

where the matrix a_{jm} is positive definite. Define

$$\lambda(x,t,u,v) = \min_{|w|=1} a_{jm}(x,t,u,v)w_jw_m > 0;$$

this exists and is obtained because

$$w \mapsto a_{jm}(x,t,u,v)w_jw_m$$

is positive and continuous for each (x,t,u,v) on the compact set $\{w \in \mathbf{R}^N : |w| = 1\}$. Then for any vector $w \in \mathbf{R}^N$, $w \neq \mathbf{0}$

$$a_{jm}(x,t,u,v)w_jw_m = a_{jm}\tfrac{w_j}{|w|}\tfrac{w_m}{|w|}|w|^2 \geq \lambda|w|^2.$$

Condition (H5) will be verified if the perturbation α_{ij} satisfies the smallness condition

$$\sum_{j=1}^{N}\sum_{i=1}^{n}|\alpha_{ij}(x,t,u,v)u_i| \leq \lambda(x,t,u,v)\sum_{j=1}^{N}\left|\sum_{i=1}^{n}u_iv_{ij}\right|.$$

Indeed, we have

$$\sum_{i,k=1}^{n}\sum_{j=1}^{N}A_{ij}(x,t,u,v)u_iu_kv_{kj}=\sum_{i,k=1}^{n}\sum_{j=1}^{N}\left[\sum_{m=1}^{N}a_{jm}v_{im}+\alpha_{ij}\right]u_iu_kv_{kj}$$
$$=\sum_{j,m=1}^{N}a_{jm}\left(\sum_{i=1}^{n}u_iv_{im}\right)\left(\sum_{k=1}^{n}u_kv_{kj}\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_{ij}u_i\right)\left(\sum_{k=1}^{n}u_kv_{kj}\right)$$
$$\geq\sum_{j=1}^{N}\left\{\lambda\left(\sum_{k=1}^{n}u_kv_{kj}\right)^2-\left|\sum_{i=1}^{n}\alpha_{ij}u_i\right|\left|\sum_{k=1}^{n}u_kv_{kj}\right|\right\}\geq 0.$$

We follow the approach of [1, 5, 6] and start with the derivation, presented in Section 2, of a local energy estimates for weak solutions to (1). We then outline, in Section 3 and Section 4 how the methods in [5] can be applied to obtain local integrability and boundedness.

We also remark that the techniques presented can be modified to handle doubly degenerate problems, where $A_{ij}(x,t,u,v)v_{ij}\geq\Phi(|u|)|v|^p-C_3|u|^\delta-\phi_o(x,t)$ for some Φ, following the same lines as the proof in [6].

2 Energy Estimates for u

2.1 Notation & Preliminaries

Let $(x_0,t_0)\in\Omega_T$, without loss of generality we can assume $(x_0,t_0)=(0,0)$. For $R>0$ we set $Q_R=B_R(0)\times(-R^p,0)$, and for $-R^p\leq\tau\leq 0$, we define $Q_R^\tau=B_R(0)\times(-R^p,\tau)$. For a fixed $0<\sigma<1$, we consider a function $\zeta\in C^\infty(\Omega_T)$ with $0\leq\zeta\leq 1$, $\zeta=1$ in $Q_{\sigma R}^\tau$, and $\zeta=0$ near $|x|=R$ or $t=-R^p$. We also require that

$$|\zeta_t|+|\nabla\zeta|^p\leq\frac{C_\sigma}{R^p}=\frac{2}{(1-\sigma)^pR^p}.$$

We denote by $\zeta_k\in C_0^\infty(Q_R^\tau)$ the elements of a sequence of functions $\zeta_k\to\zeta$ uniformly in Q_R^τ. While, for $\eta>0$, we let J_η be a smooth, symmetric, mollifying kernel in space-time, and for a given function f we use the notation $f_\eta\equiv J_\eta*f$ to represent its convolution with J_η.

Finally, for fixed $\epsilon>0$, and $\kappa>0$, we consider the function

$$f(s)=\frac{(s-\kappa)_+}{(s-\kappa)_++\epsilon}. \tag{3}$$

In the following, we will use the fact that $0\leq f(s)\leq 1$, and that

$$f'(s)=\begin{cases}0 & s<\kappa,\\ \dfrac{\epsilon}{[(s-\kappa)_++\epsilon]^2} & s>\kappa\end{cases}$$

verifies

$$0 \leq f'(s) \leq \begin{cases} 0 & s < \kappa, \\ \frac{1}{\epsilon} & \kappa < s < 2\kappa, \\ \frac{1}{s\,\kappa} & s > 2\kappa, \end{cases} \tag{4}$$

provided $0 < \epsilon < \frac{1}{2}$.

We are now ready to start the derivation of our energy estimate. Fix $\eta > 0$, $\kappa > 0$ and consider the test function $\{u_{i,\eta}(x,t) f(|u_\eta(x,t)|) \zeta_k^p(x,t)\}_\eta$. Because this is a C_0^∞ function for η sufficiently small, we can substitute it into the definition of weak solution to obtain

$$\begin{aligned} \iint_{\Omega_T} & - u_i \frac{\partial}{\partial t} \{u_{i,\eta} f(|u_\eta|) \zeta_k^p\}_\eta \; dx\, dt \\ & + \iint_{\Omega_T} A_{ij}(x,t,u,\nabla u) \frac{\partial}{\partial x_j} \{u_{i,\eta} f(|u_\eta|) \zeta_k^p\}_\eta \; dx\, dt \\ = & \iint_{\Omega_T} B_i(x,t,u,\nabla u) \{u_{i,\eta} f(|u_\eta|) \zeta_k^p\}_\eta \; dx\, dt. \end{aligned} \tag{5}$$

For convenience of notation, we rewrite (5) in compact form as $I_1 + I_2 = I_3$, and discuss each of these terms in turn.

2.2 Estimate of I_1

We begin by using the symmetry of the mollifying kernel, and integration by parts to rewrite I_1 as

$$\begin{aligned} I_1 = - \iint_{\Omega_T} u_{i,\eta} \frac{\partial}{\partial t} \{u_{i,\eta} f(|u_\eta|) \zeta_k^p\} \; dx\, dt & \\ = \iint_{Q_R^\tau} \left(\frac{\partial}{\partial t} u_{i,\eta} \right) u_{i,\eta} f(|u_\eta|) \zeta_k^p \, dx\, dt. \end{aligned}$$

We then notice that summing over the index i implies

$$\sum_i u_{i,\eta} \frac{\partial}{\partial t} u_{i,\eta} = \frac{1}{2} \sum_i \frac{\partial}{\partial t} (u_{i,\eta})^2 = \frac{1}{2} \frac{\partial}{\partial t} |u_\eta|^2 = |u_\eta| \frac{\partial}{\partial t} |u_\eta|, \tag{6}$$

and we derive

$$I_1 = \iint_{Q_R^\tau} |u_\eta| \frac{\partial |u_\eta|}{\partial t} f(|u_\eta|) \zeta_k^p \, dx\, dt.$$

If we now let $k \to \infty$, thanks to the uniform convergence of $\zeta_k \to \zeta$, and the smoothness of the mollified functions we obtain

$$\lim_{k \to \infty} I_1 = \iint_{Q_R^\tau} |u_\eta| \frac{\partial |u_\eta|}{\partial t} f(|u_\eta|) \zeta^p \, dx\, dt.$$

Proceeding in a standard fashion, we rewrite the integral on the right hand side as

$$\begin{aligned}
&\iint_{Q_R^\tau} \frac{\partial}{\partial t}\left(\int_0^{|u_\eta|} sf(s)\,ds\right)\zeta^p\,dx\,dt \\
&= \iint_{Q_R^\tau} \frac{\partial}{\partial t}\left\{\left(\int_0^{|u_\eta|} sf(s)\,ds\right)\zeta^p\right\}\,dx\,dt \\
&\qquad\qquad - p\iint_{Q_R^\tau}\left(\int_0^{|u_\eta|} sf(s)\,ds\right)\zeta^{p-1}\zeta_t\,dx\,dt,
\end{aligned}$$

and applying integration by parts, since $\zeta = 0$ on $t = -R^p$, we gather

$$\begin{aligned}
\lim_{k\to\infty} I_1 = &\int_{B_R}\left(\int_0^{|u_\eta|} sf(s)\,ds\right)\zeta^p\,dx\bigg|_{t=\tau} \\
&\qquad - p\iint_{Q_R^\tau}\left(\int_0^{|u_\eta|} sf(s)\,ds\right)\zeta^{p-1}\zeta_t\,dx\,dt.
\end{aligned}\tag{7}$$

We would like to take the limit for $\eta \downarrow 0$ in (7), and we are able to do so, since from

$$\left|\int_0^{|u_\eta|} sf(s)\,ds - \int_0^{|u|} sf(s)\,ds\right| = \left|\int_{|u|}^{|u_\eta|} sf(s)\,ds\right| \le \gamma_1\left||u_\eta|^2 - |u|^2\right|,$$

with $\gamma_1 = \frac{1}{2}(\max|f|)$, we can conclude

$$\begin{aligned}
&\left|\int_{B_R}\left(\int_0^{|u_\eta|} sf(s)\,ds - \int_0^{|u|} sf(s)\,ds\right)\zeta^p\,dx\bigg|_{t=\tau}\right| \\
&\qquad\qquad \le \gamma_1\int_{B_R}\left||u_\eta|^2 - |u|^2\right|\,dx\bigg|_{t=\tau} \xrightarrow{\eta\downarrow 0} 0
\end{aligned}$$

for a.e. τ, and

$$\begin{aligned}
&\left|\iint_{Q_R^\tau}\left(\int_0^{|u_\eta|} sf(s)\,ds - \int_0^{|u|} sf(s)\,ds\right)\zeta^{p-1}\zeta_t\,dx\,dt\right| \\
&\qquad\qquad \le \gamma_2\iint_{Q_R^\tau}\left||u_\eta|^2 - |u|^2\right|\,dx\,dt \xrightarrow{\eta\downarrow 0} 0,
\end{aligned}$$

where γ_2 is a constant that depends on σ, R and p. (Note that the above limits are zero due to the fact that $u \in L_{\infty,loc}(0,T;L_{2,loc}(\Omega))$.) In conclusion, we have the following estimate

$$\begin{aligned}
\lim_{\eta\downarrow 0}\lim_{k\to\infty} I_1 = &\int_{B_R}\left(\int_0^{|u|} sf(s)\,ds\right)\zeta^p\,dx\bigg|_{t=\tau} \\
&\qquad - p\iint_{Q_R^\tau}\left(\int_0^{|u|} sf(s)\,ds\right)\zeta^{p-1}\zeta_t\,dx\,dt.
\end{aligned}\tag{8}$$

2.3 Estimate of I_2

We start as in Section 2.2, and use the symmetry of the mollifying kernel to rewrite I_2:

$$I_2 = \iint_{Q_R^\tau} A_{ij,\eta}(x,t,u,\nabla u)\frac{\partial}{\partial x_j}\left\{u_{i,\eta} f(|u_\eta|)\zeta_k^p\right\}\, dx\, dt.$$

We then take the limit for $k \to \infty$, and by the smoothness of the mollified functions we obtain

$$\lim_{k\to\infty} I_2 = \iint_{Q_R^\tau} A_{ij,\eta}(x,t,u,\nabla u)\frac{\partial}{\partial x_j}\left\{u_{i,\eta} f(|u_\eta|)\zeta^p\right\}\, dx\, dt. \tag{9}$$

As done while deriving the estimate for I_1, we would like to consider the limit for $\eta \downarrow 0$ as well. To do so, we notice that the structure condition (H2) implies the inequality

$$\iint_{Q_R^\tau} |A_{ij}(x,t,u,\nabla u)|^{\frac{p}{p-1}}\, dx\, dt \le \gamma \iint_{Q_R^\tau} \left[|\nabla u|^p + |u|^\delta + \phi_1^{\frac{p}{p-1}}\right]\, dx\, dt.$$

From which, we have that $A_{ij}(x,t,u,\nabla u) \in L_{\frac{p}{p-1}}(Q_R^\tau)$, since $\delta < m$ and since by the classical embedding theorems for parabolic spaces we know

$$u \in L_{\infty,loc}(0,T; L_{2,loc}(\Omega)) \cap L_{p,loc}(0,T; W^1_{p,loc}(\Omega)) \hookrightarrow L_{m,loc}(\Omega_T). \tag{10}$$

Therefore, we obtain $A_{ij,\eta}(x,t,u,\nabla u) \xrightarrow{\eta\downarrow 0} A_{ij}(x,t,u,\nabla u)$ in $L_{\frac{p}{p-1}}(Q_R^\tau)$.

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial x_j}\left\{u_{i,\eta} f(|u_\eta|)\zeta^p\right\} = {}& \frac{\partial u_{i,\eta}}{\partial x_j} f(|u_\eta|)\zeta^p \\ &+ u_{i,\eta} f'(|u_\eta|)\frac{\partial |u_\eta|}{\partial x_j}\zeta^p + p u_{i,\eta} f(|u_\eta|)\zeta^{p-1}\frac{\partial \zeta}{\partial x_j};\end{aligned}$$

hence from $u_{i,\eta} \to u_i$ and $\nabla u_{i,\eta} \to \nabla u_i$ almost everywhere [3, Appendix C, Theorem 6] we conclude that

$$\frac{\partial}{\partial x_j}\left\{u_{i,\eta} f(|u_\eta|)\zeta^p\right\} \to \frac{\partial}{\partial x_j}\left\{u_i f(|u|)\zeta^p\right\} \quad a.e.$$

If next we use our estimates for f and f', we have the upper bound

$$\begin{aligned}\left|\frac{\partial}{\partial x_j}\left\{u_{i,\eta} f(|u_\eta|)\zeta^p\right\}\right|^p &\le \left\{|\nabla u_{i,\eta}| + 2\kappa\frac{1}{\epsilon}|\nabla u_\eta| + |u_\eta|\frac{1}{\kappa|u_\eta|}|\nabla u_\eta| + C|u_\eta|\right\}^p \\ &\le C\left\{|\nabla u_\eta|^p + |u_\eta|^p\right\},\end{aligned}$$

which, applying a slight generalization of Lebesgue's Dominated Convergence Theorem [4, §1.8], gives

$$\frac{\partial}{\partial x_j}\left\{u_{i,\eta} f(|u_\eta|)\zeta^p\right\} \xrightarrow{\eta\downarrow 0} \frac{\partial}{\partial x_j}\left\{u_i f(|u|)\zeta^p\right\} \quad \text{in } L_p(Q_R^\tau).$$

We then have that equation (9) yields

$$\lim_{\eta\downarrow 0}\lim_{k\to\infty} I_2 = \iint_{Q_R^\tau} A_{ij}(x,t,u,\nabla u)\frac{\partial u_i}{\partial x_j} f(|u|)\zeta^p\,dx\,dt + \iint_{Q_R^\tau} A_{ij}(x,t,u,\nabla u)u_i f'(|u|)\frac{\partial |u|}{\partial x_j}\zeta^p\,dx\,dt + \iint_{Q_R^\tau} A_{ij}(x,t,u,\nabla u)u_i f(|u|)p\,\zeta^{p-1}\frac{\partial \zeta}{\partial x_j}\,dx\,dt. \quad (11)$$

The first integral above can be estimated with the help of (H1) as follows:

$$\iint_{Q_R^\tau} A_{ij}(x,t,u,\nabla u)\frac{\partial u_i}{\partial x_j} f(|u|)\zeta^p\,dx\,dt \ge C_0 \iint_{Q_R^\tau} |\nabla u|^p f(|u|)\zeta^p\,dx\,dt - C_3 \iint_{Q_R^\tau} |u|^\delta f(|u|)\zeta^p\,dx\,dt - \iint_{Q_R^\tau} \phi_0(x,t) f(|u|)\zeta^p\,dx\,dt. \quad (12)$$

To handle the second integral, we use the parabolicity assumption (H5), and the equality $\dfrac{\partial |u|}{\partial x_j} = \dfrac{\partial u_k}{\partial x_j}\dfrac{u_k}{|u|}$, true for $u \neq 0$:

$$\iint_{Q_R^\tau} A_{ij}(x,t,u,\nabla u)u_i f'(|u|)\frac{\partial |u|}{\partial x_j}\zeta^p\,dx\,dt = \iint_{Q_R^\tau} A_{ij}(x,t,u,\nabla u)u_i u_k \frac{\partial u_k}{\partial x_j}\frac{f'(|u|)}{|u|}\zeta^p\,dx\,dt \ge 0. \quad (13)$$

For the last integral, we need (H2) to derive

$$\iint_{Q_R^\tau} A_{ij}(x,t,u,\nabla u)u_i f(|u|)p\,\zeta^{p-1}\frac{\partial \zeta}{\partial x_j}\,dx\,dt \ge -p\,C_1 \iint_{Q_R^\tau} |\nabla u|^{p-1}|u| f(|u|)\zeta^{p-1}|\nabla\zeta|\,dx\,dt - p\iint_{Q_R^\tau}\Big(C_4\,|u|^{\delta\left(1-\frac{1}{p}\right)+1} f(|u|)\,\zeta^{p-1}|\nabla\zeta| + \phi_1(x,t)|u| f(|u|)\zeta^{p-1}|\nabla\zeta|\Big)\,dx\,dt. \quad (14)$$

Finally, we combine (11), (12), (13), and (14) so to obtain the inequality:

$$\lim_{\eta\downarrow 0}\lim_{k\to\infty} I_2 \ge C_0 \iint_{Q_R^\tau} |\nabla u|^p f(|u|)\zeta^p\,dx\,dt - C_3 \iint_{Q_R^\tau} |u|^\delta f(|u|)\zeta^p\,dx\,dt - \iint_{Q_R^\tau} \phi_0(x,t) f(|u|)\zeta^p\,dx\,dt - pC_1 \iint_{Q_R^\tau} |\nabla u|^{p-1}|u| f(|u|)\zeta^{p-1}|\nabla\zeta|\,dx\,dt - p\,C_4 \iint_{Q_R^\tau} |u|^{\delta\left(1-\frac{1}{p}\right)+1} f(|u|)\zeta^{p-1}|\nabla\zeta|\,dx\,dt - p\iint_{Q_R^\tau} \phi_1(x,t)|u| f(|u|)\zeta^{p-1}|\nabla\zeta|\,dx\,dt. \quad (15)$$

2.4 Estimate of I_3

Once again, our first step is to rewrite I_3 in the form

$$I_3 = \iint_{Q_R^\tau} B_{i,\eta}(x,t,u,\nabla u)\left\{u_{i,\eta} f(|u_\eta|)\zeta_k^p\right\}\, dx\, dt,$$

and to consider the limit for $k \to \infty$:

$$\lim_{k\to\infty} I_3 = \iint_{Q_R^\tau} B_{i,\eta}(x,t,u,\nabla u)\left\{u_{i,\eta} f(|u_\eta|)\zeta^p\right\}\, dx\, dt.$$

To justify taking the limit for $\eta \downarrow 0$ in this case, we proceed by noticing that (H3) implies

$$|B_i(x,t,u,\nabla u)|^{\frac{m}{m-1}} \le C_2|\nabla u|^{p\left(1-\frac{1}{\delta}\right)\left(\frac{m}{m-1}\right)} + C_5|u|^{m\frac{\delta-1}{m-1}} + \phi_2^{\frac{m}{m-1}}(x,t).$$

Which yields $B_{i,\eta} \longrightarrow B_i$ in $L_{\frac{m}{m-1}}(Q_R^\tau)$, in view of the embedding (10), and the relations $\delta < m$, $p\left(1-\frac{1}{\delta}\right)\left(\frac{m}{m-1}\right) = p\left(\frac{1-1/\delta}{1-1/m}\right) < p$. Moreover, since we know that

$$u_{i,\eta} f(|u_\eta|)\zeta^p \longrightarrow u_i f(|u|)\zeta^p \quad \text{for a.e. } (x,t), \text{ and} \quad |u_{i,\eta} f(|u_\eta|)\zeta^p|^m \le C|u_\eta|^m;$$

we can apply the same generalization of Lebesgue's Dominated Convergence Theorem to see that

$$u_{i,\eta} f(|u_\eta|)\zeta^p \longrightarrow u_i f(|u|)\zeta^p \quad \text{in } L_m(Q_R^\tau).$$

Thus,

$$\lim_{\eta\downarrow 0}\lim_{k\to\infty} I_3 = \iint_{Q_R^\tau} B_i(x,t,u,\nabla u) u_i f(|u|)\zeta^p\, dx\, dt, \tag{16}$$

and we can use (H3) once more to conclude

$$\lim_{\eta\downarrow 0}\lim_{k\to\infty} I_3 \le C_2 \iint_{Q_R^\tau} |\nabla u|^{p\left(1-\frac{1}{\delta}\right)}|u| f(|u|)\zeta^p\, dx\, dt \\ + C_5 \iint_{Q_R^\tau} |u|^\delta f(|u|)\zeta^p\, dx\, dt + \iint_{Q_R^\tau} \phi_2(x,t)|u| f(|u|)\zeta^p\, dx\, dt. \tag{17}$$

2.5 The Energy Estimate

To derive our energy estimate (presented in Proposition 3 below), we use the intermediate result stated as Lemma 2. This is a direct consequence of equations (8), (15) and (17): starting from (5), one can use the bounds given in the previous sections, and then apply Young's inequality to treat the terms involving $|\nabla u|^{p-1}$, $|\nabla u|^{p\left(1-\frac{1}{\delta}\right)}$, and $|u|^{p\left(1-\frac{1}{\delta}\right)}$.

Lemma 2 *Let $p > 1$, let f be defined by* (3)*, and let $u \in L_{\infty,loc}(0,T; L_{2,loc}(\Omega)) \cap L_{p,loc}(0,T; W^1_{p,loc}(\Omega))$ be a weak solution of* (1)*. If the assumptions (H1)-(H5) are verified, then for any $Q^\tau_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^p, \tau) \subset\subset \Omega_T$ we have*

$$\int_{B_R} \left(\int_0^{|u|} sf(s)\, ds \right) \zeta^p \, dx \bigg|_{t=\tau} + \iint_{Q^\tau_R} |\nabla u|^p f(|u|) \zeta^p \, dx\, dt$$
$$\leq \gamma \left(\iint_{Q^\tau_R} |u|^\delta f(|u|)\zeta^p \, dx\, dt + \iint_{Q^\tau_R} |u|^p f(|u|) |\nabla \zeta|^p \, dx\, dt \right.$$
$$+ \iint_{Q^\tau_R} \left(\int_0^{|u|} sf(s)\, ds \right) \zeta^{p-1} |\zeta_t| \, dx\, dt + \iint_{Q^\tau_R} \phi_0(x,t) f(|u|)\zeta^p \, dx\, dt +$$
$$\left. \iint_{Q^\tau_R} \phi_1(x,t)|u| f(|u|)\zeta^{p-1}|\nabla\zeta| \, dx\, dt + \iint_{Q^\tau_R} \phi_2(x,t)|u| f(|u|)\zeta^p \, dx\, dt \right) \tag{18}$$

for some constant $\gamma = \gamma(C_0, C_1, C_2, C_3, C_4, C_5, p)$.

To extract useful information from Lemma 2, we need to substitute our choice of $f(s)$, and then let $\epsilon \downarrow 0$. We first note that

$$\int_0^{|u|} sf(s)\, ds = \int_0^{|u|} \frac{(s-\kappa)_+}{(s-\kappa)_+ + \epsilon} s\, ds \geq \int_0^{|u|} \frac{(s-\kappa)_+^2}{(s-\kappa)_+ + \epsilon} ds$$
$$\geq \chi[|u| > \kappa] \int_\kappa^{|u|} \frac{(s-\kappa)^2}{(s-\kappa)+\epsilon} ds \geq \int_0^{(|u|-\kappa)_+} \frac{s^2}{s+\epsilon} ds$$
$$\geq \frac{1}{2}(|u|-\kappa)_+^2 - \epsilon(|u|-\kappa)_+ + \epsilon^2 \ln\left[(|u|-\kappa)_+ + \epsilon\right] - \epsilon^2 \ln \epsilon, \tag{19}$$

which implies

$$\lim_{\epsilon\downarrow 0} \int_0^{|u|} sf(s)\, ds = \int_0^{|u|} \frac{(s-\kappa)_+}{(s-\kappa)_+ + \epsilon} s\, ds \geq \frac{1}{2}(|u|-\kappa)_+^2;$$

and hence

$$\lim_{\epsilon\downarrow 0} \int_{B_R} \left(\int_0^{|u|} sf(s)\, ds \right) \zeta^p \, dx \bigg|_{t=\tau} \geq \frac{1}{2} \int_{B_R} (|u|-\kappa)_+^2 \zeta^p \, dx \bigg|_{t=\tau}.$$

By remarking that if $\epsilon < \kappa < s$ then $\dfrac{(s-\kappa)s}{s-\kappa+\epsilon} = s - \epsilon\left[1 + \dfrac{\kappa-\epsilon}{s-\kappa+\epsilon}\right] \leq s$, one can see that

$$\int_0^{|u|} sf(s)\, ds = \int_0^{|u|} \frac{(s-\kappa)_+}{(s-\kappa)_+ + \epsilon} s\, ds = \chi[|u|>\kappa] \int_\kappa^{|u|} \frac{(s-\kappa)s}{s-\kappa+\epsilon} ds$$
$$\leq \chi[|u|>\kappa] \int_\kappa^{|u|} s\, ds \leq \frac{1}{2}|u|^2 \chi[|u|>\kappa].$$

Moreover, since

$$f(|u|) = \frac{(|u|-\kappa)_+}{(|u|-\kappa)_+ + \epsilon} \uparrow \chi[|u|>\kappa] \quad \text{as } \epsilon \downarrow 0,$$

we can use the Monotone Convergence Theorem to pass to the limit as $\epsilon \downarrow 0$ in the remaining terms of (18), and gather the bound

$$\begin{aligned}
&\frac{1}{2}\int_{B_R} (|u|-\kappa)_+^2 \zeta^p \, dx\Big|_{t=\tau} + \iint_{Q_R^\tau} |\nabla u|^p \chi[|u|>\kappa]\zeta^p \, dx\, dt \\
&\leq \gamma \Bigg(\iint_{Q_R^\tau} |u|^\delta \chi[|u|>\kappa] \, dx\, dt + \iint_{Q_R^\tau} |u|^p \chi[|u|>\kappa] |\nabla\zeta|^p \, dx\, dt \\
&\quad + \iint_{Q_R^\tau} |u|^2 \chi[|u|>\kappa]\zeta_t \, dx\, dt + \iint_{Q_R^\tau} \phi_0(x,t)\chi[|u|>\kappa] \, dx\, dt \\
&\quad + \iint_{Q_R^\tau} \phi_1(x,t)|u|\chi[|u|>\kappa]|\nabla\zeta| \, dx\, dt + \iint_{Q_R^\tau} \phi_2(x,t)|u|\chi[|u|>\kappa] \, dx\, dt \Bigg).
\end{aligned}$$

In turn, the above inequality leads to the classical local energy estimate stated in Proposition 3 below, if one takes in account the relation $\big|\nabla|u|\big|^p \leq |\nabla u|^p$.

Proposition 3 *(Local Energy Estimate) Under the hypotheses (H1)-(H5), if u is a weak solution of* (1) *then for $Q_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^p, t_0) \subset\subset \Omega_T$, $0<\sigma<1$, and $\kappa > 0$*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-R^p<\tau<0} \int_{B_{\sigma R}} (|u|-\kappa)_+^2 \, dx\Big|_{t=\tau} + \iint_{Q_{\sigma R}} |\nabla(|u|-\kappa)_+|^p \zeta^p \, dx\, dt \\
&\qquad \leq \gamma \Bigg(\iint_{Q_R} |u|^\delta \chi[|u|>\kappa] \, dx\, dt + \frac{1}{(1-\sigma)^p R^p} \iint_{Q_R} |u|^p \chi[|u|>\kappa] \, dx\, dt \\
&\qquad + \frac{1}{(1-\sigma)^p R^p} \iint_{Q_R} |u|^2 \chi[|u|>\kappa] \, dx\, dt + \iint_{Q_R} \phi_0(x,t)\chi[|u|>\kappa] \, dx\, dt \\
&\qquad + \frac{1}{(1-\sigma)R} \iint_{Q_R} \phi_1(x,t)|u|\chi[|u|>\kappa] \, dx\, dt \\
&\qquad + \iint_{Q_R} \phi_2(x,t)|u|\chi[|u|>\kappa] \, dx\, dt \Bigg).
\end{aligned} \tag{20}$$

for some constant $\gamma = \gamma(C_0, C_1, C_2, C_3, C_4, C_5, p)$.

3 Higher Integrability of u

Owing to Proposition 3, we can proceed as in [5] to show higher integrability properties for u, that is the first part of Theorem 1. In fact, thanks to the Sobolev embedding for parabolic spaces [1, Chap. 1], and hypotheses (H6) for the functions ϕ_0, ϕ_1, and ϕ_2, we have

$$\begin{aligned}\left(\iint_{Q_{\sigma R}} (|u|-\kappa)_+^{p\left(\frac{N+2}{N}\right)} \zeta^p \, dx\, dt\right)^{\frac{1}{1+\frac{p}{N}}} &\le \gamma \iint_{Q_R} |u|^\delta \chi[|u|>\kappa]\, dx\, dt \\ &+ \frac{\gamma}{(1-\sigma)^p R^p} \iint_{Q_R} |u|^p \chi[|u|>\kappa]\, dx\, dt \\ &+ \frac{\gamma}{(1-\sigma)^p R^p} \iint_{Q_R} |u|^2 \chi[|u|>\kappa]\, dx\, dt + \gamma\, ||\phi_0||_{L_\mu(Q_R)} (\text{meas}[|u|>\kappa])^{1-\frac{1}{\mu}} \\ &+ \gamma \left(\frac{||\phi_1||_{L_s(Q_R)}}{(1-\sigma)R} + ||\phi_2||_{L_s(Q_R)}\right) \iint_{Q_R} |u| \chi[|u|>\kappa]\, dx\, dt.\end{aligned} \tag{21}$$

Inequality (21) is the key link needed to obtain for our systems exactly the same higher integrability result proven in [5, Proposition 3] for single equations:

Proposition 4 *Under the hypotheses and notation of Theorem 1, we have that*

if $s, \mu \ge \frac{(N+p)}{p}$*, then* $u \in L_{q,loc}(\Omega_T)$ *for any* $q < \infty$*;*

if $s, \mu < \frac{(N+p)}{p}$*, then* $u \in L_{q,loc}(\Omega_T)$ *for any* $q < q^*$*.*

Indeed, suppose $u \in L_{\beta,loc}$. Then we can use (21) to see that

$$\begin{aligned}&\left\{\kappa^{\left(\frac{N+2}{N}\right)p} \operatorname*{meas}_{Q_{\sigma R}}[|u|>2\kappa]\right\}^{\frac{1}{1+p/N}} \\ &\qquad \le C_{\gamma,R,\sigma,p} \|u\|^\beta_{L_\beta(Q_R)} \left\{\left(\frac{1}{\kappa}\right)^{\beta-\delta} + \left(\frac{1}{\kappa}\right)^{\beta-p} + \left(\frac{1}{\kappa}\right)^{\beta-2}\right\} \\ &\qquad\qquad + C\|u\|^{\beta\left(1-\frac{1}{\mu}\right)}_{L_\beta(Q_R)} \left(\frac{1}{\kappa}\right)^{\beta\left(1-\frac{1}{\mu}\right)} + C\|u\|^{\beta\left(1-\frac{1}{s}\right)}_{L_\beta(Q_R)} \left(\frac{1}{\kappa}\right)^{\beta\left(1-\frac{1}{s}\right)-1}.\end{aligned}$$

Therefore, if

$$\alpha(\beta) = \frac{N+2}{N}p + \left(1+\frac{p}{N}\right) \min\left\{\beta-2, \beta-\delta, \beta\left(1-\frac{1}{s}\right)-1, \beta\left(1-\frac{1}{\mu}\right)\right\}$$

then $u \in L^{\text{weak}}_{\alpha(\beta),loc}$. Thus $u \in L_{q,loc}(Q_R)$ for all $q < \alpha(\beta)$, and we can iterate this process starting from $\beta_0 = \max\{2, \dfrac{N+2}{N}p, r\}$ to obtain the result. The details can be found in [5].

4 Boundedness of u

The L_∞ local estimate part of Theorem 1 is a straightforward application of DeGiorgi's technique; again the details can be found in [5]. In particular, we fix $\rho > 0$, $\sigma > 0$, so that $Q_\rho \subset\subset \Omega_T$. For each integer n, we define

$$\rho_n = \sigma\rho + \frac{(1-\sigma)}{2^n}\rho,$$

and set $Q^n = Q_{\rho_n}$. Next we fix $\kappa > 0$ to be chosen later, and set

$$\kappa_n = \kappa\left(1 - \frac{1}{2^{n+1}}\right).$$

For $\frac{N+2}{N}p > 2$, we consider

$$Y_n = \frac{1}{\operatorname{meas} Q^n}\iint_{Q^n} |u-\kappa_n|^m \, dx\, dt,$$

while for $\frac{N+2}{N}p \leq 2$, we take

$$Y_n = \frac{1}{\operatorname{meas} Q^n}\iint_{Q^n} |u-\kappa_n|^\lambda \, dx\, dt,$$

for λ sufficiently large. This is well defined thanks to the local integrability proven in Section 3. We then apply the local energy estimate (21)in a standard way to obtain an estimate of the form

$$Y_{n+1} \leq \gamma(B_1^n Y_n^{1+\epsilon_1} + B_2^n Y_n^{1+\epsilon_2} + B_3^n Y_n^{1+\epsilon_3}),$$

for positive constants $\gamma, B_1, B_2, B_3, \epsilon_1, \epsilon_2$ and ϵ_3. As final step, we choose κ sufficiently large so to have $Y_n \to 0$ as $n \to \infty$ which implies $|u| < \kappa$ in $Q_{\sigma\rho}$.

It should be clear from the above presentation how the crucial roles in the generalization of the results in [5] to system of the form (1) are played by the local energy estimate of Proposition 3, and by the fact that the techniques in [5] really depend just on $|u|$. In a similar fashion, it is an easy exercise to check that the same ingredients (Proposition 3 and replacement of u by $|u|$) lead to the more general results of [6] .

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