Electronic Journal of Qualitative Theory of Differential Equations
2004, No. **8**, 1-18; **http://www.math.u-szeged.hu/ejqtde/**

POSITIVE SOLUTIONS FOR FIRST ORDER NONLINEAR FUNCTIONAL BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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ABSTRACT. In this paper we study a boundary value problem for a first order functional differential equation on an infinite interval. Using fixed point theorems on appropriate cones in Banach spaces, we derive multiple positive solutions for our boundary value problem.

1. INTRODUCTION

Boundary value problems on infinite intervals appear in many problems of practical interest, for example in linear elasticity problems, nonlinear fluid flow problems and foundation engineering (see e.g. [1,10,16] and the references therein). This is the reason why these problems have been studied quite extensively in the literature, especially the ones involving second order differential equations. Second order boundary value problems on infinite intervals are treated with various methods, such as fixed point theorems (see e.g. [1,5,6,10,14-16,23]), upper and lower solutions method (see e.g. [2,8,9]), diagonalization method (see e.g. [1,18,20]) and others. An interesting overview on infinite interval problems, including real world examples, history and various methods of solvability, can be found in the recent book of Agarwal and O' Regan [1].

A rather less extensive study has been done on first order boundary value problems on infinite intervals. One of the major ways to deal with such problems is to use numerical methods, see for example [10,16]. In short, the basic idea in this case is to build a finite interval problem such that its solution approximates the solution of the infinite problem quite well on the finite interval. The difficulty of this method lies in setting the finite interval boundary value problem in such a way that the approximation is accurate. Another way to deal with boundary value problems on infinite intervals is to use fixed point theorems. See for example [7,9]. Using this approach one will have to reformulate the boundary value problem to an operator equation and to use an appropriate compactness criterion on infinite intervals for the corresponding operator (see Lemma 2.2 below).

In the recent years a growing interest has arisen for positive solutions of boundary value problems. See for example [10,11,17,18,23]. Also, nowadays, functional boundary value problems are extensively investigated, usually via fixed point theorems (see [6,12,21,22] and the references therein).

2000 *Mathematics Subject Classification.* 34K10, 34B40.
Key words and phrases. first order, functional differential equations, infinite interval, multiple positive solutions.

This paper is motivated by [10-12,14,18]. We will deal with a first order functional boundary value problem on an infinite interval, seeking positive non-zero solutions. In order to do that, we will use two fixed point theorems, one of them being the well known Krasnoselkii's fixed point theorem.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ and $J := [-r, 0]$ for some $r \geq 0$. If I is an interval in $\mathbb{R}$ we denote by $C(I)$ the set of all continuous real functions $\psi : I \to \mathbb{R}$. Also, we denote by $BC(I)$ the Banach space of all $\psi \in C(I)$ such that $\sup\{|\psi(s)| : s \in I\} < +\infty$ endowed with the usual sup-norm

$$\|\psi\|_I := \sup\{|\psi(s)| : s \in I\}.$$

If $x \in C(J \cup \mathbb{R}^+)$ and $t \in \mathbb{R}^+$, then we denote by x_t the element of $C(J)$ defined by

$$x_t(s) = x(t+s), \ \ s \in J.$$

The boundary value problem, which we will study, consists of the equation

$$x'(t) = f(t, x_t), \ \ t \in \mathbb{R}^+, \tag{1.1}$$

along with the nonlinear condition

$$Ax_0 - x(+\infty) = \phi, \tag{1.2}$$

where $f : \mathbb{R}^+ \times C(J) \to \mathbb{R}$, $\phi : J \to \mathbb{R}$ are continuous functions, $x(+\infty) := \lim_{t\to+\infty} x(t)$ and it holds that

(H_1) $A > 1$, $\phi(0) \geq 0$ and $\phi(t) \geq -\frac{\phi(0)}{A-1}$, $t \in J$.

As indicated in [13], since $x(+\infty)$ exists in $\mathbb{R}$, we must suppose that the following is true

$$\lim_{t\to\infty} f(t, \psi) = 0, \ \ \psi \in C(J).$$

However, this is a direct consequence of the forthcoming assumption (H_3), so we do not state it as a separate assumption.

The paper is organized as follows. In Section 1 we state the boundary value problem and in Section 2 we present the fixed point theorems, formulate the corresponding operator and prove that it is compact. Then in Sections 3 and 4 we prove our new results for the functional and the ordinary case, respectively, and in Section 5 we give an application. We must notice that even for the ordinary boundary value problem, which corresponds to the case $r = 0$, the results we present in Section 4, are new.

2. Preliminaries

Definition. *A solution of the boundary value problem* $(1.1) - (1.2)$ *is a function* $x \in C(J \cup \mathbb{R}^+)$*, continuously differentiable on* $\mathbb{R}^+$*, which satisfies equations* (1.1) *for* $t \in \mathbb{R}^+$*, and* (1.2) *for* $t \in J$*. Additionally,* x *is called positive solution if* $x(t) \geq 0$*,* $t \in J \cup \mathbb{R}^+$*.*

Searching for positive solutions of the boundary value problem $(1.1) - (1.2)$, it is necessary to reformulate this problem to an integral equation. This is done in the next lemma.

Lemma 2.1. *A function $x \in C(J \cup \mathbb{R}^+)$ is a solution of the boundary value problem (1.1) − (1.2) if and only if $x(t) = Rx(t)$, $t \in J \cup \mathbb{R}^+$, where $R : C(J \cup \mathbb{R}^+) \to C(J \cup \mathbb{R}^+)$ is such that*

$$Rx(t) = \begin{cases} \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \int_0^t f(\theta, x_\theta)d\theta, & t \in \mathbb{R}^+, \\ \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \frac{\phi(t)}{A}, & t \in J. \end{cases}$$

Proof. From (1.1), we have

$$x(t) = x(0) + \int_0^t f(\theta, x_\theta)d\theta, \ \ t \in \mathbb{R}^+, \tag{2.1}$$

so

$$x(+\infty) = \lim_{t\to+\infty} x(t) = x(0) + \int_0^{+\infty} f(\theta, x_\theta)d\theta. \tag{2.2}$$

Now, from (1.2) and (2.2) we get

$$x(s) = \frac{\phi(s)}{A} + \frac{1}{A}\left(x(0) + \int_0^{+\infty} f(\theta, x_\theta)d\theta\right), \ \ s \in J. \tag{2.3}$$

Therefore

$$x(0) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta. \tag{2.4}$$

Using (2.1) and (2.4) we have

$$x(t) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \int_0^t f(\theta, x_\theta)d\theta, \ \ t \in \mathbb{R}^+.$$

Also, combining (2.3) and (2.4) we get

$$x(s) = \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \frac{\phi(s)}{A}, \ \ s \in J.$$

So, $x(t) = Rx(t)$, $t \in J \cup \mathbb{R}^+$.

On the other hand, if $x \in C(J \cup \mathbb{R}^+)$ is such that $x(t) = Rx(t)$, $t \in J \cup \mathbb{R}^+$, then

$$x'(t) = (Rx(t))' = f(t, x_t), \ \ t \in \mathbb{R}^+.$$

Also, for any $s \in J$ we have

$$\begin{aligned} Ax_0(s) - x(+\infty) &= Ax(s) - x(+\infty) \\ &= \frac{\phi(0)}{A-1} + \frac{A}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \phi(s) \\ &\quad - \frac{\phi(0)}{A-1} - \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta - \int_0^{+\infty} f(\theta, x_\theta)d\theta \\ &= \phi(s). \end{aligned}$$

Finally, it is clear that Rx is continuous at zero. The proof is complete. $\square$

Set

$$C^+(J) := \{x \in C(J) : x(t) \geq 0, \;\; t \in J\}$$

and consider the following assumptions.

(H_2) Assume that $f(\mathbb{R}^+ \times C^+(J)) \subseteq \mathbb{R}^+$ and, for every $t \in \mathbb{R}^+$, the function $f(t, \cdot) : C(J) \to \mathbb{R}^+$ maps bounded subsets of $C(J)$ to bounded subsets of $\mathbb{R}^+$.

By (H_2), we conclude that for every $s \in \mathbb{R}^+$ and $m > 0$, the $\sup_{\|y\|_J \in [0,m]} f(s, y)$ exists in $\mathbb{R}^+$. So, we set

$$F(s, m) := \sup_{\|y\|_J \in [0,m]} f(s, y), \;\; s \in \mathbb{R}^+, \;\; m > 0,$$

and we assume that

(H_3) For every $m > 0$, it holds

$$\Theta(m) := \int_0^{+\infty} F(\theta, m) d\theta < +\infty.$$

Then for every $m > 0$ set

$$Q(m) := \max \left\{ \frac{\phi(0) + A\Theta(m)}{A - 1}, \quad \frac{\phi(0) + A\Theta(m)}{A(A - 1)} + \frac{\|\phi\|_J}{A} \right\}.$$

(H_4) Assume that there exists $\rho > 0$ such that

$$Q(\rho) < \rho.$$

(H_5) There exist $E \subseteq \mathbb{R}^+$, with $measE > 0$, and functions $u : E \to [0, r]$, continuous $v : E \to \mathbb{R}^+$ with $\sup\{v(t) : t \in E\} > 0$ and nondecreasing $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$t - u(t) \geq 0, \quad t \in E$$

and

$$f(t, y) \geq v(t)w(y(-u(t))), \quad (t, y) \in E \times C^+(J).$$

The following assumption (H_6) is the analogue of assumption (H_5), when the function w is nonincreasing.

(H_6) There exist $E \subseteq \mathbb{R}^+$, with $measE > 0$, and functions $u : E \to [0, r]$, continuous $v : E \to \mathbb{R}^+$ with $\sup\{v(t) : t \in E\} > 0$ and nonincreasing $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \geq v(t)w(y(-u(t))), \quad (t, y) \in E \times C^+(J).$$

Now, consider the Banach space

$$Y(t_0) := \{x \in C([t_0, +\infty)) : \lim_{t\to+\infty} x(t) =: l_x \in \mathbb{R}\},$$

endowed with the usual norm

$$\|x\|_{[t_0,+\infty)} := \sup\{|x(t)| : t \in [t_0, +\infty)\}.$$

The following compactness criterion, which is due to Avramescu [4], will be used to prove that R is completely continuous. Note that the classical Arzela - Ascoli Theorem cannot be applied here, since the domain of R is a space of functions with unbounded domain.

Lemma 2.2. *Let $t_0 \in \mathbb{R}$ and $M \subseteq Y(t_0)$ have the following properties*

(i) There exists $K > 0$ such that $|u(t)| \leq K$ for every $t \geq t_0$ and $u \in M$.
(ii) For every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|u(t_1) - u(t_2)| < \epsilon$ for every t_1, $t_2 \geq t_0$, with $|t_1 - t_2| < \delta(\epsilon)$, and $u \in M$.
(iii) For every $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that $|u(t) - l_u| < \epsilon$ for every $t \geq T(\epsilon)$ and $u \in M$.

Then M is relatively compact in $Y(t_0)$.

We can now prove the following.

Lemma 2.3. *Let assumptions $(H_1) - (H_3)$ hold. Then the operator $R : Y(-r) \to Y(-r)$, defined in Lemma 2.1, is completely continuous.*

Proof. Let $x \in C(J \cup \mathbb{R}^+)$. Then using (H_3), we have

$$\begin{aligned}\lim_{t\to+\infty} R(x(t)) &= \lim_{t\to+\infty}\left(\frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \int_0^t f(\theta, x_\theta)d\theta\right)\\ &= \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \int_0^{+\infty} f(\theta, x_\theta)d\theta\\ &= \frac{\phi(0)}{A-1} + \frac{A}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta\\ &< +\infty.\end{aligned}$$

Therefore $R : Y(-r) \to Y(-r)$.

Now, let $x \in Y(-r)$ and $\{x^{[n]}\}_{n\in\mathbb{N}} \subseteq Y(-r)$, such that $\|x^{[n]} - x\|_{J\cup\mathbb{R}^+} \to 0$, $n \to +\infty$. Then, for $n \in \mathbb{N}$, we have

$$\|Rx^{[n]} - Rx\|_{J\cup\mathbb{R}^+} \leq \frac{A}{A-1}\int_0^{+\infty} |f(\theta, x_\theta^{[n]}) - f(\theta, x_\theta)|d\theta.$$

Since f is continuous, for any $t \in \mathbb{R}^+$, it holds that

$$f(t, x_t^{[n]}) \to f(t, x_t), \quad n \to +\infty.$$

Also

$$|f(t, x_t^{[n]})| \leq F(t, m),$$

where $m := \sup_{n\in\mathbb{N}} \|x^{[n]}\|_{J\cup\mathbb{R}^+}$. Notice that $m < +\infty$, since $\|x^{[n]} - x\|_{J\cup\mathbb{R}^+} \to 0$, $n \to +\infty$, and $x \in Y(-r)$.

Therefore, from Lebesgue's Dominated Convergence Theorem, it follows that

$$\int_0^{+\infty} |f(\theta, x_\theta^{[n]}) - f(\theta, x_\theta)| d\theta \to 0, \ \ n \to +\infty.$$

So, $\|Rx^{[n]} - Rx\|_{J\cup\mathbb{R}^+} \to 0$, $n \to +\infty$, thus R is continuous.

Now let $V \subseteq Y(-r)$ be bounded, that is

$$\|x\|_{J\cup\mathbb{R}^+} \leq m, \ \ x \in V,$$

where $m > 0$. We will prove that $R(V)$ is relatively compact in $Y(-r)$. Indeed for every $t \in J \cup \mathbb{R}^+$ we have

$$|Rx(t)| \leq \max\left\{\frac{\phi(0)}{A-1} + \frac{A}{A-1}\Theta(m), \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\Theta(m) + \frac{\|\phi\|_J}{A}\right\},$$

i.e.

$$\|Rx\| \leq Q(m).$$

Also, for arbitrary t_1, $t_2 \in J \cup \mathbb{R}^+$, we have

- case t_1, $t_2 \in \mathbb{R}^+$, $t_1 > t_2$
$$|Rx(t_1) - Rx(t_2)| = \left|\int_{t_2}^{t_1} f(\theta, x_\theta) d\theta\right| \leq \int_{t_2}^{t_1} F(\theta, m) d\theta.$$
- case t_1, $t_2 \in J$, $t_1 > t_2$
$$|Rx(t_1) - Rx(t_2)| < |\phi(t_1) - \phi(t_2)|.$$
- case $t_1 \in \mathbb{R}^+$, $t_2 \in J$
$$\begin{aligned} |Rx(t_1) - Rx(t_2)| &= \left|\frac{\phi(0) - \phi(t_2)}{A} + \int_0^{t_1} f(\theta, x_\theta) d\theta\right| \\ &\leq \frac{|\phi(0) - \phi(t_2)|}{A} + \int_0^{t_1} f(\theta, x_\theta) d\theta \\ &\leq \frac{|\phi(0) - \phi(t_2)|}{A} + \int_0^{t_1} F(\theta, m) d\theta. \end{aligned}$$

Therefore, since ϕ is uniformly continuous, in any of the above cases and for any $\epsilon > 0$ there exists $\delta(\epsilon, \phi, F) > 0$ such that

$$|Rx(t_1) - Rx(t_2)| < \epsilon,$$

when t_1, $t_2 \in J \cup \mathbb{R}^+$, with $|t_1 - t_2| < \delta(\epsilon, \phi, F)$ and $x \in V$.

Furthermore, for $t_0 \in \mathbb{R}^+$ we have

$$|Rx(t_0) - \lim_{t\to+\infty} Rx(t)| = \left|\int_{t_0}^{+\infty} f(\theta, x_\theta)d\theta\right| \leq \int_{t_0}^{+\infty} F(\theta, m)d\theta.$$

So, having in mind assumption (H_3) we conclude that for every $\epsilon > 0$, there exists $T(\epsilon, F) > 0$ such that

$$|Rx(t_0) - \lim_{t\to+\infty} Rx(t)| < \epsilon,$$

for every $t_0 \geq T(\epsilon, F)$ and $x \in V$.

Now, we can apply Lemma 2.2 and get that $R(V)$ is relatively compact in $Y(-r)$. This completes the proof. $\square$

Definition. *Let $\mathbb{E}$ be a real Banach space. A cone in $\mathbb{E}$ is a nonempty, closed set $\mathbb{P} \subset \mathbb{E}$ such that*

(i) $\kappa u + \lambda v \in \mathbb{P}$ for all u, $v \in \mathbb{P}$ and all κ, $\lambda \geq 0$
(ii) u, $-u \in \mathbb{P}$ implies $u = 0$.

Let $\mathbb{P}$ be a cone in a Banach space $\mathbb{E}$. Then, for any $b > 0$, we denote by $\mathbb{P}_b$ the set

$$\mathbb{P}_b := \{x \in \mathbb{P} : \|x\| < b\}$$

and by $\partial\mathbb{P}_b$ the boundary of $\mathbb{P}_b$ in $\mathbb{P}$, i.e. the set

$$\partial\mathbb{P}_b := \{x \in \mathbb{P} : \|x\| = b\}.$$

Part of our results are based on the following theorem, which is an application of the fixed point theory in a cone. Its proof can be found in [3].

Theorem 2.4. *Let $g : \overline{\mathbb{P}_b} \to \mathbb{P}$ be a completely continuous map such that $g(x) \neq \lambda x$ for all $x \in \partial\mathbb{P}_b$ and $\lambda \geq 1$. Then g has a fixed point in $\mathbb{P}_b$.*

It easy to see that the condition

$$g(x) \neq \lambda x \text{ for all } x \in \partial\mathbb{P}_b \text{ and } \lambda \geq 1,$$

can be replaced by the following stricter assumption

$$\|g(x)\| < \|x\| \text{ for all } x \in \partial\mathbb{P}_b$$

So, we get the following corollary of Theorem 2.4.

Theorem 2.5. *Let $g : \overline{\mathbb{P}_b} \to \mathbb{P}$ be a compact map such that $\|g(x)\| < \|x\|$ for all $x \in \partial\mathbb{P}_b$. Then g has a fixed point in $\mathbb{P}_b$.*

Also, we will use the well known Krasnoselskii's fixed point theorem (see [13]).

Theorem 2.6. *Let $E = (E, \|\cdot\|)$ be a Banach space and $\mathbb{P} \subset E$ be a cone in $\mathbb{P}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathbb{P}$, with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$g : \mathbb{P} \cap (\overline{\Omega_2} \backslash \Omega_1) \to \mathbb{P}$$

be a completely continuous operator such that either

$$\begin{aligned} &\|g(x)\| \leq \|x\|,\ x \in \mathbb{P} \cap \partial\Omega_1, \text{ and } \|g(x)\| \geq \|x\|,\ x \in \mathbb{P} \cap \partial\Omega_2, \\ \text{or } &\|g(x)\| \geq \|x\|,\ x \in \mathbb{P} \cap \partial\Omega_1, \text{ and } \|g(x)\| \leq \|x\|,\ x \in \mathbb{P} \cap \partial\Omega_2. \end{aligned}$$

Then g has a fixed point $x \in \mathbb{P} \cap (\overline{\Omega_2} \backslash \Omega_1)$.

In this paper, we will use the following Theorem 2.7, which is a corollary of Theorem 2.6, for the special case when the sets Ω_1 and Ω_2 are balls (with common center at point zero and positive, nonequal radium).

Theorem 2.7. *Let $E = (E, \|\cdot\|)$ be a Banach space and $\mathbb{P} \subset E$ be a cone. Also, σ, τ are positive constants with $\sigma \neq \tau$. Suppose*

$$g : \overline{\mathbb{P}}_{\max\{\sigma,\tau\}} \backslash \mathbb{P}_{\min\{\sigma,\tau\}} \to \mathbb{P}$$

is a completely continuous operator and assume that conditions

(*i*) $\|g(x)\| \leq \|x\|$ *for* $x \in \partial\mathbb{P}_\sigma$,

(*ii*) $\|g(x)\| \geq \|x\|$ *for* $x \in \partial\mathbb{P}_\tau$

hold. Then g has at least a fixed point x with

$$\min\{\sigma, \tau\} < \|x\| < \max\{\sigma, \tau\}.$$

3. Positive solutions for the functional problem

Before we state our main results we set

$$\Phi := \frac{\phi(0)}{A(A-1)}, \quad \Lambda := \Phi + \frac{\phi(-r)}{A}$$

and

$$\mu := \frac{1}{A-1} \int_E v(\theta) d\theta.$$

Theorem 3.1. *Suppose that conditions $(H_1) - (H_4)$ hold. Also suppose that if $\phi = 0$, there exists $t_0 \in \mathbb{R}^+$ such that $f(t_0, 0) \neq 0$. Then the boundary value problem $(1.1) - (1.2)$ has at least one positive nonzero solution x such that*

$$\Phi \leq \|x\|_{J \cup \mathbb{R}^+} < \rho.$$

More precisely we have

$$x(t) \geq \Phi, \ \ t \in J \cup \mathbb{R}^+.$$

Proof. We will first justify why any positive solution x of the boundary value problem $(1.1) - (1.2)$, if one exists, is nonzero. By hypothesis, if $\phi = 0$, then there exists

$t_0 \in \mathbb{R}^+$ such that $f(t_0, 0) \neq 0$. Then it is clear that equation (1.1) does not have the zero solution. On the other hand, if $\phi(t_1) \neq 0$ for some $t_1 \in J$, then by (1.2) we get that $Ax(t_1) - x(+\infty) \neq 0$, which also means that $x \neq 0$.

Now, set

$$\mathbb{P} := \{x \in BC(J \cup \mathbb{R}^+) : x \geq 0\}$$

and observe that $\mathbb{P}$ is a cone in $BC(J \cup \mathbb{R}^+)$. Also, we notice that for every $x \in \mathbb{P}$ and $t \in \mathbb{R}^+$ we have $x_t \in C^+(J)$, so by (H_2) $f(t, x_t) \geq 0$ and , taking into account (H_1), we easily obtain $Rx(t) \geq 0$. This means that $R(\mathbb{P}) \subset \mathbb{P}$, so since we are looking for a positive solution of the boundary value problem $(1.1) - (1.2)$, it is enough to find a fixed point of the operator $R : \mathbb{P} \to \mathbb{P}$.

Let ρ be the constant introduced by (H_4). Then, obviously $R(\overline{\mathbb{P}_\rho}) \subseteq \mathbb{P}$ and, from Lemma 2.3 it follows that R is a completely continuous operator.

Furthermore, we will show that $\|Rx\|_{J\cup\mathbb{R}^+} < \|x\|_{J\cup\mathbb{R}^+}$, for every $x \in \partial\mathbb{P}_\rho$. Assume that this is not true. Then, for some $x \in \partial\mathbb{P}_\rho$, it holds $\|x\|_{J\cup\mathbb{R}^+} \leq \|Rx\|_{J\cup\mathbb{R}^+}$. Also, observe that, from the formula of Rx and assumptions (H_2), (H_3), for every $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
Rx(t) &= \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \int_0^t f(\theta, x_\theta)d\theta \\
&\leq \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{+\infty} F(\theta, \rho)d\theta + \int_0^t F(\theta, \rho)d\theta \\
&\leq \frac{\phi(0)}{A-1} + \frac{1}{A-1}\Theta(\rho) + \Theta(\rho) \\
&= \frac{\phi(0) + A\Theta(\rho)}{A-1} \\
&\leq Q(\rho).
\end{aligned}$$

Additionally, for every $t \in J$ we have

$$\begin{aligned}
Rx(t) &= \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \frac{\phi(t)}{A} \\
&\leq \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\int_0^{+\infty} F(\theta, \rho)d\theta + \frac{\phi(t)}{A} \\
&\leq \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\Theta(\rho) + \frac{\|\phi\|_J}{A} \\
&= \frac{\phi(0) + A\Theta(\rho)}{A(A-1)} + \frac{\|\phi\|_J}{A} \\
&\leq Q(\rho).
\end{aligned}$$

So, we have

$$\rho = \|x\|_{J\cup\mathbb{R}^+} \leq \|Rx\|_{J\cup\mathbb{R}^+} \leq Q(\rho),$$

which contradicts (H_4).

We can now apply Theorem 2.5 to obtain that the boundary value problem $(1.1) - (1.2)$ has at least one positive nonzero solution x, such that

$$0 < \|x\|_{J\cup\mathbb{R}^+} < \rho. \tag{3.1}$$

If x is a positive solution of the boundary value problem $(1.1)-(1.2)$, then, taking into account the formula of R and the fact that $A > 1$, we conclude that

$$x(t) \geq \frac{\phi(0)}{A(A-1)}, \ \ t \in J \cup \mathbb{R}^+,$$

which implies that

$$\|x\|_{J\cup\mathbb{R}^+} \geq \Phi.$$

Now it is easy to see that $\Phi \leq Q(\rho)$ and so, by (H_4), $\Phi < \rho$. Then, taking into account (3.1), we obtain

$$\Phi \leq \|x\|_{J\cup\mathbb{R}^+} < \rho$$

and the proof is complete. $\square$

Theorem 3.2. *Suppose that $(H_1)-(H_5)$ hold and $\phi \in C(J)$ is a nondecreasing function. Also suppose that there exists $\gamma > 0$ such that*

$$\gamma \leq \frac{1}{A} w(\gamma)\mu. \tag{3.2}$$

where w, v are the functions involved in (H_5). Then the boundary value problem $(1.1)-(1.2)$ has at least one positive solution x, with

$$D < \|x\|_{J\cup R^+} \leq \max\{\tau, \rho\},$$

where

$$D := \begin{cases} \rho, & \text{if } \tau > \rho \\ \max\{\tau, \Lambda\}, & \text{if } \tau < \rho, \end{cases}$$

ρ is the constant involved in (H_4) and $\rho \neq \tau := A\gamma$.

More precisely we have

$$x(t) \geq \Lambda, \ \ t \in J \cup \mathbb{R}^+.$$

Proof. Define the set

$$\mathbb{K} := \left\{ x \in BC(J\cup\mathbb{R}^+) : x \geq 0, \ \ x \text{ is nondecreasing and } \ x(0) \geq \frac{1}{A}x(+\infty) \right\}.$$

Notice that $\mathbb{K}$ is a cone in $BC(J\cup\mathbb{R}^+)$. It is clear that, by (H_1) and (H_2), for any $x \in \overline{\mathbb{K}}_d$, where $d = max\{\rho, \tau\}$, we have $Rx(t) \geq 0$ for every $t \in J\cup\mathbb{R}^+$. Also, since $x \geq 0$ we have, also by (H_2), that $(Rx)'(t) = f(t, x_t) \geq 0$, $t \in \mathbb{R}^+$. Namely $Rx|\mathbb{R}^+$ is a nondecreasing function. Also, taking into account the formula of $Rx|J$ and the fact that ϕ is nondecreasing, we get that $Rx|J$ is also nondecreasing. Since Rx is continuous at zero, we conclude that Rx is also nondecreasing on $J\cup\mathbb{R}^+$. Moreover it is clear that $ARx(0) - Rx(+\infty) = \phi(0)$. By (H_1), we have $\frac{\phi(0)}{A} \geq 0$ and thus $Rx(0) = \frac{1}{A}x(+\infty) + \frac{\phi(0)}{A} \geq \frac{1}{A}x(+\infty)$. So $R : \overline{\mathbb{K}}_d \backslash \mathbb{K}_{\min\{\rho,\tau\}} \to \mathbb{K}$. Also, from Lemma 2.3, we get that R is completely continuous.

Furthermore, as we did in Theorem 3.1, we can prove that $\|Rx\|_{J\cup\mathbb{R}^+} < \|x\|_{J\cup\mathbb{R}^+}$ for $x \in \partial\mathbb{K}_\rho$.

Now we will prove that $\|Rx\|_{J\cup\mathbb{R}^+} \geq \|x\|_{J\cup\mathbb{R}^+}$ for every $x \in \partial\mathbb{K}_\tau$. For this purpose it suffices to prove that $Rx \geq \tau$ for every $x \in \partial\mathbb{K}_\tau$. By (H_1) we have

$$\begin{aligned} Rx(-r) &= \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta + \frac{\phi(-r)}{A} \\ &\geq \frac{1}{A-1}\int_0^{+\infty} f(\theta, x_\theta)d\theta. \end{aligned}$$

So, using (H_5) and the fact that x is nondecreasing, we obtain

$$\begin{aligned} Rx(-r) &\geq \frac{1}{A-1}\int_E v(\theta)w(x_\theta(-u(\theta)))d\theta \\ &= \frac{1}{A-1}\int_E v(\theta)w(x(\theta - u(\theta)))d\theta \\ &\geq \frac{1}{A-1}\int_E v(\theta)w(x(0))d\theta. \end{aligned}$$

However, $x(0) \geq \frac{1}{A}x(+\infty)$ and $x(+\infty) = \|x\|_{J\cup\mathbb{R}^+}$, since x is nondecreasing. Therefore, taking into account (3.2) and the fact that $\|x\|_{J\cup\mathbb{R}^+} = \tau = A\gamma$, we get

$$\begin{aligned} Rx(-r) &\geq w\left(\frac{1}{A}\|x\|_{J\cup\mathbb{R}^+}\right)\frac{1}{A-1}\int_E v(\theta)d\theta \\ &= w(\gamma)\mu \\ &\geq A\gamma = \tau = \|x\|_{J\cup\mathbb{R}^+} \ . \end{aligned}$$

Hence, since Rx is nondecreasing, we have

$$Rx(t) \geq \tau = \|x\|_{J\cup\mathbb{R}^+}, \ \ t \in J \cup \mathbb{R}^+.$$

Therefore, for every $x \in \partial\mathbb{K}_\tau$ we have $Rx \geq \|x\|_{J\cup\mathbb{R}^+}$ and so $\|Rx\|_{J\cup\mathbb{R}^+} \geq \|x\|_{J\cup\mathbb{R}^+}$.

Thus, we can apply Theorem 2.7 to get that the boundary value problem (1.1) – (1.2) has at least one positive solution x, such that

$$\min\{\tau, \rho\} < \|x\|_{J\cup\mathbb{R}^+} < \max\{\tau, \rho\}. \tag{3.3}$$

Now, if x is a positive solution of the boundary value problem (1.1) – (1.2), then, taking into account the formula of R and the facts that $A > 1$ and x is nondecreasing, we conclude that

$$x(t) \geq x(-r) = Rx(-r) \geq \Lambda, \ \ t \in J \cup \mathbb{R}^+,$$

which implies

$$\|x\|_{J\cup\mathbb{R}^+} \geq \Lambda.$$

Also, taking into account (3.3), we obtain

$$\max\{\min\{\tau, \rho\}, \Lambda\} < \|x\|_{J\cup\mathbb{R}^+} < \max\{\tau, \rho\}. \tag{3.4}$$

Now we observe that for every $\theta > 0$ we have $\Lambda \leq Q(\theta)$. So, since $Q(\rho) < \rho$, we have $\Lambda < \rho$ and if $\tau > \rho$, then $\max\{\min\{\tau, \rho\}, \Lambda\} = \max\{\rho, \Lambda\} = \rho$. On the other hand, if $\tau < \rho$, then $\max\{\min\{\tau, \rho\}, \Lambda\} = \max\{\tau, \Lambda\}$. Therefore (3.4) takes the form

$$D \leq \|x\|_{J\cup\mathbb{R}^+} < \max\{\tau, \rho\}$$

and the proof is complete. $\square$

Theorem 3.3. *Suppose that* $(H_1)-(H_4)$, (H_6) *hold and, moreover,* ϕ *is a nondecreasing function. Also suppose that there exists* $\tau > 0$ *such that*

$$\tau \leq w(\tau)\mu. \tag{3.5}$$

Then the boundary value problem $(1.1)-(1.2)$ *has at least one positive solution* x*, such that*

$$D \leq \|x\|_{J\cup\mathbb{R}^+} < \max\{\tau,\rho\},$$

where D *is defined in Theorem* 3.2*,* ρ *is the constant involved in* (H_4) *and* $\rho \neq \tau$*.*

More precisely we have

$$x(t) \geq \Lambda, \ \ t \in J\cup\mathbb{R}^+.$$

Proof. Define the set

$$\mathbb{K} := \left\{x \in BC(J\cup\mathbb{R}^+) : x \geq 0, \ \ x \text{ is nondecreasing}\right\}.$$

It holds that $R : \overline{\mathbb{K}}_d \backslash \mathbb{K}_{\min\{\rho,\tau\}} \to \mathbb{K}$ and that R is a completely continuous operator. The proof is similar to the one we used in Theorem 3.2.

Furthermore, as we did in Theorem 3.1, we can prove that $\|R(x)\|_{J\cup\mathbb{R}^+} < \|x\|_{J\cup\mathbb{R}^+}$ for $x \in \partial\mathbb{K}_\rho$.

Now we will prove that $\|Rx\|_{J\cup\mathbb{R}^+} \geq \|x\|_{J\cup\mathbb{R}^+}$ for every $x \in \partial\mathbb{K}_\tau$. For this purpose it suffices to prove that $Rx \geq \tau$ for every $x \in \partial\mathbb{K}_\tau$. As in Theorem 3.2, using (H_1) and (H_6), we obtain

$$Rx(-r) \geq \frac{1}{A-1}\int_E v(\theta)w(x(\theta - u(\theta)))d\theta.$$

Therefore, taking into account the fact that w is nonincreasing and (3.5) we have

$$\begin{aligned} Rx(-r) &\geq \frac{1}{A-1}w(\tau)\int_E v(\theta)d\theta \\ &= w(\tau)\mu \\ &\geq \tau. \end{aligned}$$

Hence, since Rx is nondecreasing, we have

$$Rx(t) \geq \tau = \|x\|_{J\cup\mathbb{R}^+}, \ \ t \in J\cup\mathbb{R}^+.$$

Therefore, for every $x \in \partial\mathbb{K}_\tau$ we have $Rx \geq \|x\|_{J\cup\mathbb{R}^+}$ and hence $\|Rx\|_{J\cup\mathbb{R}^+} \geq \|x\|_{J\cup\mathbb{R}^+}$.

So, applying Theorem 2.7 we get that there exists at least one positive solution x of the boundary value problem $(1.1)-(1.2)$, such that (3.3) holds. Finally, as in the previous Theorem 3.2, we can prove that (3.3) takes the form of (3.4) and the proof is complete. $\square$

Combining Theorems 3.1 and 3.2 (*resp.* 3.3) we can prove easily the following theorem, which ensures the existence of two positive solutions for the boundary value problem $(1.1)-(1.2)$.

Theorem 3.4. *Suppose that conditions* $(H_1)-(H_5)$ *(resp.* $(H_1)-(H_4),(H_6)$*) hold and* ϕ *is a nondecreasing function. Also, suppose that if* $\phi=0$*, there exists* $t_0\in\mathbb{R}^+$ *such that* $f(t_0,0)\neq 0$ *and, additionally, there exists* $\gamma>0$ *such that* (3.2) *(resp.* (3.5)*) holds. Then, if* $\rho<A\gamma$ *(resp.* $\rho<\gamma$*), the boundary value problem* $(1.1)-(1.2)$ *has at least two positive solutions* x_1*,* x_2 *such that*

$$\Lambda<\|x_1\|_{J\cup\mathbb{R}^+}<\rho<\|x_2\|_{J\cup\mathbb{R}^+}<\tau,$$

where

$$\tau:=\begin{cases} A\gamma, & \text{if } (H_5) \text{ holds,}\\ \gamma, & \text{if } (H_6) \text{ holds.}\end{cases}$$

Moreover we have

$$x_i(t)\geq\Lambda,\ \ t\in J\cup\mathbb{R}^+,\ \ i=1,2.$$

Going a step further, the following theorem, which ensure the existence of countably infinite positive solutions of the boundary value problem $(1.1)-(1.2)$. Its proof can be easily obtained through the combination of the results of Theorems 3.2 and 3.3.

Theorem 3.5. *Assume that* $(H_1)-(H_3)$*,* (H_5) *(resp.* $(H_1)-(H_3)$*,* (H_6)*) hold,* ϕ *is a nondecreasing function and there exist two strictly increasing real sequences* $(\rho_\nu)_{\nu\in\mathbb{N}}$*,* $(\gamma_\nu)_{\nu\in\mathbb{N}}$ *(*$\mathbb{N}$ *is the set of natural numbers) such that*

$$\rho_\nu<\tau_\nu:=A\gamma_\nu<\rho_{\nu+1},\ \ \nu\in\mathbb{N}$$

$$(\textit{resp. }\ \rho_\nu<\tau_\nu:=\gamma_\nu<\rho_{\nu+1},\ \ \nu\in\mathbb{N}).$$

Moreover, assume that (H_4) *is satisfied for all* ρ_ν *in place of* ρ *and* (3.2) *(resp.* (3.5)*) is also satisfied for all* γ_ν *in place of* γ*. Then, the boundary value problem* $(1.1)-(1.2)$ *has a sequence of positive solutions* $(x_\nu)_{\nu\in\mathbb{N}}$ *such that*

$$\rho_\nu<\|x_\nu\|_{J\cup\mathbb{R}^+}<\tau_\nu<\|x_{\nu+1}\|_{J\cup\mathbb{R}^+}<\rho_{\nu+1},\quad \nu\in\mathbb{N}.$$

Moreover we have

$$x_\nu(t)\geq\Lambda,\ \ t\in J\cup\mathbb{R}^+,\ \ \nu\in\mathbb{N}.$$

Remark 3.6. It is clear that the assumption:

there exists $\gamma>0$ such that (3.2) (resp. (3.5)) holds

in Theorem 3.4, can be replaced by the following:

$$\limsup_{\theta\to+\infty}\frac{w(\theta)}{\theta}>\frac{A}{\mu}\ \ (\text{resp.}\limsup_{\theta\to+\infty}\frac{w(\theta)}{\theta}>\frac{1}{\mu}).$$

Indeed if $\limsup_{\theta\to+\infty}\frac{w(\theta)}{\theta}>\frac{A}{\mu}$ (resp. $\limsup_{\theta\to+\infty}\frac{w(\theta)}{\theta}>\frac{1}{\mu}$), then there exists $\gamma>\frac{\rho}{A}$ (resp. $\gamma>\rho$) such that $\frac{w(\gamma)}{\gamma}>\frac{A}{\mu}$ (resp. $\frac{w(\gamma)}{\gamma}>\frac{1}{\mu}$).

4. Positive solutions for the ordinary problem

If we choose $r = 0$ then we no longer have a functional boundary value problem, but an ordinary one instead. In this case, the boundary value problem is formed as follows

$$x'(t) = f(t, x(t)), \quad t \in \mathbb{R}^+ \tag{4.1}$$

$$Ax(0) - x(+\infty) = C \tag{4.2}$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and A, C are real numbers satisfying the following:

($\widehat{H}_1$) $A > 1$ and $C \geq 0$

Definition. *A solution of the boundary value problem* (4.1) − (4.2) *is a function* $x \in C(\mathbb{R}^+)$ *which satisfies equations* (4.1) − (4.2). *Additionally, x is called positive solution if $x(t) \geq 0$, $t \in \mathbb{R}^+$.*

It is clear that a function $x \in C(\mathbb{R}^+)$ continuously differentiable on $\mathbb{R}^+$, is a solution of the boundary value problem (4.1) − (4.2) if and only if it satisfies the equation $x = \widehat{R}x$, where the operator $\widehat{R} : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ is given by the formula

$$\widehat{R}x(t) := \frac{C}{A-1} + \frac{1}{A-1}\int_0^{+\infty} f(\theta, x(\theta))d\theta + \int_0^t f(\theta, x(\theta))d\theta, \quad t \in \mathbb{R}^+.$$

In this, ordinary, case assumptions (H_2)-(H_4) are replaced by the following:

($\widehat{H}_2$) Assume that $f(\mathbb{R}^+ \times \mathbb{R}^+) \subseteq \mathbb{R}^+$ and for every $t \in \mathbb{R}^+$ the function $f(t, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ maps bounded subsets of $\mathbb{R}^+$ into bounded subsets of $\mathbb{R}^+$.

It is obvious that, under assumption ($\widehat{H}_2$), for every $s \in \mathbb{R}^+$ and $m > 0$, the $\sup_{y \in [0,m]} f(s, y)$ exists in $\mathbb{R}^+$. Then we set

$$\widehat{F}(s, m) := \sup_{y \in [0,m]} f(s, y), \quad s \in \mathbb{R}^+, \quad m > 0.$$

($\widehat{H}_3$) Assume that for every $m > 0$, it holds

$$\widehat{\Theta}(m) := \int_0^{+\infty} \widehat{F}(\theta, m)d\theta < +\infty.$$

Now set

$$\widehat{Q}(m) := \frac{C + A\widehat{\Theta}(m)}{A-1}$$

and assume the following:

($\widehat{H}_4$) There exists $\rho > 0$ such that

$$\widehat{Q}(\rho) < \rho.$$

Also, observe that the analogues of assumptions $(H_5), (H_6)$, for the ordinary case, can be unified in the following:

$(\widehat{H}_5)$ There exist $E \subseteq \mathbb{R}^+$, with $measE > 0$, and functions $v : E \to \mathbb{R}^+$ continuous, with $\sup\{v(t) : t \in E\} > 0$ and monotonous $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \geq v(t)w(y), \quad (t, y) \in E \times \mathbb{R}.$$

Finally we set

$$\widehat{\Phi} := \frac{C}{A - 1}$$

and then we have the following theorems, which correspond to Theorems $3.1 - 3.5$, respectively, for the ordinary case. The proofs of these theorems are omitted, since they can be easily derived from the proofs of Theorems $3.1-3.5$, with some obvious modifications. Also it is easy to see that the constant Λ in the present ordinary case is equal to the constant $\widehat{\Phi}$.

Theorem 4.1. *Suppose that conditions $(\widehat{H}_1) - (\widehat{H}_4)$ hold. Also suppose that, in case $C = 0$, there exists $t_0 \in \mathbb{R}^+$ such that $f(t_0, 0) \neq 0$. Then the boundary value problem $(4.1) - (4.2)$ has at least one positive nonzero solution x, such that*

$$\widehat{\Phi} \leq \|x\|_{\mathbb{R}^+} < \rho.$$

More precisely we have

$$x(t) \geq \widehat{\Phi}, \quad t \in \mathbb{R}^+.$$

Theorem 4.2. *Suppose that $(\widehat{H}_1) - (\widehat{H}_5)$ hold and that there exists $\gamma > 0$ such that*

$$\gamma < \frac{1}{A} w(\gamma)\mu, \tag{4.3}$$

if w is nondecreasing, or

$$\gamma < w(\gamma)\mu, \tag{4.4}$$

if w is nonincreasing. Then the boundary value problem $(4.1) - (4.2)$ has at least one positive solution x, with

$$\widehat{D} < \|x\|_{\mathbb{R}^+} < \max\{\tau, \rho\}.$$

where

$$\widehat{D} := \begin{cases} \rho, & \text{if } \tau > \rho \\ \max\{\tau, \widehat{\Phi}\}, & \text{if } \tau < \rho, \end{cases}$$

ρ is the constant involved in $(\widehat{H}_4)$, $\rho \neq \tau$ and $\tau := A\gamma$, if w in nondecreasing, or $\tau := \gamma$, if w in nonincreasing. More precisely, in any case we have

$$x(t) \geq \widehat{\Phi}, \quad t \in \mathbb{R}^+.$$

Theorem 4.3. *Suppose that conditions* $(\widehat{H}_1) - (\widehat{H}_5)$ *hold. Also, suppose that if* $C = 0$, *there exists* $t_0 \in \mathbb{R}^+$ *such that* $f(t_0, 0) \neq 0$ *and, additionally, there exists* $\gamma > 0$ *such that* (4.3), *if* w *is nondecreasing, or* (4.4), *if* w *is nonincreasing, holds. Then, if* $\rho < A\gamma$, *in case* w *is nondecreasing, or* $\rho < \gamma$, *in case* w *is nonincreasing, the boundary value problem* (4.1) $-$ (4.2) *has at least two positive solutions* x_1, x_2 *such that*

$$\widehat{\Phi} < \|x_1\|_{\mathbb{R}^+} < \rho < \|x_2\|_{\mathbb{R}^+} < \tau$$

where $\tau := A\gamma$, *if* w *in nondecreasing, or* $\tau := \gamma$, *if* w *in nonincreasing. Moreover, we have*

$$x_i(t) \geq \widehat{\Phi}, \ \ t \in \mathbb{R}^+, \ \ i = 1, 2.$$

Theorem 4.4. *Assume that* $(\widehat{H}_1) - (\widehat{H}_3)$, $(\widehat{H}_5)$ *hold and there exist two strictly increasing real sequences* $(\rho_\nu)_{\nu \in \mathbb{N}}$, $(\gamma_\nu)_{\nu \in \mathbb{N}}$ ($\mathbb{N}$ *is the set of natural numbers) such that*

$$\rho_\nu < \tau_\nu := A\gamma_\nu < \rho_{\nu+1}, \quad \nu \in \mathbb{N},$$

if w *is nondecreasing, or*

$$\rho_\nu < \tau_\nu := \gamma_\nu < \rho_{\nu+1}, \quad \nu \in \mathbb{N},$$

if w *is nonincreasing. Moreover, assume that* $(\widehat{H}_4)$ *is satisfied for all* ρ_ν *in place of* ρ *and* (4.3), *if* w *is nondecreasing, or* (4.4), *if* w *is nonincreasing, is also satisfied for all* γ_ν *in place of* γ. *Then, the boundary value problem* (4.1) $-$ (4.2) *has a sequence of positive solutions* $(x_\nu)_{\nu \in \mathbb{N}}$ *such that*

$$\rho_\nu < \|x_\nu\|_{\mathbb{R}^+} < \tau_\nu < \|x_{\nu+1}\|_{\mathbb{R}^+} < \rho_{\nu+1}, \quad \nu \in \mathbb{N}.$$

Moreover we have

$$x_\nu(t) \geq \widehat{\Phi}, \ \ t \in \mathbb{R}^+, \ \ \nu \in \mathbb{N}.$$

5. An application

Consider the boundary value problem

$$x'(t) = \frac{x_t^2(-\frac{1}{5})}{10(1+t^2)}, \ \ t \in \mathbb{R}^+, \tag{5.1}$$

$$10x_0 - x(+\infty) = \phi, \tag{5.2}$$

where $\phi(t) = \frac{20}{9}t + 1$, $t \in J := [-\frac{1}{5}, 0]$.

In this case, function f is defined as follows

$$f(t, y) = \frac{y^2(-\frac{1}{5})}{10(1+t^2)}, \ \ t \in \mathbb{R}^+, \ \ y \in C([-\frac{1}{5}, 0]).$$

Now observe that $\phi(0) = 1 \geq 0$, $A = 10$ and

$$\phi(t) = \frac{20}{9}t + 1 \geq -\frac{1}{9} = -\frac{\phi(0)}{A-1}, \ \ t \in [-\frac{1}{5}, 0].$$

So assumption (H_1) holds. Also it is obvious that assumption (H_2) holds. Furthermore, we can choose $F(t,m) = \frac{m^2}{10(1+t^2)}$, $t \in \mathbb{R}^+$, $m > 0$, so we have

$$\begin{aligned}\int_0^{+\infty} F(t,m)dt &= \frac{m^2}{10}\int_0^{+\infty}\frac{dt}{1+t^2}\\ &= \frac{m^2}{10}\lim_{\zeta\to+\infty}\int_0^{\zeta}\frac{dt}{1+t^2}\\ &= \frac{m^2}{10}\lim_{\zeta\to+\infty}(arctan\zeta)\\ &= \frac{m^2\pi}{20} < +\infty.\end{aligned}$$

Therefore, condition (H_3) is true. Also, we have just proved that $\Theta(m) = \frac{m^2\pi}{20}$. Now we have for $m > 0$

$$Q(m) = \max\left\{\frac{1+\frac{\pi}{2}m^2}{9}, \frac{1+\frac{\pi}{2}m^2}{90} + \frac{1}{10}\right\} = \frac{1+\frac{\pi}{2}m^2}{9}$$

and so (H_4) holds for $\rho = 1$.

Also, we obtain

$$f(t,y) \geq v(t)w(y(-u(t))),$$

where $v(t) = \frac{1}{10(1+t^2)}$, $t \in E = [\frac{1}{5}, +\infty)$, $w(t) = t^2$, $t \in \mathbb{R}^+$ and $u(t) = \frac{1}{5}$, $t \in E$. Obviously $t - u(t) \geq 0$, $t \in E$, and $\sup\{v(t) : t \in E\} = \frac{5}{52} > 0$. Additionally, inequality (3.2) is formed as follows

$$\gamma \leq \frac{1}{90}\gamma^2\int_{\frac{1}{5}}^{+\infty}\frac{dt}{10(1+t^2)} = \frac{\frac{\pi}{2} - \arctan\frac{1}{5}}{900}\gamma^2,$$

which is satisfied for any $\gamma \geq \frac{900}{\frac{\pi}{2}-\arctan\frac{1}{5}}$. Also, notice that ϕ is nondecreasing and that since $\gamma \geq \frac{900}{\frac{\pi}{2}-\arctan\frac{1}{5}}$, it holds

$$1 = \rho < \tau = A\gamma = 10\gamma.$$

Consequently, from Theorem 3.4, for $\gamma = \frac{900}{\frac{\pi}{2}-\arctan\frac{1}{5}}$ we have that the boundary value problem (5.1) – (5.2) has at least two positive solutions x_1, x_2 such that

$$\frac{1}{15} < \|x_2\|_{J\cup\mathbb{R}^+} < 1 < \|x_1\|_{J\cup\mathbb{R}^+} < \frac{9000}{\frac{\pi}{2} - \arctan\frac{1}{5}} \approx 6553.077\ .$$

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