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A GLOBAL BIFURCATION RESULT OF A NEUMANN PROBLEM WITH INDEFINITE WEIGHT

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ABSTRACT. This paper is concerned with the bifurcation result of nonlinear Neumann problem

$$\begin{cases} -\Delta_p u = & \lambda m(x)|u|^{p-2}u + f(\lambda, x, u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} \quad = & 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases}$$

We prove that the principal eigenvalue λ_1 of the corresponding eigenvalue problem with $f \equiv 0$, is a bifurcation point by using a generalized degree type of Rabinowitz.

1. INTRODUCTION

The purpose of this paper is to study a bifurcation phenomenon for the following nonlinear elliptic problem

$$(\mathcal{P}) \qquad \begin{cases} -\Delta_p u = & \lambda m(x)|u|^{p-2}u + f(\lambda, x, u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} \quad = & 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $I\!R^N$, $N \geq 1$, with smooth boundary and ν is the unit outward normal vector on $\partial\Omega$; the weight function m belongs to $L^\infty(\Omega)$ and λ is a parameter. We assume that $\int_\Omega m(x)dx < 0$ and $|\Omega^+| \neq 0$ with $\Omega^+ = \{x \in \Omega;\ m(x) > 0\}$, where $|\,.\,|$ is the Lebesgue measure of $I\!R^N$. The so-called p-Laplacian is defined by $-\Delta_p u = -\nabla.(|\nabla u|^{p-2}\nabla u)$ which occurs in many mathematical models of physical processes as glaciology, nonlinear diffusion and filtration problem, see [18], power-low materials [2], the mathematical modelling of non-Newtonian fluids [1]. For a discussion of some physical background, see [10]. In this context and for certain physical motivations, see for example [17]. Observe that in the particular case $f \equiv 0$ and $p = 2$, $(\mathcal{P})$ cames linear.

The nonlinearity f is a function satisfying some conditions to be specified later.

Classical Neumann problems involving the p-Laplacian operator have been studied by many authors. Senn and Hess [14, 15] studied an eigenvalue problem with Neumann boundary condition. Bandele, Pizio and Tesei

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[4] studied the existence and uniqueness of positive solutions of some nonlinear Neumann problems; we cite also the paper [6] where the authors studied the role played by the indefinite weight on the existence of positive solution. In [16], the author shows that the first positive eigenvalue λ_1, of

$$(\mathcal{E}) \qquad \begin{cases} -\Delta_p u = & \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} \quad = & 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

is well defined and if $\int_\Omega m(x)\,dx < 0$, it is simple and isolated. These fundamental properties will be used in proof of our main bifurcation result.

In general, variational method and bifurcation theory have been used in pure and applied mathematics to establish the existence, multiplicity and structure of solutions to Partial Differential Equations. However, the relationship between these two methods have remained largely unrecognized and searchers have tended to use one method or the other. The present paper gives an example of nonlinear partial differential equation with Neumann boundary condition, expanding variational and bifurcation methods to occur the connection between these two distinct "arguments".

In recent years, bifurcation problems with a particular with a particular nonlinearity were studied by several authors, with the right hand side of the first equation of the form f and the Direchlet boundary condition. In fact, bifurcation Direchlet boundary condition problems with other conditions on m and f were studied on bounded smooth domains by [5] and [9]. These results were extended for any bounded domain and m is only locally bounded by [11] and [12]. The authors considered the bifurcation phenomena, namely on the interior of domain. The case $\Omega = \mathbb{R}^N$ was treated by Dràbek and Huang [13] under some appropriate hypotheses.

The purpose of this paper is to study the bifurcation phenomenon from the first eigenvalue of $(\mathcal{E})$ when $\displaystyle\int_\Omega m(x)\,dx < 0$, by using a combination of topological and variational methods. Our main result is formulated by Theorem 3.2, where we investigate the situation improving the conditions of the nonlinearity f for Neumann boundary condition. In Proposition 3.1, we give a characterization of the bifurcation points of $(\mathcal{P})$ related to the spectrum of $(\mathcal{E})$. We establish the existence of a global branch of nonlinear solutions pairs (λ, u), with $u \neq 0$, bifurcating from the trivial branch at $\lambda = \lambda_1$. Bifurcation here means that there is a sequence of nontrivial solutions (λ, u), with $u \neq 0$, going to zero as λ approaches the right eigenvalues.

The rest of the paper is organized as follows: Section 2 is devoted to statement of some assumptions and notations which we use later and prove some technical preliminaries; in Section 3 we verify that the topological degree is well defined for our operators in order to be able to show that this degree has a jump, when λ crosses λ_1, which implies the bifurcation result.

In fact, we may employ the global bifurcation result of that of Rabinowitz [19].

2. Assumptions and Preliminaries

We first introduce some basic definitions, assumptions and notations. Here $p > 1$, Ω is a bounded domain in $I\!R^N$, $(N \geq 1)$ with a smooth boundary. $W^{1,p}(\Omega)$ is the usual Sobolev space, equipped with the standard norm

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}, \quad u \in W^{1,p}(\Omega).$$

2.1. **Assumptions.** We make the following assumptions:
$f : \Omega \times I\!R \times I\!R \longrightarrow I\!R$ is a Carathéodory's function, satisfying the homogenization condition type

$$f(\lambda, x, s) = o(|s|^{p-1}), \ \text{ as } \ s \to 0 \tag{2.1}$$

uniformly a.e. with respect to $x \in \Omega$ and uniformly with respect to λ in any bounded subset of $I\!R$. Moreover f satisfies the asymptotic condition:
There is $q \in (p, \ p^*)$ such that

$$\lim_{|s| \to +\infty} \frac{f(\lambda, x, s)}{|s|^{q-1}} = 0, \tag{2.2}$$

uniformly a.e. with respect to $x \in \Omega$ and uniformly with respect to λ in any bounded subset of $I\!R$. Here p^* is the critical Sobolev exponent defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad N > p \\ +\infty & \text{if} \quad N \leq p, \end{cases}$$

2.2. **Definitions.** 1. By a solution of $(\mathcal{P})$, we understand a pair (λ, u) in $I\!R \times W^{1,p}(\Omega)$ satisfying $(\mathcal{P})$ in the weak sense, i.e.,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \lambda \int_\Omega m(x)|u|^{p-2} uv \, dx + \int_\Omega f(\lambda, x, u) v \, dx, \tag{2.3}$$

for all $v \in W^{1,p}(\Omega)$. This is equivalent to saying that u is a critical point of the energy functional corresponding to $(\mathcal{P})$ defined as

$$\frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{p} \int_\Omega m(x)|u|^p dx - \int_\Omega F(\lambda, x, u) dx,$$

where F denoted the Nemitskii operator associated to f. In other words, F is the primitive of f with respect to the third variable, i.e., $F(\lambda, x, u) = \int_0^u f(\lambda, x, s) \, ds$. We note that the pair $(\lambda, 0)$ is a solution of $(\mathcal{P})$ for every $\lambda \in I\!R$. The pairs of this form will be called the trivial solutions of $(\mathcal{P})$. We say that $P = (\mu, 0)$ is a bifurcation point of $(\mathcal{P})$, if in any neighborhood of P in $I\!R \times W^{1,p}(\Omega)$ there exists a nontrivial solution of $(\mathcal{P})$.

2. Throughout, we shall denote by X a real reflexive Banach space and by X' stand for its dual with respect to the pairing $\langle .,. \rangle$. We shall deal with mapping T acting from X into X'. T is demicontinuous at u in X, if $u_n \to u$ strongly in X, implies that $Tu_n \rightharpoonup Tu$ weakly in X'. T is said to belong to the class (S_+), if for any sequence u_n weakly convergent to u in X and $\limsup_{n\to+\infty} \langle Tu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$ strongly in X.

2.3. **Degree theory.** If $T \in (S_+)$ and T is demicontinuous, then it is possible to define the degree $Deg[T; D, 0]$, where $D \subset X$ is a bounded open set such that $Tu \neq 0$ for any $u \in \partial D$. Its properties are analogous to the ones of the Leray-Schauder degree (cf. [7]).

Assume that T is a potential operator, i.e., for some continuously differentiable functional $\Phi : X \to I\!R$, $\Phi'(u) = Tu$, $u \in X$. A point $u_0 \in X$ will be called a critical point of Φ if $\Phi'(u_0) = 0$. We say that u_0 is an isolated critical point of Φ if there exists $\epsilon > 0$ such that for any $u \in B_\epsilon(u_0)$, $\Phi'(u) \neq 0$ if $u \neq u_0$. Then, the limit

$$Ind(T, u_0) = \lim_{\epsilon\to 0^+} Deg[\Phi'; B_\epsilon(u_0), 0]$$

exists and is called the index of the isolated critical point u_0, where $B_r(w)$ denotes the open ball of radius r in X centered at w.

Now, we can formulate the following two lemmas which we can find in [20].

Lemma 2.1. *Let u_0 be a local minimum and an isolated critical point of Φ. Then*

$$Ind(\Phi', u_0) = 1.$$

Lemma 2.2. *Assume that $\langle \Phi'(u), u\rangle > 0$ for all $u \in X$, $\|u\|_X = r$. Then*

$$Deg[\Phi'; B_\rho(0), 0] = 1.$$

2.4. **Preliminaries.** Let us define, for $(u, v) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega)$, the operators

$$A_p,\ G :\ W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$$

and

$$F : I\!R \times W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$$

$$\langle A_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx$$

$$\langle Gu, v\rangle = \int_\Omega m(x)|u|^{p-2}uv\, dx$$

$$\langle F(\lambda, u), v\rangle = \int_\Omega f(\lambda, x, u)v\, dx$$

Remark 2.1. *(i) Due to (2.3) a function u is a weak solution of $(\mathcal{P})$ if, and only if,*

$$A_p u - \lambda G u - F(\lambda, u) = 0 \quad \text{in} \quad (W^{1,p}(\Omega))' \tag{2.4}$$

(ii) The operator A_p has the following properties:

(a) A_p is odd, $(p-1)$-homogeneous and strictly monotone, i.e.,

$$\langle A_p u - A_p v, u - v \rangle > 0 \quad \forall\, u \neq v.$$

(b) $A_p \in (S_+)$.

Lemma 2.3. *G is well defined, compact, odd and $(p-1)$-homogeneous.*

Proof. The definition and compactness of G are required by the compactness of Sobolev embedding

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega).$$

The oddness and $(p-1)$ homogeneity of G are obvious. Thus, the lemma is proved. ∎

Lemma 2.4. *For any $\lambda \in I\!R$, the Nemitskii operator $F(\lambda, .)$ is well defined, compact and $F(\lambda, 0) = 0$. Moreover, we have*

$$\lim_{\|u\|_{1,p} \to 0} \frac{F(\lambda, u)}{\|u\|_{1,p}^{p-1}} = 0 \ \text{ in } \ (W^{1,p}(\Omega))', \tag{2.5}$$

uniformly for λ in any bounded subset of $I\!R$.

Proof. Conditions (2.1) and (2.2) imply that for any $\epsilon > 0$, there are two reals $\delta = \delta(\epsilon)$ and $M = M(\delta) > 0$ such that for a.e., $x \in \Omega$, we have

$$|f(\lambda, x, s)| \leq \epsilon |s|^{p-1} \quad \text{for } |s| \leq \delta \tag{2.6}$$

and

$$|f(\lambda, x, s)| \leq M |s|^{q-1} \quad \text{for } |s| \geq \delta. \tag{2.7}$$

Therefore, for $0 < \epsilon \leq 1$, we get by integration on Ω that

$$\int_\Omega |f(\lambda, x, u(x))|^{q'}\, dx \leq \int_\Omega |u(x)|^{q'(p-1)}\, dx + M \int_\Omega |u(x)|^q\, dx. \tag{2.8}$$

We have $q'(p-1) \leq p'(p-1) = p < q$. Thus $L^q(\Omega) \hookrightarrow L^{q'(p-1)}(\Omega)$ and there is a constant $c > 0$ such that

$$\int_\Omega |u(x)|^{q'(p-1)} dx \leq c \int_\Omega |u(x)|^q dx. \tag{2.9}$$

Inserting (2.9) in (2.7), we deduce the estimate

$$\int_\Omega |f(\lambda, x, u(x))|^{q'} dx \leq (c + M) \int_\Omega |u(x)|^q\, dx. \tag{2.10}$$

Thus $F(\lambda, .)$ maps $L^q(\Omega)$ into its dual $L^{q'}(\Omega)$ continuously (for more detail on the properties of Nemitskii operator the reader can see [8]). Moreover, if $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ then $u_n \to u$ in $L^q(\Omega)$, because $p < q < p^*$ and $F(\lambda, u_n) \to F(\lambda, u)$ in $L^{q'}(\Omega)$. Since $L^{q'}(\Omega) \hookrightarrow (W^{1,p}(\Omega))'$, it follows that $F(\lambda, u_n) \to F(\lambda, u)$ in $(W^{1,p}(\Omega))'$. This implies that $F(\lambda, .)$ is compact. It is not difficult to verify that $F(\lambda, 0) = 0$, for all $\lambda \in \mathbb{R}$.

In virtue of (2.1), we have $\dfrac{F(\lambda, u)}{\|u\|_{1,p}^{p-1}} \to 0$ in $L^{q'}(\Omega)$, as $u \to 0$ in $W^{1,p}(\Omega)$. Indeed, set $\overline{u} = \frac{u}{\|u\|_{1,p}}$. Hence

$$\frac{F(\lambda, u)}{\|u\|_{1,p}^{p-1}} = \frac{F(\lambda, u)}{|u|^{p-1}} |\overline{u}|^{p-1}. \tag{2.11}$$

From this and Hölder's inequality, we deduce that

$$\int_\Omega \left| \frac{F(\lambda, u)}{\|u\|_{1,p}^{p-1}} \right|^{q'} dx \leq \left[\int_\Omega \left(\frac{|F(\lambda, u(x))|}{|u(x)|^{p-1}} \right)^{q't} dx \right]^{\frac{1}{t}} \left[\int_\Omega |\overline{u}|^{(p-1)q't'} dx \right]^{\frac{1}{t'}}, \tag{2.12}$$

for some $t > 0$ which satisfies

$$\frac{q'(p-1)}{p^*} < \frac{1}{t} < \frac{p^* - (p-1)q'}{p^*}. \tag{2.13}$$

This is always possible, since $p < q < p^*$. By (2.6) and (2.7), we conclude that

$$\left\| \left| \frac{F(\lambda, u)}{|u|^{p-1}} \right|^{q'} \right\|_t^t \leq \epsilon |\Omega| + M^{q't} \int_\Omega |u|^{q't(q-p)} dx, \quad \forall \epsilon > 0. \tag{2.14}$$

From this inequality and the fact that $u \to 0$ in $W^{1,p}(\Omega)$, we have the limit

$$\left\| \left| \frac{F(\lambda, u)}{|u|^{p-1}} \right|^{q'} \right\|_t^t \to 0 \ \text{ as } u \to 0 \text{ in } W^{1,p}(\Omega).$$

On the other hand, $\overline{u}$ belongs to $L^{p^*}(\Omega)$ (because $\displaystyle\int_\Omega |\overline{u}|^{p^*} dx \leq c$). Then, we find a constant $c > 0$ such that

$$\| |\overline{u}|^{(p-1)q'} \|_{t'} \leq c,$$

since $q't'(p-1) < p^*$ by (2.13). This completes the proof. ∎

Remark 2.2. *Note that every continuous map $T : X \longrightarrow X'$ is also demi-continuous. Note also, that if $T \in (S_+)$ then $(T+K) \in (S_+)$ for any compact operator $K : X \longrightarrow X'$.*

Remark 2.3. *λ is an eigenvalue of $(\mathcal{E})$ if and only if, $u \in W^{1,p}(\Omega) \backslash \{0\}$ solves*

$$A_p u - \lambda G u = 0. \tag{2.15}$$

Now, define an operator $T_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$; by $T_\lambda u = A_p u - \lambda G u - F(\lambda, u)$.
In view of Lemma 2.3, Lemma 2.4, Remark 2.1 and Remark 2.2, it follows that, for $\epsilon > 0$, sufficiently small, the degree

$$Deg[T_\lambda; B_\epsilon(0), 0], \tag{2.16}$$

is well defined for any $\lambda \in I\!R$ such that $T_\lambda u \neq 0$ for any $\|u\|_{1,p} = \epsilon$ (here $B_\epsilon(0)$ is the open ball of radius ϵ in $W^{1,p}(\Omega)$ centered at 0).
By using the same argument as used in proof of Lemma 2.3, we can state the following proposition which plays a crucial role in our bifurcation result.

Proposition 2.1. *If $(\mu, 0)$ is a bifurcation point of problem $(\mathcal{P})$, then μ is an eigenvalue of $(\mathcal{E})$.*

Proof. Fix $\mu \in I\!R$. Since $(\mu, 0) \in I\!R \times W^{1,p}(\Omega)$ is a bifurcation of $(\mathcal{P})$ there exists a sequence $\{(\lambda_j, u_j)\}_j \subset I\!R \times W^{1,p}(\Omega)$ of nontrivial solutions of the problem $(\mathcal{P})$ such that

$$\lambda_j \to \mu \text{ in } I\!R \text{ and } u_j \to 0 \quad \text{in} \quad W^{1,p}(\Omega), \tag{2.17}$$

as $j \to +\infty$.
(λ_j, u_j) solve the equation (2.4). Therefore, by $(p-1)-$homogeneity we have

$$A_p v_j - \lambda_j G v_j = \frac{F(\lambda_j, u_j)}{\|u_j\|_{1,p}^{p-1}},$$

where $v_j = \dfrac{u_j}{\|u_j\|_{1,p}^{p-1}}$. The sequence $(v_j)_j$ is bounded in $W^{1,p}(\Omega)$. Thus, there is a function $v \in W^{1,p}(\Omega)$ such that $v_j \rightharpoonup v$ in $W^{1,p}(\Omega)$ (for a subsequence if necessary). Then, by combining Remark 2.1, Lemma 2.3 and Lemma 2.4, we obtain that $v_j \to v$ strongly in $W^{1,p}(\Omega)$ and

$$0 = A_p v_j - \lambda_j G v_j - \frac{F(\lambda_j, u_j)}{\|u_j\|_{1,p}^{p-1}} \to A_p v - \mu G v. \tag{2.19}$$

Then

$$A_p v = \mu G v \text{ in } (W^{1,p}(\Omega))'. \tag{2.20}$$

It is clear that $v \neq 0$, because $\| v_j \|_{1,p} = 1, \forall j \in I\!N^*$. Hence (2.20) proves that μ is an eigenvalue of $(\mathcal{E})$ in view of Remark 2.3. This clearly concludes the proof. ∎

3. Main Results

The goal of this section it to prove our main bifurcation results. In order to do so, we shall introduce further notations and some properties of the principal positive eigenvalue of the eigenvalue problem $(\mathcal{E})$ which will be used in our analysis. For this purpose, consider the variational characterization

of λ_1.
We recall that λ_1 can be characterized variationally as follows

$$\lambda_1 = inf\left\{\int_\Omega |\nabla u|^p\,dx;\ u\in W^{1,p}(\Omega),\ \int_\Omega m(x)|u|^p dx = 1\right\}. \tag{3.1}$$

In fact, we have the following theorem.

Theorem 3.1. [16] *Let us suppose that $m \in L^\infty(\Omega)$ such that $mes\{x \in \Omega/m(x) > 0\} \neq 0$, then we have*
(i) λ_1 is effectively an eigenvalue of $(\mathcal{E})$ with weight m; and $0 < \lambda_1 < +\infty$ if and only if m changes sign and $\int_\Omega m(x)\,dx < 0$.
(ii) λ_1 is simple, namely, if u and v are two eigenfunctions associated to λ_1 then $u = kv$ for some k.
(iii) If u is an eigenfunction associated with λ_1, then $\min_{\overline{\Omega}} |u| > 0$.
(iv) λ_1 is isolated.

Lemma 3.1. *Let $(u_n)_n$ be a sequence in $W^{1,p}(\Omega)$ such that*

$$\|\nabla u_n\|_p^p - \lambda\int_\Omega m(x)|u_n|^p\,dx \le c, \tag{$*$}$$

for some $0 < \lambda < \lambda_1$ and positive constant c independent on n. Then $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$.

Proof. From $(*)$, we deduce that

$$\|\nabla u_n\|_p^p - \lambda\|m\|_\infty\|u_n\|_p^p \le c.$$

That is,

$$\|\nabla u_n\|_p^p \le c + \lambda\|m\|_\infty\|u_n\|_p^p,\ \forall n \in I\!N^*.$$

Thus it suffices to show that $(\|u_n\|_p)_n$ is bounded. Suppose by contradiction that $\|u_n\|_p \to \infty$ (for a suitable subsequence if necessary). We distinguish two cases:

- $\underline{\|\nabla u_n\|_p \text{ is bounded.}}$ Set $v_n = \frac{u_n}{\|u_n\|_p}$, $\forall n \in I\!N^*$. Thus $(v_n)_n$ is bounded in $W^{1,p}(\Omega)$. Consequently, by compactness there exists a subsequence (noted also $(v_n)_n$) such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$, $v_n \to v$ in $L^p(\Omega)$ and $v_n \to v$ almost everywhere in Ω, for some function $v \in W^{1,p}(\Omega)$. It is clear that $\|v\|_p = 1$ (because $\|v_n\|_p = 1$, $\forall n$) and

$$\|v\|_{1,p} \le \liminf_{n\to+\infty}\|v_n\|_{1,p}$$

which implies that

$$\|\nabla v\|_p \le \liminf_{n\to+\infty}\frac{\|\nabla u_n\|_p}{\|u_n\|_p} = 0.$$

This yields that v is, almost everywhere, in Ω, equal to a nonzero constant d.
Now, dividing $(*)$ by the quantity $\|u_n\|_p^p$ and letting $n \to +\infty$, we obtain

$$-\lambda \mid d \mid \int_\Omega m(x)\, dx \le 0.$$

So, we get that $\int_\Omega m(x)\, dx \ge 0$ which is a contradiction.

- $\underline{\|\nabla u_n\|_p \text{ is unbounded}}$. We can suppose that, for n large enough, $\|\nabla u_n\|_p > c^{\frac{1}{p}}$, where c is given by $(*)$.
From $(*)$, we deduce that

$$-\lambda \int_\Omega m(x) \mid u_n \mid^p \, dx < 0,$$

for all n large enough. That is

$$\int_\Omega m(x) \mid u_n \mid^p \, dx > 0.$$

As n tends to plus infinity, u_n is admissible in the variational characterization of λ_1 given by (3.1). Thus

$$\lambda_1 \int_\Omega m(x) \mid u_n \mid^p \, dx \le \|\nabla u_n\|_p^p.$$

Set $v_n = \frac{u_n}{\|\nabla u_n\|_{1,p}}$. Thus $(v_n)_n$ is bounded in $W^{1,p}(\Omega)$ and consequently there is a function $v \in W^{1,p}(\Omega)$ such that v_n converges to v weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ (for a subsequence if necessary).
Dividing $(*)$ by $\|\nabla u_n\|_{1,p}^p$ and combining with last inequality, we arrive at

$$\lambda_1 \int_\Omega m(x) \mid v_n \mid^p \, dx - \lambda \int_\Omega m(x) \mid v_n \mid^p \, dx \le \frac{c}{\|\nabla u_n\|_{1,p}^p}. \qquad (**)$$

Let n goes to $+\infty$ in $(**)$, we conclude that

$$(\lambda_1 - \lambda) \int_\Omega m(x) \mid v \mid^p \, dx \le 0.$$

This and the fact that $\int_\Omega m(x) \mid v \mid^p dx \ge 0$ imply that

$$\lambda_1 - \lambda \le 0.$$

Which is a contradiction.

Finally, from the both above cases, we conclude that $(\|\nabla u_n\|_p)_n$ is bounded and the proof of the lemma is achieved. ■

Let $E = I\!R \times W^{1,p}(\Omega)$ be equipped with the norm

$$\|(\mu, u)\| = \left(|\mu|^2 + \|u\|_{1,p}^2\right)^{\frac{1}{2}}$$

Definition 3.1. *We say that*

$$C = \{(\lambda, u) \in E : \ (\lambda, u)) \ \textit{solves} \ (\mathcal{P}),\, u \neq 0\}$$

is a continuum *(or branch) of nontrivial solutions of* $(\mathcal{P})$*, if it is a connected subset in* E*.*

Theorem 3.2. *Under the assumptions* (2.1) *and* (2.2)*, the pair* $(\lambda_1, 0)$ *is a bifurcation point of* $(\mathcal{P})$*. Moreover, there is a continuum of nontrivial solutions* C *of* $(\mathcal{P})$ *such that* $(\lambda_1, 0) \in \overline{C}$ *and* C *is either unbounded in* E *or there is* $\mu \neq \lambda_1$*, an eigenvalue of* $(\mathcal{P})$*, with* $(\mu, 0)) \in \overline{C}$*.*

Proof. We shall employ the homotopy invariance principle of the considered degree to deduce that

$$Deg[A_p - \lambda G; B_\epsilon(0), 0] \tag{3.2}$$

jumps from 1 to -1, as λ crosses λ_1. If this fact is proved, then Theorem 3.1 follows exactly as in the classical global bifurcation result of Rabinowitz [19]. Choose $\delta = \delta(\lambda) > 0$ small enough, so that $\lambda_1 - \delta > 0$ and the interval $(\lambda_1 - \delta,\, \lambda_1 + \delta)$ does not contain any eigenvalue of $(\mathcal{E})$ different of λ_1. A such δ exists because λ_1 is isolated in the spectrum. Then, the variational characterization (3.1) of λ_1 and Lemma 2.2 yield

$$Deg[A_p - \lambda G; B_\epsilon(0), 0] = 1, \tag{3.3}$$

when $\lambda \in (\lambda_1 - \delta, \lambda_1)$. To evaluate (3.2) for $\lambda \in (\lambda_1,\, \lambda_1 + \delta)$, we use a similar argument developed in [11] (see also [12]). Fix a number $t_0 > 0$ and define a continuously differentiable function $h : I\!R \longrightarrow I\!R$ by

$$h(t) = \begin{cases} 0 & \text{for} \quad t \leq t_0 \\ a(t - 2t_0) & \text{for} \quad t \geq 3t_0, \end{cases}$$

where $a > \frac{\delta}{\lambda_1}$ and h is positive and strictly convex in $(t_0,\, 3t_0)$. Now, define an auxiliary functional

$$\begin{aligned} \Phi_\lambda(u) &= \tfrac{1}{p}\|\nabla u\|_p^p - \tfrac{\lambda}{p}\int_\Omega m(x)|u|^p dx + h(\tfrac{1}{p}\|\nabla u\|_p^p) \\ &= \tfrac{1}{p}\langle A_p u, u\rangle - \tfrac{\lambda}{p}\langle Gu, u\rangle + h(\tfrac{1}{p}\|\nabla u\|_p^p). \end{aligned}$$

Then Φ_λ is continuously Frêchet differentiable. It is not difficult to show that any critical point $w \in W^{1,p}(\Omega)$ of Φ_λ solves the equation

$$A_p w - \frac{\lambda}{1 + h'(\frac{1}{p}\|\nabla w\|_p^p)} G w = 0.$$

However, the only nontrivial critical points of Φ_λ occur if

$$h'\left(\frac{1}{p}\|\nabla u\|_p^p\right) = \frac{\lambda}{\lambda_1} - 1. \tag{3.4}$$

Because $\lambda \neq \lambda_1$ and by definition of h, we must have $\frac{1}{p}\|\nabla u\|_p^p \in (t_0, 3t_0)$.
Due to the simplicity of λ_1, either $u_0 = \mp\alpha u_1$ for some $\alpha \in I\!R_*^+$, where $u_1 > 0$ is the principal eigenfunction normalized by $\|u_1\|_{1,p} = 1$ corresponding to λ_1. Thus, for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the derivative Φ'_λ possesses precisely three isolated critical points $-\alpha u_1$, 0, αu_1 .
Now, to complete the proof, it suffices to prove that these points are local minimums of Φ_λ, so that we can apply Lemma 2.1. For this, we argue by variational method. We claim that:
$(C1)$ Φ_λ is weakly lower semicontinuous.
$(C2)$ Φ_λ is coercive, i.e.

$$\lim_{\|u\|_{1,p}\to+\infty} \|\Phi_\lambda(u)\|_{1,p} = \infty.$$

Indeed, for $(C1)$ let due to the definition of $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$. Thus, Lemma 2.3 implies

$$\langle Gu_n, u_n\rangle \longrightarrow \langle Gu, u\rangle. \tag{3.5}$$

Thanks to the weakly lower semicontinuity of the norm, we obtain

$$\|\nabla u\|_p \leq \lim_{n\to+\infty} \|\nabla u_n\|_p. \tag{3.6}$$

Using (3.5), (3.6) and the fact that h is increasing in $(3t_0,\ \infty)$, we deduce that

$$\liminf_{n\to\infty} \Phi_\lambda(u_n) \geq \Phi_\lambda(u).$$

We deal now with $(C2)$, $\Phi_\lambda(u)$ is coercive otherwise, there exist a sequence $(u_n)_n$ in $W^{1,p}(\Omega)$ and a constant $c > 0$ such that $\lim_{n\to\infty} \|u_n\|_{1,p} = \infty$ and $\Phi_\lambda(u_n) \leq c$.
Therefore

$$\begin{aligned}
\Phi_\lambda(u_n) &= \tfrac{1}{p}\langle A_p u_n, u_n\rangle - \tfrac{\lambda}{p}\langle Gu_n, u_n\rangle + h(\tfrac{1}{p}\|\nabla u_n\|_p^p) \\
&= \frac{1}{p}\int_\Omega |\nabla u_n|^p\,dx - \frac{\lambda}{p}\int_\Omega m(x)|u_n|^p\,dx + h(\frac{1}{p}\|\nabla u_n\|_p^p) \\
&\geq \tfrac{1+a}{p}\int_\Omega |\nabla u_n|^p\,dx - \frac{\lambda}{p}\int_\Omega m(x)|u_n|^p\,dx - 2at_0.
\end{aligned}$$

It follows that

$$\|\nabla u_n\|_p^p - \frac{\lambda}{1+a}\int_\Omega m(x)|u_n|^p\,dx \leq c + 2at_0.$$

Since $\lambda_1 > \frac{\lambda}{1+a}$ then Lemma 3.1 implies that $\|u_n\|_{1,p}$ is bounded which is contradiction.
Consequently, from (C_1) and (C_2) and the fact that Φ_λ is even, there are precisely two points at which the minimum of Φ_λ is achieved: $-\alpha u_1$ and αu_1 for some $\alpha \in I\!R_*^+$, in view of [3]. The point origin 0 is obviously an isolated critical point of "the saddle type".
From Lemma 2.1, we have

$$Ind(\Phi'_\lambda, -\alpha u_1) = Ind(\Phi'_\lambda,\ \alpha u_1) = 1. \tag{3.7}$$

Simultaneously, we have

$$\langle \Phi'_\lambda(u), u\rangle\ > 0,$$

for any $\|u\|_{1,p} = R$, with $R > 0$ large enough. Indeed, it is easy to verify that

$$\langle A_p u, u\rangle > \lambda\langle Gu, u\rangle \ \text{ and }\ \langle A_p u, u\rangle > 3t_0,$$

for $\|u\|_{1,p}$ large enough. Therefore,

$$\begin{aligned} \langle \Phi'_\lambda(u), u\rangle &\geq (\lambda_1 - \lambda)\langle Gu, u\rangle + a\langle A_p u, u\rangle \\ &\geq \tfrac{-\delta}{\lambda_1}\langle A_p u, u\rangle + a\langle A_p u, u\rangle \\ &\geq (a - \tfrac{\delta}{\lambda_1})\|\nabla u\|_p^p. \end{aligned}$$

That is $\langle \Phi'_\lambda(u), u\rangle \to \infty$, as $\|u\|_{1,p} \to \infty$, due to the choice of a and Lemma 3.1.
Lemma 2.2 implies that

$$Deg[\Phi'_\lambda; B_R(0), 0] = 1. \tag{3.8}$$

We choose R so large such that $\|\alpha u_1\|_{1,p} = R$, i.e, $\alpha u_1 \in \partial B_R(0)$.
Now, thanks to additivity property of the degree, (3.7) and (3.8), we deduce that

$$Deg[L - \lambda G; B_\epsilon(0), 0] = -1. \tag{3.9}$$

On the other hand, it is clear that

$$\langle A_p u, u\rangle - \lambda\langle Gu, u\rangle \to 0,$$

as $\|u\|_{1,p} \to 0$. Then, by the definition of h, we obtain

$$Deg[A_p u - \lambda G; B_\epsilon(0), 0] = Ind(\Phi'_\lambda, 0), \tag{3.10}$$

for $\epsilon > 0$ small enough. Which implies from (2.3) and the homotopy invariance principle of the degree, that for $\epsilon > 0$ small enough,

$$Deg[T_\lambda; B_\epsilon(0), 0] = Deg[A_p u - \lambda G; B_\epsilon(0), 0)], \tag{3.11}$$

for $\lambda \in (\lambda_1 - \delta,\ \lambda_1 + \delta)\backslash\{\lambda_1\}$. Consequently, we conclude from (3.3), (3.9), (3.10) and (3.11) that

$$Deg[T_\lambda; B_\epsilon(0), 0] = 1 \ \text{ for } \lambda \in (\lambda_1 - \delta, \lambda_1),$$

$$Deg[T_\lambda; B_\epsilon(0), 0] = -1 \quad \text{for } \lambda \in (\lambda_1, \lambda_1 + \delta)$$

for $\epsilon > 0$ sufficiently small. The "jump" of the degree is established and the proof is completed. ■

Remark 3.1. *We can extend the bifurcation result above to any eigenvalue λ_n which is isolated in the spectrum and of odd multiplicity in order to be able to apply the above argument.*

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