Electronic Journal of Qualitative Theory of Differential Equations
2004, No. **10**, 1-22; **http://www.math.u-szeged.hu/ejqtde/**

Oscillation Criteria For Second Order Superlinear Neutral Delay Differential Equations

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Abstract. New oscillation criteria for the second order nonlinear neutral delay differential equation $[y(t)+p(t)y(t-\tau)]''+q(t)\,f(y(g(t)))=0$, $t\geq t_0$ are given. The relevance of our theorems becomes clear due to a carefully selected example.

Keywords and Phrases: Oscillation, second order nonlinear neutral delay differential equations.

2000 AMS Subject Classification: 34K11, 34K40.

1. Introduction

Consider the second order nonlinear neutral delay differential equation

$$(E)\qquad [y(t)+p(t)y(t-\tau)]''+q(t)f(y(g(t)))=0,\ t\in[t_0,\infty),$$

where

$$\left.\begin{array}{c}\tau \geq 0,\ q \in C([t_0,\infty),R^+),\ p \in C^1([t_0,\infty),R^+),\ 0 \leq p(t) < 1,\\ g \in C^2([t_0,\infty),R^+),\ g(t) \leq t,\ g^{'}(t) > 0,\ \lim_{t\to\infty} g(t) = \infty,\end{array}\right\} \tag{1}$$

(F_1) $\quad f \in C(R),\ f' \in C(R\backslash\{0\}),\quad u\,f(u) > 0,\ f'(u) \geq 0,\quad$ for $\quad u \neq 0\,.$

Our attention is restricted to those solutions of (E) that satisfies $sup\{|y(t)| : t \geq T\} > 0$. We make a standing hypothesis that (E) does possess such solutions. By a solution of (E) we mean a function $y(t) : [t_0,\infty) \to R$, such that $y(t)+p(t)y(t-\tau) \in C^2(t \geq t_0)$ and satisfies (E) on $[t_0,\infty)$. For further question concerning existence and uniqueness of solutions of neutral delay differential equations see Hale [18].

A solution of Eq. (E) is said to be oscillatory if it is defined on some ray $[T,\infty)$ and has an infinite sequence of zeros tending to infinity; otherwise it called nonoscillatory. An equation itself is called oscillatory if all its solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions of second order linear and nonlinear neutral delay differential equations. (See for example [5]-[17],[23], [24], [26] and the references quoted therein). Most of these papers considered the equation (E) under the assumption that $f'(u) \geq k > 0$ for u$\neq$ 0, which is not applicable for $f(u) = |u|^\lambda \operatorname{sgn} u$ - classical Emden-Fowler case. Very recently, the results of Atkinson [3] and Belhorec [4] for Emden-Fowler differential equations have been extended to the equation (E) by Wong [26] under the assumption that the nonlinear function $f(u)$ satisfies the sublinear condition

$$0 < \int_{0+}^{\varepsilon} \frac{du}{f(u)},\ \int_{0-}^{-\varepsilon} \frac{du}{f(u)} < \infty \quad \text{for all} \quad \varepsilon > 0$$

as well as the superlinear condtion

$$0 < \int_{\varepsilon}^{+\infty} \frac{du}{f(u)},\ \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty \quad \text{for all} \quad \varepsilon > 0.$$

The special case where $f(u) = |u|^\gamma \operatorname{sgn} u$, $u \in R$, $(0 < \gamma \neq 1)$ is of particular interest. In this case, the differential equation (E) becomes

(EF) $\qquad [y(t)+p(t)y(t-\tau)]^{''} + q(t)\, y^\gamma(g(t))\, \operatorname{sgn} y(g(t)) = 0,\ t \geq t_0\,.$

The equation (EF) is sublinear for $\gamma \in (0,1)$ and it is superlinear for $\gamma > 1$.

Established oscillation criteria have been motivated by classical averaging criterion of Kamenev, for the linear differential equation $x''(t) + q(t)x(t) = 0$. More recently, Philos [25] introduced the concept of general means and obtained further extensions of the Kamenev type oscillation criterion for the linear differential equation. The subject of extending oscillation criteria for the linear differential equation to that of the Emden-Fowler equation and the more general equation $x''(t) + q(t)f(x(t)) = 0$ has been of considerable interest in the past 30 years.

The object of this paper is to prove oscillation criteria of Kamenev's and Philos's type for the equation (E).

For other oscillation results of various functional differential equation we refer the reader to the monographs [1, 2, 7, 17, 21].

2. Main Results

In this section we will establish some new oscillation criteria for oscillation of the superlinear equation (E) subject to the nonlinear condition

$$(F_2) \qquad f'(u)\,\Lambda(u) \geq \lambda > 1, \quad u \neq 0,$$

where

$$\Lambda(u) = \begin{cases} \displaystyle\int_u^{\infty} \frac{ds}{f(s)}, & u > 0 \\ \displaystyle\int_u^{-\infty} \frac{ds}{f(s)}, & u < 0 \end{cases}$$

Considering the special case where $f(u) = |u|^{\gamma} \operatorname{sgn} u$, it is easy to see that (F_1) holds when $\gamma > 1$.

It will be convenient to make the following notations in the remainder of this paper. Let $\Phi(t, t_0)$ denotes the class of positive and locally integrable functions, but not integrable, which contains all the bounded functions for $t \geq t_0$. For arbitrary functions $\varrho \in C^1[[t_0, \infty), R^+]$ and $\phi \in \Phi(t, t_0)$, we define

$$Q(t) = q(t)f[(1 - p(g(t))], \quad \alpha(t, T) = \int_T^t \phi(s)ds,$$

$$\nu(t,T)=\frac{1}{\phi(t)}\int\limits_T^t \frac{\varrho(s)\,\phi^2(s)}{g'(s)}ds,\quad \nu_1(t,T)=\frac{1}{\phi(t)}\int\limits_T^t \frac{\phi^2(s)}{g'(s)}ds,$$

$$A_{\varrho,\phi}(t,T)=\frac{1}{\alpha(t,T)}\int\limits_T^t \phi(s)\int\limits_T^s \varrho(u)Q(u)\,du\,ds,$$

$$B_\phi(t,T)=\frac{1}{\alpha(t,T)}\int\limits_T^t \phi(s)\int\limits_T^s Q(u)\,du\,ds.$$

2.1. Kamenev's Type Oscillation Criteria

Theorem 2.1 *Assume that* (1) *holds and let the function* $f(u)$ *satisfies the assumptions* (F_1) *and* (F_2). *Suppose that there exist* $\phi\in\Phi(t,t_0)$ *and* $\varrho\in C^1[[t_0,\infty),R^+]$ *such that*

$$(C_1)\qquad \varrho'(t)\geq 0,\quad \left(\frac{\varrho'(t)}{g'(t)}\right)'\leq 0,\quad for\ \ t\geq t_0,$$

$$(C_2)\qquad \int\limits_{t_0}^{\infty}\frac{\alpha^\mu(s,T)}{\nu(s,T)}ds=\infty,\quad 0<\mu<1.$$

The superlinear equation (E) *is oscilatory if*

$$(C_3)\qquad \lim_{t\to\infty}A_{\varrho,\phi}(t,T)=\infty,\quad T\geq t_0\,.$$

Proof. Let $y(t)$ be a nonoscillatory solution of Eq.(E). Without loss of generality, we assume that $y(t)\neq 0$ for $t\geqslant t_0$. Further, we suppose that there exists a $t_1\geq t_0$ such that $y(t)>0$, $y(t-\tau)>0$ and $y(g(t))>0$ for $t\geqslant t_1$, since the substitution $u=-y$ transforms Eq. (E) into an equation of the same form subject to the assumption of Theorem. Let

$$x(t)=y(t)+p(t)y(t-\tau)\tag{2}$$

By (1) we see that $x(t)\geq y(t)>0$ for $t\geqslant t_1$, and from (E) it follows that

$$x''(t)=-q(t)f(y(g(t)))<0,\qquad \text{for}\quad t\geqslant t_1.\tag{3}$$

Therefore $x^{'}(t)$ is decreasing function. Now, as $x(t) > 0$ and $x^{''}(t) < 0$ for t$\geq t_1$, in view of Kiguradze Lemma [21], we have immediately that $x^{'}(t) > 0$, for $t \geqslant t_1$. Consequently,

$$x(t) > 0,\ x'(t) > 0,\ x''(t) < 0, \quad \text{for} \quad t \geqslant t_1, \tag{4}$$

and there exists positive constant K_1 and $T \geq t_1$ such that

$$x(g(t)) \geq K_1 \quad \text{for} \quad t \geq T\,. \tag{5}$$

Now using (4) in (2), we have

$$\begin{aligned} y(t) &= x(t) - p(t)y(t-\tau) = x(t) - p(t)[x(t-\tau) - p(t-\tau)y(t-2\tau)] \\ &\geq x(t) - p(t)x(t-\tau) \geq (1-p(t))x(t). \end{aligned}$$

Thus

$$y(g(t)) \geq (1 - p(g(t))x(g(t)) \quad \text{for} \quad t \geq t_1\,. \tag{6}$$

Using (F_2), we get

$$f(y(g(t)) \geq f[1 - p(g(t))] \cdot f[x(g(t))] \quad \text{for} \quad t \geq t_1\,. \tag{7}$$

Then, from (6) it follows that

$$x^{''}(t) + Q(t)\, f[x(g(t))] \leq 0, \quad t \geq t_1. \tag{8}$$

Define the function

$$w(t) = \varrho(t)\frac{x^{'}(t)}{f[x(g(t))]}, \quad \text{for} \quad t \geqslant t_1. \tag{9}$$

Then $w(t) > 0$. Differentiating (9) and using (8), we have

$$\begin{aligned} w^{'}(t) &\leq -\varrho(t)Q(t) + \varrho^{'}(t)\frac{x^{'}(t)}{f[x(g(t))]} \\ &\quad - \varrho(t)g^{'}(t)\frac{f'[x(g(t))]}{f^2[x(g(t))]}\, x^{'}(t)x^{'}(g(t)),\ t \geq t_1 \end{aligned} \tag{10}$$

Since f is nondecreasing function, $g(t) \leq t$, taking into account (4), we have

$$x(t) \geq x(g(t)),\ x'(t) \leq x'(g(t)),\ f(x(t)) \geq f[x(g(t))], \quad \text{for} \quad t \geq t_1\,.$$

Also, since Λ is nonincreasing function, from (5), we conclude that there exists positive constant K such that

$$\Lambda[x(g(t))] \leq K \quad \text{for} \quad t \geq T\,,$$

which together with (F_2), implies that

$$f'[x(g(t))] \geq \frac{\lambda}{K} = \Omega \quad \text{for} \quad t \geq T\,. \tag{11}$$

Then from (10), we get

$$w^{'}(t) \leq -\varrho(t)Q(t) + \frac{\varrho^{'}(t)}{\varrho(t)}\,w(t) - \Omega\,\varrho(t)\,g'(t)\left(\frac{x^{'}(t)}{f[x(g(t))]}\right)^2,\ \ t \geq T. \tag{12}$$

Integrate (12) from T to t, so that we have

$$\begin{aligned} w(t) = \varrho(t)\frac{x^{'}(t)}{f[x(g(t))]} \quad &\leq \quad C - \int_T^t \varrho(s)Q(s)ds + \int_T^t \frac{\varrho^{'}(s)}{\varrho(s)}\,w(s)\,ds \\ &\quad -\Omega \int_T^t \varrho(s)\,g'(s)\left(\frac{x^{'}(s)}{f[x(g(s))]}\right)^2 ds \end{aligned} \tag{13}$$

where $C = \varrho(T)\,x^{'}(T)/\,f[x(g(T))]$.

Using the fact that $\varrho'(t)/g'(t)$ is positive, nonincreasing function, by Bonnet Theorem, there exists a $\zeta \in [T, t]$, so that

$$\begin{aligned} \int_T^t \varrho'(s)\,\frac{x'(s)}{f[x(g(s))]}\,ds \quad &\leq \quad \int_T^t \frac{\varrho^{'}(s)}{g^{'}(s)}\frac{x^{'}(g(s))g^{'}(s)}{f[x(g(s))]}\,ds \\ &= \quad \frac{\varrho^{'}(T)}{g^{'}(T)}\int_T^\zeta \frac{x^{'}(g(s))g^{'}(s)}{f[x(g(s))]}\,ds \\ &= \quad \frac{\varrho^{'}(T)}{g^{'}(T)}\int_{x(g(T))}^{x(g(\zeta))} \frac{du}{f(u)} \leq \frac{\varrho^{'}(T)}{g^{'}(T)}\,\Lambda[x(g(T))] = M\,. \end{aligned}$$

Thus, for $t \geq T$, we find from (13), that

$$w(t) + \Omega \int_T^t \frac{g'(s)}{\varrho(s)}\,w^2(s)\,ds \leq L - \int_T^t \varrho(s)Q(s)ds \tag{14}$$

where $L = C + M$. We multiply (14) by $\phi(t)$ and integrate from T to t, we get

$$\int_T^t \phi(s)w(s)ds + \Omega \int_T^t \phi(s) \int_T^s \frac{g'(u)}{\varrho(u)}\, w^2(u)duds \le \alpha(t,T)[L_1 - A_{\varrho,\phi}(t,T)].$$

Using the condition (C_3), there exists a $t_3 \ge T$ such that $L_1 - A_{\varrho,\phi}(t,T) \le 0$ for $t \ge t_3$. Then, for every $t \ge t_3$

$$G(t) = \Omega \int_T^t \phi(s) \int_T^s \frac{g'(u)}{\varrho(u)}\, w^2(u)duds \le -\int_T^t \phi(s)w(s)ds\,.$$

Since G is nonnegative, we have

$$G^2(t) \le \left(\int_T^t \phi(s)w(s)ds\right)^2, \quad t \ge t_3. \tag{15}$$

By Schwarz inequality, we obtain

$$\begin{aligned} \left\{\int_T^t \phi(s)\, w(s)ds\right\}^2 &\le \left(\frac{1}{\phi(t)} \int_T^t \frac{\varrho(s)\, \phi^2(s)}{g'(s)}\, ds\right) \left(\phi(t) \int_T^t \frac{g'(s)}{\varrho(s)} w^2(s)ds\right) \\ &= \nu(t,T)\, \frac{G'(t)}{\Omega}, \quad t \ge t_3. \end{aligned} \tag{16}$$

Now,

$$\begin{aligned} G(t) &= \Omega \int_T^t \phi(s) \int_T^s \frac{g'(u)}{\varrho(u)} w^2(u)duds \\ &\ge \Omega \left(\int_T^{t_3} \frac{g'(u)}{\varrho(u)} w^2(u)du\right) \int_T^t \phi(s)\, ds = \Omega\, Q \cdot \alpha(t,T)\,. \end{aligned} \tag{17}$$

From (15), (16) and (17), for all $t \ge t_3$ and some $\mu,\ 0 < \mu < 1$, we get

$$\Omega^{\mu+1}\, Q^{\mu}\, \frac{\alpha^{\mu}(t,T)}{\nu(t,T)} \le G^{\mu-2}(t)G'(t)\,. \tag{18}$$

Integrating (18) from t3 to t, we obtain

$$\Omega^{\mu+1}\, Q^{\mu} \int\limits_{t_3}^{t} \frac{\alpha^{\mu}(s,T)}{\nu(s,T)} ds \leq \left[\frac{1}{1-\mu} \frac{1}{G^{1-\mu}(t_3)} \right] < \infty$$

and this contradicts the assumption (C_2). Therefore, the equation (E) is oscillatory. ■

Since the differentiable function $\varrho(t) \equiv 1$ satisfied the condition (C_1), we have the following Corollary.

Corollary 2.1 *Let the following condition holds*

$$\int\limits_{t_0}^{\infty} \frac{\alpha^{\mu}(s,T)}{\nu_1(s,T)}\, ds = \infty, \quad 0 < \mu < 1\,.$$

The superlinear equation (E) *is oscillatory if*

$$\lim_{t\to\infty} B_{\phi}(t, t_0) = \infty\,.$$

Example 2.1 Consider the following delay differential equation

$$(E_1) \quad \left(y(t) + \frac{1}{2}\, y\Big(t - \frac{\pi}{2}\Big) \right)' + \frac{2+\cos t}{t\,\sqrt{t}}\, y^{\lambda}\Big(\frac{t}{2}\Big)\ \operatorname{sgn} y\Big(\frac{t}{2}\Big) = 0, \qquad t \geq \frac{\pi}{2}.$$

where $\lambda > 1$. Here $g(t) = \dfrac{t}{2}$ and $Q(t) = \dfrac{2+\cos t}{2^{\lambda}\, t\,\sqrt{t}}$. We choose $\varrho(t) = \frac{t}{2}$ and $\phi(t) = \frac{1}{t}$. Then

$$\alpha(t, \frac{\pi}{2}) = \ln \frac{2t}{\pi}, \quad \nu(t, \frac{\pi}{2}) = t\, \ln \frac{2t}{\pi}\,.$$

Now, we see that

$$\int\limits_{\pi/2}^{t} \frac{\alpha^{\mu}(s,\pi/2)}{\nu(s,\pi/2)}\, ds = \int\limits_{\pi/2}^{t} \frac{1}{s} \left(\ln \frac{2s}{\pi} \right)^{\mu-1} ds = \frac{1}{\mu} \left(\ln \frac{2t}{\pi} \right)^{\mu},$$

so that for $\mu > 0$

$$\lim_{t\to\infty} \int\limits_{\pi/2}^{t} \frac{\alpha^{\mu}(s,\pi/2)}{\nu(s,\pi/2)}\, ds = \infty\,.$$

Further,

$$\begin{aligned}
A_{\varrho,\phi}(t,\pi/2) &= \frac{1}{\alpha(t,T)}\int_T^t \phi(s)\int_T^s \varrho(u)Q(u)\,du\,ds\\
&= \frac{1}{\ln\frac{2t}{\pi}}\int_{\pi/2}^t \frac{1}{s}\int_{\pi/2}^s \frac{u}{2}\,\frac{2+\cos u}{2^\lambda u\,\sqrt{u}}\,du\,ds \geq \frac{1}{2^{\lambda+1}\ln\frac{2t}{\pi}}\int_{\pi/2}^t \frac{1}{s}\int_{\pi/2}^s \frac{1}{\sqrt{u}}\,du\,ds\\
&= \frac{1}{2^\lambda\ln\frac{2t}{\pi}}\left[2\sqrt{t}-2\sqrt{\frac{\pi}{2}}-\sqrt{\frac{\pi}{2}}\ln\frac{2t}{\pi}\right],
\end{aligned}$$

so that

$$\limsup_{t\to\infty} A_{\varrho,\phi}(t,\pi/2)=\infty\,.$$

Consequently, all conditions of Theorem 2.1 are satisfied, and hence the equation (E_1) is oscillatory.

Theorem 2.2 *Let the function ϱ satisfies the condition (C_1). If there exists a positive constant Ω and $T\geq t_0$ such that*

$$(C_4)\qquad \limsup_{t\to\infty}\int_T^t\left(\varrho(s)Q(s)-\frac{1}{4\,\Omega}\,\frac{\varrho'^{\,2}(s)}{\varrho(s)g'(s)}\right)ds=\infty$$

then the superlinear equation (E) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of Eq. (E). Without loss of generality, we assume that $y(t)\neq 0$ for $t\geqslant t_0$. Further, we suppose that there exists a $t_1\geq t_0$ such that $y(t)>0$, $y(t-\tau)>0$ and $y(g(t))>0$ for $t\geqslant t_1$. Next consider the function $w(t)$ defined with (9). Then, as in the proof of Theorem 2.1 we have that there exists a constant $\Omega>0$ and $T\geq t_1$, such that (12) is satisfied. From (12) we get

$$w'(t)+\varrho(t)Q(t)\leq\frac{\varrho'(t)}{\varrho(t)}w(t)-\Omega\,\frac{g'(t)}{\varrho(t)}\,w^2(t),\quad t\geqslant T \tag{19}$$

or

$$w'(t)+\varrho(t)Q(t)-\frac{1}{4\,\Omega}\,\frac{\varrho'^{\,2}(t)}{\varrho(t)g'(t)}\leq-\Omega\,\frac{g'(t)}{\varrho'(t)}\left(w(t)-\frac{1}{2\Omega}\,\frac{\varrho'(t)}{g'(t)}\right)^2<0$$

Thus, integrating the former inequality from T to t, we are lead to

$$\int_T^t \left(\varrho(s)Q(s) - \frac{1}{4\,\Omega}\,\frac{\varrho'^{\,2}(s)}{\varrho(s)g'(s)} \right) ds < w(T) - w(t) < w(T) < \infty$$

and this contradicts (C_4). Then every solution of Eq. (E) oscillates. ∎

By choosing $\varrho(t) \equiv 1$, we get the following Corollary.

Corollary 2.2 *The superlinear equation (E) is oscillatory if*

$$\limsup_{t\to\infty} \int_T^t Q(s)ds = \infty\,.$$

2.2. Philos's Type Oscillation Criteria

Next, we present some new oscillation results for Eq. (E), by using integral averages condition of Philos-type. Following Philos [25], we introduce a class of functions $\Re$. Let

$$D_0 = \{(t,s) : \mathrm{t}{>}\mathrm{s} \geq \mathrm{t}_0\} \;\text{ and }\; \mathrm{D} = \{(\mathrm{t},\mathrm{s}) : \mathrm{t} \geq \mathrm{s} \geq \mathrm{t}_0\}.$$

The function $H \in C(D,R)$ is said to belongs to the class $\Re$ if

(I) $H(t,t) = 0$ for $t \geq t_0$; $H(t,s) > 0$ for $(t,s) \in D_0$;

(II) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable such that

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \;\text{ forall }\; (\mathrm{t},\mathrm{s}) \in \mathrm{D}_0.$$

Theorem 2.3 *The superlinear equation (E) is oscillatory if there exist the functions $\varrho \in C^1[[t_0,\infty),R^+]$, $H \in \Re$ and the constant $\Omega > 0$, such that*

$$(C_5) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\varrho(s)Q(s) - \frac{\eta(s)G^2(t,s)}{4\,\Omega} \right] ds = \infty.$$

where

$$G(t,s) = \frac{\varrho'(s)}{\varrho\,(s)}\,\sqrt{H(t,s)} - h(t,s)\,, \quad \eta(t) = \frac{\varrho\,(t)}{g'(t)}\,.$$

Proof. Let $y(t)$ be a nonoscillatory solution of Eq. (E). Without loss of generality, we assume that $y(t) \neq 0$ for $t \geqslant t_0$. Further,we suppose that there exists a $t_1 \geq t_0$ such that $y(t) > 0$, $y(t-\tau) > 0$ and $y(g(t)) > 0$ for $t \geqslant t_1$. Consider the function $w(t)$ defined with (9). Then, as in the proof of Theorem 2.1, we obtain (12). Consequently, we get

$$\begin{aligned}\int_T^t H(t,s)\varrho(s)Q(s)\,ds \;&\leq\; -\int_T^t H(t,s)w^{'}(s)ds + \int_T^t H(t,s)\,\frac{\varrho'(s)}{\varrho(s)}\,w(s)\,ds\\ &\quad -\Omega\int_T^t H(t,s)\,\frac{g'(s)}{\varrho(s)}w^2(s)\,ds,\;\; t\geq T\,,\end{aligned}$$

which for all $t \geq T$, implies

$$\begin{aligned}\int_T^t H(t,s)\varrho(s)Q(s)ds &\leq H(t,T)w(T) - \Omega\int_T^t \frac{H(t,s)}{\eta(s)}\,w^2(s)\,ds\\ &\quad + \int_T^t \left(H(t,s)\,\frac{\varrho'(s)}{\varrho(s)} + \frac{\partial H}{\partial s}(t,s)\right)\,w(s)\,ds\end{aligned}$$

Hence,

$$\begin{aligned}\int_T^t H(t,s)\varrho(s)Q(s)ds &\leq H(t,T)w(T)\\ &\quad -\int_T^t \left[\sqrt{\Omega\,\frac{H(t,s)}{\eta(s)}}\,w(s) - \frac{\sqrt{\eta(s)}\,G(t,s)}{2\sqrt{\Omega}}\right]^2 ds \qquad (20)\\ &\quad + \frac{1}{4\,\Omega}\int_T^t \eta(s)\,G^2(t,s)\,ds\,, \quad t\geq T\,.\end{aligned}$$

Thereby, including (II), we conclude that

$$\begin{aligned}\int_{t_0}^t H(t,s)\varrho(s)Q(s)\,ds &= \int_{t_0}^T H(t,s)\varrho(s)Q(s)\,ds + \int_T^t H(t,s)\varrho(s)Q(s)\,ds\\ &\leq H(t,t_0)\int_{t_0}^T \varrho(s)Q(s)\,ds + H(t,t_0)w(T) + \frac{1}{4\,\Omega}\int_{t_0}^t \eta(s)\,G^2(t,s)\,ds\,.\end{aligned}$$

Accordingly, we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \left[H(t,s)\varrho(s)Q(s) - \frac{\eta(s)G^2(t,s)}{4\,\Omega} \right] ds$$
$$\leq \int\limits_{t_0}^{T} \varrho(s)Q(s)\,ds + w(T) < \infty\,.$$

Thus, we come to a contradiction with assumption (C_5). ■

From Theorem 2.3 we get the following Corollaries.

Corollary 2.3 *The superlinear equation* (E) *is oscillatory if there exists the function* $H \in \Re$ *and the constant* $\Omega > 0$*, such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \left[H(t,s)\,Q(s) - \frac{h^2(t,s)}{4\,\Omega\,g'(s)} \right] ds = \infty.$$

Corollary 2.4 *The superlinear equation* (E) *is oscillatory if there exist the functions* $\varrho \in C^1[[t_0,\infty), R^+]$*,* $H \in \Re$*, such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} H(t,s)\varrho(s)Q(s)\,ds = \infty\,,$$
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \eta(s)G^2(t,s)\,ds < \infty\,.$$

Corollary 2.5 *The superlinear equation* (E) *is oscillatory if there exists the function* $H \in \Re$*, such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} H(t,s)\,Q(s)\,ds = \infty\,,$$
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \frac{h^2(t,s)}{g'(s)}\,ds < \infty\,.$$

Example 2.2 Consider the following delay differential equation

$$(E_2) \qquad \left(y(t)+\frac{1}{t^2}\,y\Big(t-\frac{\pi}{2}\Big)\right)'+\frac{(t-1)^6}{t^4\,(t-2)^3}\,y^3\Big(t-1\Big)=0, \qquad t\geq 2.$$

Then, functions $p(t)=\dfrac{1}{t^2}$, $q(t)=\dfrac{(t-1)^6}{t^4\,(t-2)^3}$, $g(t)=t-1$ satisfy conditions (1), so as the function $f(u)=u^3$ satisfies conditions (F_1). Moreover, $Q(t)=\dfrac{1}{t}$, $\eta(t)=\dfrac{1}{t}$. By taking $\varrho(t)=\dfrac{1}{t}$ and $H(t,s)=\ln^2\dfrac{t}{s}$ for $t\geq s\geq 2$, we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,2)}\int_2^t H(t,s)\varrho(s)Q(s)\,ds=\limsup_{t\to\infty}\frac{1}{\ln^2\frac{t}{2}}\int_2^t \ln^2\frac{t}{s}\cdot\frac{ds}{s^2}=\infty$$

$$\liminf_{t\to\infty}\frac{1}{H(t,2)}\int_2^t \eta(s)G^2(t,s)\,ds=\liminf_{t\to\infty}\frac{1}{\ln^2\frac{t}{2}}\int_2^t \frac{1}{s^3}\left(2+\ln\frac{t}{s}\right)^2=\frac{1}{8}.$$

Conditions od Corollary 2.4 are satisfied and hence the equation (E_2) is *oscillatory.*

The following two oscillation criteria (Theorem 2.4 and 2.5) treat the cases when it is not possible to verify easily condition (C_5).

Theorem 2.4 *Let H belongs to the class $\Re$ and assume that*

$$(III) \qquad 0<\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\leq\infty.$$

Let the functions $\varrho\in C^1[[t_0,\infty),R^+]$ be such that

$$(C_6) \qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t G^2(t,s)\,\eta(s)\,ds<\infty,$$

where $G(t,s)$ and $\eta(t)$ are defined as in Theorem 2.3. The superlinear equation (E) is oscillatory if there exist a continuous function ψ on $[t_0,\infty)$, such that for every $T\geq t_0$ and for every $\Omega>0$

$$(C_7) \qquad \limsup_{t\to\infty}\int_{t_0}^t \frac{\psi_+^2(s)}{\eta(s)}\,ds=\infty,$$

and

$$(C_8) \qquad \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\varrho(s)Q(s) - G^2(t,s)\,\frac{\eta(s)}{4\,\Omega} \right) ds \geq \psi(T).$$

where $\psi_+(t) = \max\{\psi(t), 0\}$.

Proof. We suppose that there exists a solution $y(t)$ of the equation (E), such that $y(t) > 0$ on $[T_0, +\infty)$ for some $T_0 \geq t_0$. Defining the function $w(t)$ by (9) in the same way as in the proof of Theorem 2.3, we obtain the inequality (20). By (20), we have for $t > T \geq T_0$

$$\begin{aligned}
&\frac{1}{H(t,T)} \int_T^t \left[H(t,s)\varrho(s)Q(s) - \frac{\eta(s)\,G^2(t,s)}{4\,\Omega} \right] ds \\
&\quad \leq w(T) - \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\Omega\,\frac{H(t,s)}{\eta(s)}}\, w(s) - \frac{\sqrt{\eta(s)}\,G(t,s)}{2\,\sqrt{\Omega}} \right]^2 ds \\
&\quad \leq w(T) - \frac{\Omega}{H(t,T)} \int_T^t H(t,s)\,\frac{w^2(s)}{\eta(s)}\, ds \qquad (21) \\
&\qquad - \frac{1}{H(t,T)} \int_T^t \sqrt{H(t,s)}\, G(t,s)\, w(s)\, ds
\end{aligned}$$

Hence, for T$\geq T_0$ we get

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\varrho(s)Q(s) - \frac{\eta(s)\,G^2(t,s)}{4\,\Omega} \right] ds \\
&\quad \leq w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\Omega\,\frac{H(t,s)}{\eta(s)}}\, w(s) - \frac{\sqrt{\eta(s)}\,G(t,s)}{2\,\sqrt{\Omega}} \right]^2 ds
\end{aligned}$$

By the condition (C_7) and the previous inequality, we see that

$$\psi(T) \leq w(T) \quad \text{for every } T \geq t_0, \qquad (22)$$

Define the functions $\alpha(t)$ and $\beta(t)$ as follows

$$\alpha(t) = \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\, \frac{w^2(s)}{\eta(s)}\, ds,$$

$$\beta(t) = \frac{1}{H(t,T_0)} \int_{T_0}^{t} \sqrt{H(t,s)}\, G(t,s)\, w(s)\, ds\,.$$

Then, (21) implies that

$$\liminf_{t\to\infty} [\,\alpha(t)+\beta(t)\,]$$
$$\le w(T_0) - \limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} \left[H(t,s)\varrho(s)Q(s) - \frac{\eta(s)\, G^2(t,s)}{4\,\Omega} \right] ds$$

which together with the condition (C_7) gives that

$$\liminf_{t\to\infty} [\,\alpha(t)+\beta(t)\,] \le w(T_0) - \psi(T_0) < \infty\,. \tag{23}$$

In order to show that

$$\int_{T_0}^{\infty} \frac{w^2(s)}{\eta(s)}\, ds < \infty, \tag{24}$$

we suppose on the contrary, that (24) fails, i.e. there exists a $T_1 > T_0$ such that

$$\int_{T_0}^{t} \frac{w^2(s)}{\eta(s)}\, ds \ge \frac{\mu}{\zeta} \quad \text{for all } t \ge T_1\,, \tag{25}$$

where μ is an arbitrary positive number and ζ is a positive constant satisfying

$$\inf_{s\ge t_o} \left[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)} \right] > \zeta > 0. \tag{26}$$

Using integration by parts and (25), we have for all $t \ge T_1$

$$\alpha(t) = \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\, d\left(\int_{T_0}^{s} \frac{w^2(u)}{\eta(u)}\, du \right)$$

$$
\begin{aligned}
&= -\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{\partial H}{\partial s}(t,s)\left(\int_{T_0}^{s}\frac{w^2(u)}{\eta(u)}\,du\right)ds\\
&\geq -\frac{1}{H(t,T_1)}\int_{T_0}^{t}\frac{\partial H}{\partial s}(t,s)\left(\int_{T_0}^{s}\frac{w^2(u)}{\eta(u)}\,du\right)ds\\
&\geq -\frac{\mu}{\zeta\,H(t,T_0)}\int_{T_0}^{t}\frac{\partial H}{\partial s}(t,s)\,ds=\frac{\mu\,H(t,T_1)}{\zeta\,H(t,T_0)}\geq\frac{\mu\,H(t,T_1)}{\zeta\,H(t,t_0)}
\end{aligned}
$$

By (26), there is a $T_2 \geq T_1$ such that $H(t,T_1)/H(t,t_0) \geq \zeta$ for all $t \geq T_2$, and accordingly $\alpha(t) \geq \mu$ for all $t \geq T_2$. Since μ is an arbitrary constant, we conclude that

$$\lim_{t\to\infty}\alpha(t)=\infty. \tag{27}$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty} \in [T_0,\infty)$ with $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty}[\alpha(t_n)+\beta(t_n)]=\liminf_{t\to\infty}[\alpha(t)+\beta(t)].$$

In view of (23), there exists a constant μ_2 such that

$$\alpha(t_n)+\beta(t_n)\leq\mu_2, \tag{28}$$

for all sufficiently large n. It follows from (27) that

$$\lim_{n\to\infty}\alpha(t_n)=\infty, \tag{29}$$

and (28) implies

$$\lim_{n\to\infty}\beta(t_n)=-\infty. \tag{30}$$

Then, by (28) and (30), for n large enough we derive

$$1+\frac{\beta(t_n)}{\alpha(t_n)}\leq\frac{\mu_2}{\alpha(t_n)}<\frac{1}{2}\,.$$

Thus

$$\frac{\beta(t_n)}{\alpha(t_n)}\leq-\frac{1}{2}\quad\text{for all large } n\,.$$

which together with (30) implies that

$$\lim_{n\to\infty} \frac{\beta^2(t_n)}{\alpha(t_n)} = \infty\,. \tag{31}$$

On the other hand by Schwarz inequality, we have

$$\begin{aligned}
\beta^2(t_n) &= \left[\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n} \sqrt{H(t_n,s)}G(t_n,s)w(s)ds\right]^2 \\
&\leq \left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n} G^2(t_n,s)\,\eta(s)\,ds\right\} \\
&\qquad \times\left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n} H(t_n,s)\,\frac{w^2(s)}{\eta(s)}\,ds\right\} \\
&\leq \alpha(t_n)\left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n} G^2(t_n,s)\,\eta(s)\,ds\right\}.
\end{aligned}$$

Then, by (26), for large enough n we get

$$\frac{\beta^2(t_n)}{\alpha(t_n)} \leq \frac{1}{\zeta H(t_n,t_0)}\int_{T_0}^{t_n} G^2(t_n,s)\,\eta(s)\,ds.$$

Because of (31), we have

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n} G^2(t_n,s)\,\eta(s)\,ds = \infty\,, \tag{32}$$

which gives

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\int_{t_0}^{t} G^2(t_n,s)\,\eta(s)\,ds = \infty\,. \tag{33}$$

But the latter contradicts the assumption (C_5). Thus, (24) holds. Finally, by (22), we obtain

$$\int_{T_0}^{\infty} \frac{\psi_+^2(s)}{\eta(s)}\,ds \leq \int_{T_0}^{\infty} \frac{w^2(s)}{\eta(s)}\,ds < \infty \tag{34}$$

which contradicts the assumption (C_7). This completes the proof. ■

Theorem 2.5 *Let H belongs to the class $\Re$ satisfying the condition (III) and $\varrho \in C^1[[t_0,\infty), R^+]$ be such that*

$$(C_9) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\varrho(s)Q(s)\,ds < \infty,$$

The superlinear equation (E) is oscillatory if there exist a continuous function ψ on $[t_0,\infty)$, such that for every $T \ge t_0$ and for every $\Omega > 0$, (C_7) and

$$(C_{10}) \qquad \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)\varrho(s)Q(s) - G^2(t,s)\,\frac{\eta(s)}{4\,\Omega} \right) ds \ge \psi(T)$$

are satisfied.

Proof. For the nonoscillatory solution $y(t)$ of the equation (E), such that $y(t) > 0$ on $[T_0,\infty)$ for some $T_0 \ge t_0$, as in the proof of Theorem 2.4, (21) is satisfied. We conclude by (21) that for $T \ge T_0$

$$\limsup_{t\to\infty} [\,\alpha(t) + \beta(t)\,]$$
$$\le w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\varrho(s)Q(s) - \frac{\eta(s)\,G^2(t,s)}{4\,\Omega} \right] ds$$

where $\alpha(t)$ and $\beta(t)$ are defined as in the proof of Theorem 2.4. Together with the condition (C_9) we conclude that inequality (22) holds and

$$\limsup_{t\to\infty} [\,\alpha(t) + \beta(t)\,] \le w(T) - \psi(T) < \infty\,. \tag{35}$$

From the condition (C_{10}) it follows that

$$\begin{aligned}
\psi(T) \;&\le\; \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)\varrho(s)Q(s) - G^2(t,s)\,\frac{\eta(s)}{4\,\Omega} \right) ds \\
&\le\; \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\varrho(s)Q(s)\,ds \\
&\qquad - \frac{1}{4\,\Omega}\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} G^2(t,s)\,\eta(s)\,ds\,,
\end{aligned}$$

so that (C_9) implies that

$$\liminf_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} G^2(t,s)\,\eta(s)\,ds < \infty\,. \tag{36}$$

By (36), there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in $[T_0,\infty)$ with $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \frac{1}{H(t,t_n)} \int_{T}^{t_n} G^2(t,s)\,\eta(s)\,ds = \liminf_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} G^2(t,s)\,\eta(s)\,ds\,. \tag{37}$$

Suppose now that (24) fails to hold. As in Theorem 2.4. we conlude that (27) holds. By (35), there exists a natural number N such that (28) is verified for all $n \geq N$. Proceeding as in the proof of Theorem 2.4. we obtain (32), which contradicts (37). Therefore, (24) holds. Using (22), we conclude by (24) and using the procedure of the proof of , we conclude that (34) is satisfied, which contradicts the assumption (C_7). Hence, the superlinear equation (E) is oscillatory. ■

Remark. With the appropriate choice of functions H and h, it is possible to derive a number of oscillation criteria for Eq. (E). Defining, for example, for some integer n >1, the function H(t,s) by

$$H(t,s) = (t-s)^n, \quad \mathrm{(t,s)} \in \mathrm{D}. \tag{38}$$

we can easily check that H$\in \Re$ as well as that it satisfies the condition (III). Therefore, as a consequence of Theorems 2.3.-2.5. we can obtain a number of oscillation criteria.

Of course, we are not limited only to choice of functions H defined by (38), which has become standard and goes back to the well known Kamenev-type conditions.With a different choice of these functions it is possible to derive from Theorems 2.3.-2.5. other sets of oscillation criteria. In fact, another possibilities are to choose the function H as follows:

$$1) \qquad H(t,s) = \left(\int_{s}^{t} \frac{du}{\theta(u)} \right)^{\gamma}, \quad t \geq s \geq t_0,$$

where $\gamma > 1$ and $\theta : [t_0, \infty) \to R^+$ is a continuous function satisfying condition

$$\lim_{t\to\infty} \int\limits_{t_0}^{t} \frac{du}{\theta(u)} = \infty;$$

for example, taking $\theta(u) = u^\beta$, $\beta \geq 1$ we get

$$1a) \qquad H(t,s) = \begin{cases} \dfrac{[t^{1-\beta} - s^{1-\beta}]^\gamma}{(1-\beta)^\gamma}, & \beta < 1 \\ \left(\ln \dfrac{t}{s}\right)^\gamma, & \beta = 1 \end{cases} ;$$

$$2) \qquad H(t,s) = [A(t) - A(s)]^\gamma, \qquad \text{for} \quad t \geq s \geq t_0, \ \gamma > 1$$

$$3) \qquad H(t,s) = \left(\log \frac{A(t)}{A(s)}\right)^\gamma, \qquad \text{for} \quad t \geq s \geq t_0, \ \gamma > 1$$

where $a \in C([t_0, \infty); (0, \infty))$ and $A'(t) = \dfrac{1}{a(t)}$, $t \geq t_0$; for example, taking $a(t) = e^{-t}$, we get

$$2a) \qquad H(t,s) = (e^t - e^s)^\gamma, \qquad t \geq s \geq t_0, \ \gamma > 1$$

$$4) \qquad H(t,s) = \left(\ln \frac{A_1(s)}{A_1(t)}\right)^\gamma A_1(s), \qquad \text{for} \quad t \geq s \geq t_0, \ \gamma > 1,$$

$$5) \qquad H(t,s) = \left(\frac{1}{A_1(t)} - \frac{1}{A_1(s)}\right)^\gamma A_1^2(s), \qquad \text{for} \quad t \geq s \geq t_0, \ \gamma > 1,$$

where $a \in C([t_0, \infty); (0, \infty))$ and $A_1(t) = \displaystyle\int\limits_t^\infty \frac{ds}{a(s)} < \infty$. It is a simple matter to check that in all these cases (1)–(5), assumptions (I), (II) and (III) are verified.

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(Received November 3, 2003)