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# Fredholmness of an Abstract Differential Equation of Elliptic Type

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**Abstract**

In this work, we obtain algebraic conditions which assure the Fredholm solvability of an abstract differential equation of elliptic type. In this respect, our work can be considered as an extension of Yakubov's results to the case of boundary conditions containing a linear operator. Although essential technical, this extension is not straight forward as we show it below. The obtained abstract result is applied to a non regular boundary value problem for a second order partial differential equation of an elliptic type in a cylindrical domain. It is interesting to note that the problems considered in cylindrical domains are not coercive.

**Keywords:**Ellipticity, Abstract Differential Equation, Fredholmness, Non regular elliptic problems.

**AMS Subject Classification:**35J05, 35P20, 34L10, 47E05.

## Contents



## 1 Introduction

Many works are devoted to the study of hyperbolic or parabolic abstract equations [12, 13, 7]. In [12, 15] regular boundary value problems for elliptic abstract equations are considered. A few works are concerned with non regular problems, essentially done by Yakubov but he did not consider boundary conditions containing linear operators. We shall be interested to this type of problems, in particular, differential operator equation with a linear operator in the boundary conditions, for which we shall prove the Fredholmness in spite of non regularity. The non regularity here means that the boundary conditions do not satisfy the Shapiro-Lopatinski conditions, because they are not local, the problem is in fact a two-points problem.

Consider in the space $L_p(0,1;H)$ where $H$ is a Hilbert space, the boundary value problem for the second order abstract differential equation.

$$L(D)u \quad = \quad -u''(x) + Au(x) + A(x)u(x) \quad = \quad f(x) \qquad x \in (0,1), \tag{1}$$

$$\begin{array}{rcl} L_1 u & = & \delta u(0) \qquad\qquad\quad = \quad f_1 \\ L_2 u & = & u'(1) + Bu(0) \quad = \quad f_2 \end{array} \tag{2}$$

$A$, $A(x)$, $B$ are linear operators and $\delta$ is a complex number. The idea is to write (1), (2) as a sum of two problems, where the first one is a principal problem with a parameter which gives an isomorphism and we establish a non coercive estimates for its solution and the second one is a compact perturbation of the first. Then we use the Kato perturbation theorem to conclude. Finally, we apply the abstract results obtained to a non regular boundary value problem for a concrete partial differential equation in a cylinder.

This paper is organized as follows. In the second section, we give some preliminaries. In the third section, we study an homogeneous abstract differential equation, we prove the isomorphism and the non coercive estimates for the solution, an estimates which is not explicit with respect to the spectral parameter. In the fourth section, we consider a non homogeneous abstract differential equation, splitting the solution into two parts, the first part is the solution of a homogeneous problem and the second part is a solution of an abstract differential equation on the whole axis. Using results of the previous section and the Fourier multipliers we obtain analogous results as in the previous section. In the fifth section, we consider a general problem, then we use the Kato perturbation theorem. In the sixth section, we apply the obtained abstract results to a boundary value problem for a concrete partial differential equation.

# 2 Preliminaries

Let $H_1, H$ be Hilbert spaces such that $H_1 \subset H$ with continuous injection. We define the space

$$W_p^2(0,1;H_1,H) = \{u; u \in L_p(0,1,H_1), u" \in L_p(0,1,H)\}$$

provided with the finite norm

$$\|u\|_{W_p^2(0,1;H_1,H)} = \|u\|_{L_p(0,1;H_1)} + \|u''\|_{L_p(0,1;H)}$$

Let $E_0, E_1$ be two Banach spaces, continuously embedded in the Banach space $E$, the pair $\{E_0, E_1\}$ is said an interpolation couple. Consider the Banach space

$$E_0 + E_1 = \{u : u \in E, \ \exists u_j \in E_j, \ j = 0,1, \text{ with } u = u_0 + u_1\},$$

$$\|u\|_{E_0+E_1} = \inf_{u=u_0+u_1, \ u_j \in E_j} \left(\|u_0\|_{E_0} + \|u_1\|_{E_1}\right),$$

and the functional

$$K(t,u) = \inf_{u=u_0+u_1, \ u_j \in E_j} \left(\|u_0\|_{E_0} + t\,\|u_1\|_{E_1}\right).$$

The interpolation space for the couple $\{E_0, E_1\}$ is defined, by the K-method, as follows

$$(E_0,E_1)_{\theta,p} = \left\{u : u \in E_0 + E_1, \|u\|_{\theta,p} = \left(\int_0^\infty t^{-1-\theta p} K^p(t,u)dt\right)^{\frac{1}{p}} < \infty\right\},$$

$0 < \theta < 1, \quad 1 \le p \le \infty.$

$$(E_0,E_1)_{\theta,\infty} = \left\{u : u \in E_0 + E_1, \|u\|_{\theta,\infty} = \sup_{t\in(0,\infty)} t^{-\theta} K(t,u)dt < \infty\right\},$$

$0 < \theta < 1$.

Let $A$ be a closed operator in $H$. $H(A)$ is the domain of $A$ provided with the hilbertian norm

$$\|u\|^2_{H(A)} = \|Au\|^2 + \|u\|^2, u \in D(A).$$

# 3 Homogeneous Abstract differential equation

Looking to the principal part of the problem $(1), (2)$ with a parameter

$$L(D)u = -u''(x) + (A + \lambda I)u(x) = 0 \qquad x \in (0, 1), \tag{3}$$

$$\begin{array}{rcccl} L_1 u & = & \delta u(0) & = & f_1 \\ L_2 u & = & u'(1) + Bu(0) & = & f_2 \end{array} \tag{4}$$

**Theorem 3.1** *Assume that the following conditions are satisfied*

1. *$A$ is positive linear closed operator in $H$.*
2. *$\delta \neq 0$.*
3. *$B$ is continuous from $H(A)$ in $H(A^{1/2})$ and from $H(A^{1/2})$ in $H$.*

*Then the problem $(3), (4)$ for $f_1 \in (H, H(A))_{1-\frac{1}{2p},p}$, $f_2 \in (H, H(A))_{\frac{1}{2}-\frac{1}{2p},p}$ and $\lambda$ such that $|\arg \lambda| \leq \phi < \pi$, $|\lambda| \longrightarrow \infty$, has a unique solution in the space $W^2_p(0, 1; H(A), H)$, with $p \in (1, \infty)$, and for the solution of the problem $(3), (4)$ the following non coercive estimate holds*

$$\|u''\|_{L_p(0,1;H)} + \|Au\|_{L_p(0,1;H)}$$
$$\leq C(\lambda) \left( \|f_1\|_{(H,H(A))_{1-\frac{1}{2p},p}} + \|f_2\|_{f_2 \in (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}} \right). \tag{5}$$

**Proof.**
The solution $u$, belonging to $W^2_p(0, 1; H(A), H)$, of the equation $(3)$ is in the form

$$u(x) = e^{-xA_\lambda^{\frac{1}{2}}} g_1 + e^{-(1-x)A_\lambda^{\frac{1}{2}}} g_2 \tag{6}$$

with $A_\lambda = A + \lambda I$, $g_1$ and $g_2$ belong to $(H, H(A))_{1-1/2p,p}$. Indeed let $u \in W^2_p(0, 1; H(A), H)$ be a solution of $(3)$. Then we have

$$\left(D - A_\lambda^{\frac{1}{2}}\right)\left(D + A_\lambda^{\frac{1}{2}}\right) u(x) = 0.$$

Note by

$$v(x) = \left(D + A_\lambda^{\frac{1}{2}}\right) u(x).$$

From [17, p.168] $v \in W^1_p(0, 1; H(A^{\frac{1}{2}}), H)$ and

$$\left(D - A_\lambda^{\frac{1}{2}}\right) v(x) = 0. \tag{7}$$

So

$$v(x) = e^{-(1-x)A_\lambda^{\frac{1}{2}}} v(1). \tag{8}$$

where, according to [14, p.44],

$$v(1) \in \left(H(A^{\frac{1}{2}}), H\right)_{\frac{1}{p},p} = \left(H, H(A^{\frac{1}{2}})\right)_{1-\frac{1}{p},p}.$$

From $(7)\,,(8)$ we have

$$u(x)=e^{-xA_\lambda^{\frac{1}{2}}}u(0)+\int\limits_0^x e^{-(x-y)A_\lambda^{\frac{1}{2}}}e^{-(1-y)A_\lambda^{\frac{1}{2}}}v(1)dy$$

$$=e^{-xA_\lambda^{\frac{1}{2}}}u(0)+\frac{1}{2}A_\lambda^{-\frac{1}{2}}\left\{e^{-(1-x)A_\lambda^{\frac{1}{2}}}-e^{-xA_\lambda^{\frac{1}{2}}}e^{-A_\lambda^{\frac{1}{2}}}\right\}v(1),$$

where $u(0)\in(H(A),H)_{\frac{1}{2p},p}$ [14, p.44]. Now,

$$A^{\frac{1}{2}}:(H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}\to(H,H(A))_{\frac{p-1}{2p},p}=(H,H(A))_{1-\frac{1}{2p},p}$$

is an isomorphism. Consequently the last inequality is in the form $(6)$.

Let us show the reverse, i.e. the function $u$ in the form $(6)$ with $g_1$ and $g_2$ in $(H,H(A))_{1-1/2p,p}$, belongs to $W_p^2(0,1;H(A),H)$. From interpolation spaces properties see [11], [14, p.96] and the expression $(6)$ of the function $u$ we have

$$\begin{aligned}\|u\|_{W_p^2(0,1;H(A),H)}\ \le\ & \left(\left\|AA_\lambda^{-1}\right\|+1\right)\left\{\left(\int\limits_0^1\left\|A_\lambda e^{-xA_\lambda^{\frac{1}{2}}}g_1\right\|^p dx\right)^{\frac{1}{p}}\right.\\ & \left.+\left(\int\limits_0^1\left\|A_\lambda e^{-(1-x)A_\lambda^{\frac{1}{2}}}g_2\right\|^p dx\right)^{\frac{1}{p}}\right\}\\ & \le C\left(\|g_1\|_{(H,H(A_\lambda))_{1-1/2p,p}}+\|g_2\|_{(H,H(A_\lambda))_{1-1/2p,p}}\right)\\ & \le C(\lambda)\left(\|g_1\|_{(H,H(A))_{1-1/2p,p}}+\|g_2\|_{(H,H(A))_{1-1/2p,p}}\right).\end{aligned}\tag{9}$$

The function $u$ satisfies the boundary conditions $(2)$ if

$$\begin{cases}\delta g_1+\delta e^{-A_\lambda^{\frac{1}{2}}}g_2 & = & f_1\\ -A_\lambda^{\frac{1}{2}}e^{-A_\lambda^{\frac{1}{2}}}g_1+Bg_1+A_\lambda^{\frac{1}{2}}g_2+Be^{-A_\lambda^{\frac{1}{2}}}g_2 & = & f_2\end{cases}$$

which we can write in matrix form as:

$$\left[\begin{pmatrix}\delta I & 0\\ B & A_\lambda^{\frac{1}{2}}\end{pmatrix}+\begin{pmatrix}0 & \delta e^{-A_\lambda^{\frac{1}{2}}}\\ -A_\lambda^{\frac{1}{2}}e^{-A_\lambda^{\frac{1}{2}}} & Be^{-A_\lambda^{\frac{1}{2}}}\end{pmatrix}\right]\begin{pmatrix}g_1\\ g_2\end{pmatrix}=\begin{pmatrix}f_1\\ f_2\end{pmatrix}.\tag{10}$$

The first matrix of operators is invertible, its inverse is

$$\begin{pmatrix}\frac{1}{\delta}I & 0\\ -\frac{1}{\delta}A_\lambda^{-\frac{1}{2}}B & A_\lambda^{-\frac{1}{2}}\end{pmatrix}.\tag{11}$$

Multiplying the two members of $(10)$ by the matrix inverse $(11)$, we get the following system:

$$\begin{cases}g_1+e^{-A_\lambda^{\frac{1}{2}}}g_2 & = & \frac{1}{\delta}f_1\\ e^{-A_\lambda^{\frac{1}{2}}}g_1+g_2 & = & -\frac{1}{\delta}A_\lambda^{-\frac{1}{2}}Bf_1+A_\lambda^{-\frac{1}{2}}f_2\end{cases}$$

we can solve it by Cramer's method, because the coefficients of the linear system are bounded linear operators. The determinant is given by $I+e^{-2A_\lambda^{\frac{1}{2}}}$ which is invertible as a little perturbation of unity, in fact $\left\|e^{-2A_\lambda^{\frac{1}{2}}}\right\|\le q<1$.

Hence the solution is written as

$$\begin{cases} g_1 &= \frac{1}{\delta} f_1 + R_{11}(\lambda) f_1 + R_{12}(\lambda) f_2 \\ g_2 &= -\frac{1}{\delta} \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B f_1 + \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} f_2 + R_{21}(\lambda) f_1 \end{cases} \tag{12}$$

where $R_{ij}(\lambda)$ are given by

$$\begin{cases} R_{11}(\lambda) &= -\frac{1}{\delta} \left(I + T(\lambda)\right) e^{-2A_\lambda^{\frac{1}{2}}} + \frac{1}{\delta} \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} e^{-A_\lambda^{\frac{1}{2}}} B \\ R_{12}(\lambda) &= -\left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} e^{-A_\lambda^{\frac{1}{2}}} \\ R_{21}(\lambda) &= \frac{1}{\delta} \left(I + T(\lambda)\right) e^{-A_\lambda^{\frac{1}{2}}} \end{cases}$$

and satisfy $\|R_{ij}(\lambda)\| \to 0$ when $|\lambda| \to \infty$. $(I + T(\lambda))$ is the inverse of $I + e^{-2A_\lambda^{\frac{1}{2}}}$ obtained from the corresponding Neumann series.
Finally the solution $u$ is given by

$$u(x) = e^{-xA_\lambda^{\frac{1}{2}}} \left(\frac{1}{\delta} f_1 + R_{11}(\lambda) f_1 + R_{12}(\lambda) f_2\right)$$

$$+ e^{-(1-x)A_\lambda^{\frac{1}{2}}} \left(-\frac{1}{\delta} \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B f_1 + \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} f_2 + R_{21}(\lambda) f_1\right).$$

From the assumptions of theorem $(3.1)$ and the properties of interpolation spaces, the following applications are continuous,

$$\left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B : \quad (H, H(A))_{1-1/2p,p} \longmapsto (H, H(A))_{1-1/2p,p} \ ,$$

$$\left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} : \quad (H, H(A))_{1/2-1/2p,p} \longmapsto (H, H(A))_{1-1/2p,p} \ .$$

Then we have the estimates

$$\left\| \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B f_1 \right\|_{(H,H(A))_{1-1/2p,p}} \leq C \left\| f_1 \right\|_{(H,H(A))_{1-1/2p,p}}$$

and

$$\left\| \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} f_2 \right\|_{(H,H(A))_{1-1/2p,p}} \leq C \left\| f_2 \right\|_{(H,H(A))_{1/2-1/2p,p}} .$$

This give a bound of $g_k$ in function of $f_k$,namely

$$\|g_k\|_{(H,H(A))_{1-1/2p,p}} \leq C \left( \|f_1\|_{(H,H(A))_{1-1/2p,p}} + \|f_2\|_{(H,H(A))_{1/2-1/2p,p}} \right).$$

Finally, taking into account of the estimate $(9)$ we obtain the following non coercive estimate

$$\|u\|_{W_p^2(0,1;H(A),H)} \leq C(\lambda) \left( \|f_1\|_{(H,H(A))_{1-1/2p,p}} + \|f_2\|_{(H,H(A))_{1/2-1/2p,p}} \right).$$

# 4 Non homogeneous abstract differential equation

Consider, now, the principal problem for the non homogeneous equation with a parameter

$$L_0(\lambda, D)u = -u''(x) + A_\lambda u(x) = f(x) \qquad x \in (0,1), \tag{13}$$

$$\begin{matrix} L_{10} u &=& \delta u(0) &=& f_1 \\ L_{20} u &=& u'(1) + Bu(0) &=& f_2 \end{matrix} \tag{14}$$

We have the result.

**Theorem 4.1** *Suppose the following conditions satisfied*

1. *$A$ is a positive linear closed operator in $H$.*
2. *$B$ is continuous from $H(A)$ in $H(A^{1/2})$ and from $H(A^{1/2})$ in $H$.*
3. *$\delta \neq 0$.*

*Then the problem* $(13)$,$(14)$,*for $f$, $f_1$ and $f_2$ in $L_p(0,1;H)$, $(H,H(A))_{1-\frac{1}{2p},p}$ and $(H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}$ respectively, and for $\lambda$ such that $|\arg\lambda| \leq \phi < \pi, |\lambda| \longrightarrow \infty$, has a unique solution belonging to the space $W_p^2(0,1;H(A),H)$, for $p \in (1,\infty)$, and the following non coercive estimate holds*

$$\|u''\|_{L_p(0,1;H)} + \|Au\|_{L_p(0,1;H)} \leq$$

$$C(\lambda)\left\{\|f\|_{L_p(0,1;H)} + \|f_1\|_{(H,H(A))_{1-\frac{1}{2p},p}} + \|f_2\|_{f_2\in(H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}}\right\} \tag{15}$$

**Proof.**
In the theorem $(3.1)$ we proved the uniqueness. Let us show, now, that the solution of the problem $(13)$, $(14)$ belonging to $W_p^2(0,1;H(A),H)$ can be written in the form $u(x) = u_1(x) + u_2(x)$, $u_1(x)$ is the restriction to $[0,1]$ of $\widetilde{u}_1(x)$ where $\widetilde{u}_1(x)$ is the solution of the equation

$$L_0(\lambda,D)\widetilde{u}_1(x) = \widetilde{f}(x), \qquad x \in \mathbb{R}, \tag{16}$$

with $\widetilde{f}(x) = f(x)$ if $x \in [0,1]$ and $\widetilde{f}(x) = 0$ otherwise. $u_2(x)$ is the solution of the problem

$$L_0(\lambda,D)u_2 = 0,\ L_{10}u_2 = f_1 - L_{10}u_1,\ L_{20}u_2 = f_2 - L_{20}u_1. \tag{17}$$

The solution of the equation $(16)$ is given by the formula

$$\widehat{u}_1(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\mu x}L_0(\lambda,i\mu)^{-1}\widehat{\widetilde{f}}(\mu)d\mu \tag{18}$$

where $\widehat{\widetilde{f}}$ is the Fourier transform of the function $\widetilde{f}(x)$, $L_0(\lambda,s)$ is the characteristic pencil of the equation $(16)$ i.e. $L_0(\lambda,s) = -s^2I + A + \lambda I$.
From $(18)$ it follows that:

$$\|\widehat{u}_1\|_{W_p^2(0,1;H(A),H)} \leq \|\widehat{u}_1\|_{L_p(R;H(A))} + \|\widehat{u}_1''\|_{L_p(\mathbb{R};H)}$$

$$\leq \left\|F^{-1}L_0(\lambda,i\mu)^{-1}F\widetilde{f}(\mu)\right\|_{L_p(R;H(A))} + \left\|F^{-1}(i\mu)^2L_0(\lambda,i\mu)^{-1}F\widetilde{f}(\mu)\right\|_{L_p(\mathbb{R};H)}$$

$$\leq \left\|F^{-1}AL_0(\lambda,i\mu)^{-1}F\widetilde{f}(\mu)\right\|_{L_p(R;H)} + \left\|F^{-1}(i\mu)^2L_0(\lambda,i\mu)^{-1}F\widetilde{f}(\mu)\right\|_{L_p(\mathbb{R};H)} \tag{19}$$

where $F$ is the Fourier transform. Let's prove that the functions

$$T_{k+1}(\lambda,\mu) = (i\mu)^2A^{1-k}L_0(\lambda,i\mu)^{-1}, k = 0,1; \tag{20}$$

are Fourier multipliers of type $(p,p)$ in $H$.
From condition (1) of theorem $(4.1)$ for $|\arg\lambda| \leq \phi < \pi$, $|\lambda| \to \infty$ and $\mu \in \mathbb{R}$ we have

$$\left\|L_0(\lambda,i\mu)^{-1}\right\| = \left\|(A+\lambda I+\mu^2I)^{-1}\right\| \leq C(1+\left|\lambda+\mu^2\right|)^{-1} \leq C\,|\mu|^{-2} \tag{21}$$

$$\left\|AL_0(\lambda,i\mu)^{-1}\right\| = \left\|A(A+\lambda I+\mu^2I)^{-1}\right\| \leq C \tag{22}$$

From $(20)$, $(21)$, $(22)$ we have

$$\|T_1(\lambda,\mu)\|_{B(H)} \leq C\left\|AL_0(\lambda,i\mu)^{-1}\right\|_{B(H)} \leq C \tag{23}$$

$$\|T_2(\lambda,\mu)\|_{B(H)} \le C\mu^2 \left\|L_0(\lambda,i\mu)^{-1}\right\|_{B(H)} \le C \tag{24}$$

Since

$$\frac{\partial}{\partial\mu}T_{k+1}(\lambda,\mu) = 2ki^{2k}\mu^{2k-1}A^{1-k}L_0(\lambda,i\mu)^{-1}$$

$$-i^{2k+1}\mu^{2k}A^{1-k}L_0(\lambda,i\mu)^{-1}\frac{\partial}{\partial\mu}L_0(\lambda,i\mu)L_0(\lambda,i\mu)^{-1},$$

then

$$\left\|\frac{\partial}{\partial\mu}T_{k+1}(\lambda,\mu)\right\|_{B(H)} \le C\,|\mu|^{-1}. \tag{25}$$

According to Mikhlin-Schwartz theorem [6, p.1181], from $(23)\,,(24)\,,(25)$ it follows that the functions $\mu \to T_{k+1}(\lambda,\mu)$ are Fourier multipliers of type $(p,p)$ in $H$.
Then, from $(19)$ we have

$$\|\widetilde{u}_1\|_{W_p^2(\mathbb{R};H(A),H)} \le C\left\|\widetilde{f}\right\|_{L_p(\mathbb{R},H)}$$

hence $u_1 \in W_p^2(0,1;H(A),H)$ and from [14, p.44] we have

$$u_1'(0) \in (H(A),H)_{\frac{1}{2}+\frac{1}{2p},p} = (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p},$$

$$u_1(0) \in (H,H(A))_{1-\frac{1}{2p},p}.$$

Therefore $L_{10}u_1 \in (H,H(A))_{1-\frac{1}{2p},p}$ and $L_{20}u_1 \in (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}$.
From theorem $(3.1)$, the problem $(17)$ when $|\arg\lambda| \le \phi < \pi$, $|\lambda| \to \infty$ has a solution $u_2(x)$ which is in $W_p^2(0,1;H(A),H)$. So, from the theorem $(3.1)$ and the estimate $(5)$ we obtain the desired inequality $(15)$.

# 5 Fredholmness for an abstract differential equation

We can, now, find conditions for Fredholm solvability the problem (1), (2). It is more practical to formulate this in terms of unbounded Fredholm operators in Banach spaces.

Let's define $\mathbb{L}$ by

$$\begin{array}{llll} \mathbb{L}: & W_p^2(0,1;H(A),H) & \to & L_p(0,1;H)\oplus(H,H(A))_{1-\frac{1}{2p}}\oplus(H,H(A))_{\frac{1}{2}-\frac{1}{2p}} \\ & u & \to & (L(D)u, L_1u, L_2u) \end{array}$$

where

$$D(\mathbb{L}) = \big\{u; u\in W_p^2(0,1;H(A),H); L(D)u \in L_p(0,1;H),$$

$$L_1u \in (H,H(A))_{1-\frac{1}{2p},p}, L_2u \in (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}\Big\}$$

$L(D)u$, $L_1u$ and $L_2u$ are defined previously..

**Theorem 5.1** *Suppose the following conditions satisfied*

1. *$A$ is a linear closed positive operator in $H$. The injection of $H(A)$ in $H$ is compact.*
2. *$B$ is continuous from $H(A)$ in $H(A^{1/2})$ and from $H(A^{1/2})$ in $H$.*
3. *$\delta \neq 0$.*
4. *$A(x): H(A) \to H(A^{\frac{1}{2}})$ is compact for almost all $x \in [0,1]$.*
   *$\forall\varepsilon > 0$, for almost all $x \in [0,1]$,*

   $$\|A(x)u\|_{H(A^{\frac{1}{2}})} \le \varepsilon\,\|u\|_{H(A)} + C(\varepsilon)\,\|u\|_H\,, u \in D(A).$$

   *The functions $A(x)u$, when $u \in H$, are measurable from $[0,1]$ in $H(A^{\frac{1}{2}})$.*

*Then*

***a*** - $\forall u \in W_p^2(0,1;H(A),H)$, *and for* $|\arg\lambda| \leq \phi < \pi, |\lambda| \longrightarrow \infty$, $p \in (1,\infty)$, *the non coercive estimate holds*

$$\|u''\|_{L_p(0,1;H)} + \|Au\|_{L_p(0,1;H)} \leq C(\lambda)\left\{\|L(D)u\|_{L_p(0,1;H)}\right.$$

$$\left. + \|L_1u\|_{(H,H(A))_{1-\frac{1}{2p},p}} + \|L_2u\|_{f_2\in(H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}} + \|u\|_{L_p(0,1;H)}\right\}. \tag{26}$$

***b*** - $\mathbb{L} : u \longrightarrow (L(D)u, L_1u, L_2u)$ *from* $W_p^2(0,1;H(A),H)$ *to* $L_p(0,1;H) \oplus (H,H(A))_{1-\frac{1}{2p}} \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p}}$ *is a Fredholm.*

**Proof.**
**a**- Let $u$ be a solution of $(1),(2)$ belonging to $W_p^2(0,1;H(A),H)$. Then $u$ is a solution of the problem

$$L_0(\lambda,D)u = f(x) + \lambda u(x) - A(x)u(x) \qquad x \in (0,1), \tag{27}$$

$$\begin{array}{rcccl} L_{10}u &=& \delta u(0) &=& f_1 \\ L_{20}u &=& u'(1) + Bu(0) &=& f_2. \end{array} \tag{28}$$

From theorem $(4.1)$ we get the estimate

$$\|u''\|_{L_p(0,1;H)} + \|Au\|_{L_p(0,1;H)} \leq C(\lambda)\left\{\|f(.) + \lambda u(.) - A(.)u(.)\|_{L_p(0,1;H)}\right.$$

$$\left. + \|f_1\|_{(H,H(A))_{1-\frac{1}{2p},p}} + \|f_2\|_{f_2\in(H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}}\right\}. \tag{29}$$

Condition (4) of theorem and [16, lemme 1.2, p.162] yield the compacity of the operator

$$u \to \lambda u - A(.)u$$

$$W_p^2(0,1;H(A),H) \to L_p(0,1;H).$$

And from [16, lemme 2.7, p.53] $\forall\varepsilon > 0$ we have

$$\|f(.) + \lambda u(.) - A(.)u(.)\|_{L_p(0,1;H)} \leq$$

$$C\left[\|f\|_{L_p(0,1;H)} + \varepsilon\left(\|u\|_{L_p(0,1;H)} + \|Au\|_{L_p(0,1;H)}\right) + C(\varepsilon)\,\|u\|_{L_p(0,1;H)}\right].$$

Substituting in (29) we obtain the desired estimate.
**b**- The operator $\mathbb{L}$ can be written in the form

$$\mathbb{L} = \mathbb{L}_{0\lambda} + \mathbb{L}_{1\lambda} \tag{30}$$

where

$$D(\mathbb{L}_{0\lambda}) = \left\{u; u \in W_p^2(0,1;H(A),H); L_0(\lambda,D)u \in L_p(0,1;H),\right.$$

$$\left. L_{10}u \in (H,H(A))_{1-\frac{1}{2p},p}, L_{20}u \in (H,H(A))_{\frac{1}{2}-\frac{1}{2p},p}\right\}$$

$$\mathbb{L}_{0\lambda}u = (L_0(\lambda,D)u, L_{10}u, L_{20}u)$$

$$L_0(\lambda,D)u = -u''(x) + (A+\lambda I)u(x), \quad x \in [0,1]$$

$$L_{10}u = \delta u(0)$$

$$L_{20}u = u'(1) + Bu(0)$$

and

$$\mathbb{L}_{1\lambda}u = (-\lambda u(x) + A(x)u(x), 0, 0)\,,$$

condition $(4)$ implies that $D(\mathbb{L}) = D(\mathbb{L}_{0\lambda})$. According to the theorem (4.1), when $\lambda \to \infty$,the operator

$$\mathbb{L}_{0\lambda} : W_p^2(0,1;H(A),H) \to L_p(0,1;H) \oplus (H, H(A))_{1-\frac{1}{2p}} \oplus (H, H(A))_{\frac{1}{2}-\frac{1}{2p}}$$

has an inverse. Always, from condition $(4)$ of the theorem and from [16, lemme 1.2, p.162] we show that the operator $\mathbb{L}_{1\lambda}u$ from $W_p^2(0,1;H(A),H)$ in $L_p(0,1;H) \oplus (H,H(A))_{1-\frac{1}{2p}} \oplus (H,H(A))_{\frac{1}{2}-\frac{1}{2p}}$ is compact. Applying the perturbation theorem on Fredholm operators (see [12] or [10, p.238]), to the operator (30) we get the desired result.

# 6 Fredholmness for non regular elliptic problems

The Fredholm property for regular elliptic boundary value problems has been proved, in particular in, [1, 2, 3, 8]. In the case of non regular problems it has been done for a class of elliptic problems which still coercive, in [4, 5]. Here, we establish the same property, but for a class of elliptic problems which are not coercive. In [17, 18] other non regular boundary conditions for the same type of equations are considered and analogous results are proved.

Consider in the domain $[0,1] \times [0,1]$ the non regular boundary value problem for a second order elliptic equation

$$\begin{aligned} L(x,y,D_x,D_y)u &= D_x^2u(x,y) + D_y(a(y)D_yu(x,y)) + A(x)u(x,.)|_y \\ &= f(x,y), \quad (x,y) \in [0,1] \times [0,1]\,, \end{aligned} \tag{31}$$

$$\begin{aligned} P_1u &= D_y^{m_1}u(x,0) + \sum_{j \le m_1-1} a_jD_y^ju(x,0) = 0, \quad x \in [0,1]\,, \\ P_2u &= D_y^{m_2}u(x,1) + \sum_{j \le m_2-1} b_jD_y^ju(x,1) = 0, \quad x \in [0,1]\,, \end{aligned} \tag{32}$$

$$\begin{aligned} L_1u &= \delta u(0,y) = f_1(y), \quad y \in [0,1]\,, \\ L_2u &= D_xu(1,y) + Bu(0,y) = f_2(y), \quad y \in [0,1]\,, \end{aligned} \tag{33}$$

where $0 \le m_1, m_2 \le 1, \quad a_j, b_j, \delta \in \mathbb{C}\,, \quad D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}$.
Note, as in [14, p.186,317], $\left(L_2(G), W_2^2(G)\right)_{\theta,p} = B_{2,p}^{2\theta}(G)$ where $G \subset \mathbb{R}^r$ is a bounded domain.

**Theorem 6.1** *Suppose*

1. $a \in C^1(0,1)\,, \exists \rho > 0$ *such that* $a(y) \ge \rho$.
2. $A(x) : W_2^2(0,1;P_su = 0)$ *in* $W_2^1(0,1;P_su = 0, m_s = 0)$ *are compact for almost all* $x \in [0,1]\,; \forall \varepsilon > 0$ *and for almost all* $x \in [0,1]$ $\|A(x)u\|_{W_2^1(0,1)} \le \varepsilon \|u\|_{W_2^2(0,1)} + C(\varepsilon)\|u\|_{L_2(0,1)}$ $\forall u \in W_2^2(0,1;P_su = 0)$; *the function* $x \to A(x)u$ *from* $[0,1]$ *in* $W_2^1(0,1)$ *are measurable.*
3. $B$ *is continuous from* $H(A)$ *in* $H(A^{1/2})$ *and from* $H(A^{1/2})$ *in* $H$.

*Then the operator*

$$\mathcal{L}: u \to (L(x,y,D_x,D_y)u, L_1u, L_2u)$$

*from*

$$W_p^2(0,1;W_2^2(0,1;P_su=0),L_2(0,1))$$

*in*

$$L_p(0,1;W_2^1(0,1;P_su=0,m_s=0))$$

$$\bigoplus_{k=1}^{2} B_{2,p}^{k+1-\frac{1}{p}}\left(0,1;P_su=0,m_s< k+\frac{1}{2}-\frac{1}{p};P_s(a(y)u'(y))'=0,m_s< k-\frac{3}{2}-\frac{1}{p}\right)$$

*is a Fredholm.*

**Proof.**

Set $H=L_2(0,1)$, and define the operators $A$ and $\mathbb{A}(x)$ by the equalities

$$D_A=W_2^2(0,1;P_su=0)\quad Au=-D_y(a(y)D_yu(y)),$$

$$D_{\mathbb{A}(x)}=W_2^1(0,1;P_su=0,m_s=0)\quad \mathbb{A}(x)u=A(x)u|_y.$$

The problem $(31)$, $(32)$, $(33)$ can be written as

$$u"(x)-Au(x)+\mathbb{A}(x)u(x)=f(x), \tag{34}$$

$$\delta u'(0)=f_1$$
$$u'(1)+Bu(0)=f_2 \tag{35}$$

with $u(x)=u(x,.)$, $f(x)=f(x,.)$ and $f_k=f_k(.)$ are functions with values in the Hilbert space $H=L_2(0,1)$.

Applying the theorem $(5.1)$ to the problem $(34)$ $(35)$. From condition $(1)$ the operator $A$ is positively defined in $H$ and from [14, p.344] the injection $W_2^2(0,1;P_su=0)\subset L_2(0,1)$ is compact, then the condition $(1)$ of the theorem $(5.1)$ is satisfied. From [9] (see also [14, p.321]) we have

$$\begin{aligned}(H,H(A))_{\theta,p} &= \left(L_2(0,1),W_2^2(0,1;P_su=0)\right)_{\theta,p}\\ &= B_{2,p}^{2\theta}\left(0,1;P_ku=0,m_k<2\theta-1/2\right).\end{aligned}$$

Therefore

$$\begin{aligned}(H,H(A))_{1/2} &= B_{2,2}^{1}\left(0,1;P_ku=0,m_k=0\right)\\ &= W_2^{1}\left(0,1;P_ku=0,m_k=0\right),\end{aligned}$$

and

$$(H,H(A))_{\frac{2p-jp-1}{2p},p}=B_{2,p}^{2-j-1/p}\left(0,1;P_ku=0,m_k<3/2-j-1/p\right).$$

So, for the problem $(34)$ $(35)$, all the conditions of theorem $(5.1)$ are satisfied. From which it may be concluded the assertion of the theorem $(6.1)$.

Consider, now, in the cylindrical domain $[0,1]\times G$ where $G\subset\mathbb{R}^r$ is a bounded domain, the boundary value problem for the second order elliptic equation

$$\begin{aligned}L(x,y,D_x,D_y)u &= D_x^2u(x,y)+\sum_{i,j=1}^{r}D_i(a_{ij}(y)D_ju(x,y))+A(x)u(x,.)|_y\\ &= f(x,y),\quad (x,y)\in[0,1]\times G, \end{aligned}\tag{36}$$

$$Pu = \sum_{|\eta| \leq m} b_\eta D_y^\eta u(x, y') = 0, \quad (x, y') \in [0,1] \times \Gamma, \tag{37}$$

$$\begin{aligned} L_1 u &= \delta u(0, y) = f_1(y), \quad y \in [0,1], \\ L_2 u &= D_x u(1, y) + Bu(0, y) = f_2(y), \quad y \in [0,1], \end{aligned} \tag{38}$$

where $m \leq 1$; $\delta \in \mathbb{C}^*$, $D_x = \frac{\partial}{\partial x}$, $\Gamma = \partial G$ is the boundary of $G$. $D_x = \frac{\partial}{\partial x} D_y^\eta = D_1^{\eta_1} \cdots D_r^{\eta_r}$, $D_j^{\eta_j} = \frac{\partial}{\partial y_j}$, $\eta = (\eta_1, \cdots, \eta_r)$, $|\eta| = \eta_1 + \cdots + \eta_r$.

**Theorem 6.2** *Suppose*

1. $a_{ij} \in C^1\left(\overline{G}\right), b_\eta \in C^{2-m}\left(\overline{G}\right), \Gamma \in C^2$.
2. $a_{ij}(y) = \overline{a_{ji}(y)}$; $\exists \gamma > 0$ *such that* $\sum_{k,j=1}^{r} a_{kj}(y)\sigma_k\sigma_j \geq \gamma \sum_{k=1}^{r} \sigma_k^2, y \in \overline{G}, \sigma \in \mathbb{R}^r$
3. $A(x) : W_2^2(G; Pu = 0)$ *in* $W_2^1(G; Pu = 0, m = 0)$ *are compact for almost all* $x \in G$; $\forall \varepsilon > 0$ *and for almost all* $x \in [0,1]$ *we have* $\|A(x)u\|_{W_2^1(G)} \leq \varepsilon \|u\|_{W_2^2(G)} + C(\varepsilon) \|u\|_{L_2(G)}$ $\forall u \in W_2^2(G; Pu = 0)$; *the functions* $x \to A(x)u$ *from* $[0,1]$ *in* $W_2^1(G)$ *are measurable.*
4. $B$ *is continuous from* $H(A)$ *in* $H(A^{1/2})$ *and from* $H(A^{1/2})$ *in* $H$.

*Then the operator*

$$\mathcal{L} : u \to (L(x, y, D_x, D_y)u, L_1 u, L_2 u)$$

*from*

$$W_p^2(0,1; W_2^2(G; Pu = 0), L_2(G))$$

*in*

$$L_p(0,1; W_2^1(G; Pu = 0, m = 0))$$

$$\bigoplus_{k=1}^{2} B_{2,p}^{k+1-\frac{1}{p}} \left( G; Pu = 0, m < k + \frac{1}{2} - \frac{1}{p}; P \sum_{i,j=1}^{r} D_i(a_{ij}(y) D_j u(y)) = 0, m < k - \frac{3}{2} - \frac{1}{p} \right)$$

*is a Fredholm.*

**Proof.**

Set $H = L_2(G)$, the operators $A$ are $\mathbb{A}(x)$ defined by the equalities

$$D_A = W_2^2(G; Pu = 0) \quad Au = -\sum_{i,j=1}^{r} D_i(a_{ij}(y) D_j u(y)),$$

$$D_{\mathbb{A}(x)} = W_2^1(G; Pu = 0, m = 0) \quad \mathbb{A}(x)u = A(x)u|_y.$$

The problem $(36), (37), (38)$ can be written in the form $(34), (35)$ i.e.

$$u"(x) - Au(x) + \mathbb{A}(x)u(x) = f(x),$$

$$\delta u'(0) = f_1$$

$$u'(1) + Bu(0) = f_2$$

with $u(x) = u(x, .)$, $f(x) = f(x, .)$ and $f_k = f_k(.)$ are functions with values in the Hilbert space $H = L_2(0,1)$. The rest of the proof is similar of the one of the theorem $(6.1)$.

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