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Representations of mild solutions of time-varying linear stochastic equations and the exponential stability of periodic systems

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Abstract

The main object of this paper is to give a representation of the covariance operator associated to the mild solutions of time-varying, linear, stochastic equations in Hilbert spaces. We use this representation to obtain a characterization of the uniform exponential stability of linear stochastic equations with periodic coefficients.

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1 Preliminaries

Let H, V be separable real Hilbert spaces and let $L(H,V)$ be the Banach space of all bounded linear operators from H into V (If $H=V$ then $L(H,V) \overset{not}{=} L(H)$). We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators. We denote by $a \otimes b, a, b \in H$ the bounded linear operator of $L(H)$ defined by $a \otimes b(h) = \langle h, b \rangle a$ for all $h \in H$. The operator $A \in L(H)$ is said to be nonnegative and we write $A \geq 0$, if A is self-adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We denote by $\mathcal{E}$ the Banach subspace of $L(H)$ formed by all self-adjoint operators, by $L^+(H)$ the cone of all nonnegative operators of $\mathcal{E}$ and by I the identity operator on H.

Let $P \in L^+(H)$ and $A \in L(H)$. We denote by $P^{1/2}$ the square root of P and by $|A|$ the operator $(A^*A)^{1/2}$. We put $\|A\|_1 = Tr(|A|) \leq \infty$ and we denote by $C_1(H)$ the set $\{A \in L(H)/ \|A\|_1 < \infty\}$ (the *trace class* of operators).

If $A \in C_1(H)$ we say that A is *nuclear* and it is not difficult to see that A is compact.

The definition of nuclear operators introduced above is equivalent with that given in [6] and [9].

It is known (see [6]) that $C_1(H)$ is a Banach space endowed with the norm $\|.\|_1$ and for all $A \in L(H)$ and $B \in C_1(H)$ we have AB, $BA \in C_1(H)$.

If $\|A\|_2 = (TrA^*A)^{1/2}$ we can introduce the *Hilbert Schmidt class* of operators, namely $C_2(H) = \{A \in L(H)/ \|A\|_2 < \infty\}$ (see [5]).

$C_2(H)$ is a Hilbert space with the inner product $\langle A, B\rangle_2 = TrA^*B$ ([5]).

We denote by $\mathcal{H}_2$ the subspace of $C_2(H)$ of all self-adjoint operators.

Since $\mathcal{H}_2$ is closed in $C_2(H)$ with respect to $\|.\|_2$ we deduce that it is a Hilbert space, too. It is known (see [9]) that for all $A \in C_1(H)$ we have

$$\|A\| \leq \|A\|_2 \leq \|A\|_1 . \tag{1}$$

For each interval $J \subset \mathbf{R}_+ (\mathbf{R}_+ = [0, \infty))$ we denote by $C_s(J, L(H))$ the space of all mappings $G(t) : J \to L(H)$ that are strongly continuous.

If E is a Banach space we also denote by $C(J, E)$ the space of all mappings $G(t) : J \to E$ that are continuous.

In the subsequent considerations we assume that the families of operators $\{A(t)\}_{t\in\mathbf{R}_+}$ and $\{G_i(t)\}_{t\in\mathbf{R}_+}, i = 1, ..., m$ satisfied the following hypotheses:

P1 : *a) $A(t)$, $t \in [0, \infty)$ is a closed linear operator on H with constant domain D dense in H.*

b) there exist $M > 0$, $\eta \in (\frac{1}{2}\pi, \pi)$ and $\delta \in (-\infty, 0)$ such that $S_{\delta,\eta} = \{\lambda \in C; |\arg(\lambda - \delta)| < \eta\} \subset \rho(A(t))$, for all $t \geq 0$ and

$$\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda - \delta|}$$

for all $\lambda \in S_{\delta,\eta}$ where we denote by $\rho(A)$, $R(\lambda, A)$ the resolvent set of A and respectively the resolvent of A.

c) there exist numbers $\alpha \in (0, 1)$ and $\widetilde{N} > 0$ such that

$\|A(t)A^{-1}(s) - I\| \leq \widetilde{N} |t - s|^{\alpha}, t \geq s \geq 0.$

P2 $: G_i \in C_s(\mathbf{R}_+, L(H)), i = 1, ..., m.$

It is known that if P1 holds then the family $\{A(t)\}_{t\in\mathbf{R}_+}$ generates the evolution operator $U(t,s), t \geq s \geq 0$ (see [4]). For any $n \in \mathbf{N}$ we have $n \in \rho(A(t))$. The operators $A_n(t) = n^2R(n, A(t)) - nI$ are called the Yosida approximations of $A(t)$.

If we denote by $U_n(t,s)$ the evolution operator generates by $A_n(t)$, then it is known (see [4]) that for each $x \in H$, one has $\lim\limits_{n\to\infty} U_n(t,s)x = U(t,s)x$ uniformly on any bounded subset of $\{(t,s); t \geq s \geq 0\}$.

Let $(\Omega, F, \mathcal{F}_t, t \in [0,\infty), P)$ be a stochastic basis and $L_s^2(H) = L^2(\Omega, \mathcal{F}_s, P, H)$. We consider the stochastic equation

$$\begin{aligned} dy(t) &= A(t)y(t)dt + \sum_{i=1}^{m} G_i(t)y(t)dw_i(t) \\ y(s) &= \xi \in L_s^2(H), \end{aligned} \tag{2}$$

where the coefficients $A(t)$ and $G_i(t)$ satisfy the hypothesis P1, P2 and w_i's are independent real Wiener processes relative to $\mathcal{F}_t$.

Let us consider $T > 0$. It is known (see [2]) that (2) has a unique mild solution in $C([s,T]; L^2(\Omega; H))$ that is adapted to $\mathcal{F}_t$; namely the solution of

$$y(t) = U(t,s)\xi + \sum_{i=1}^{m} \int_s^t U(t,r)G_i(r)y(r)dw_i(r). \tag{3}$$

We associate to (2) the approximating system:

$$\begin{aligned} dy_n(t) &= A_n(t)y_n(t)dt + \sum_{i=1}^{m} G_i(t)y_n(t)dw_i(t) \\ y_n(s) &= \xi \in L_s^2(H), \end{aligned} \tag{4}$$

where $A_n(t), n \in \mathbf{N}$ are the Yosida approximations of $A(t)$.

By convenience, we denote by $y(t,s;\xi)$ (resp. $y_n(t,s;\xi)$) the solution of (2) (resp. (4)) with the initial condition $y(s) = \xi$ (resp. $y_n(s) = \xi$), $\xi \in L_s^2(H)$.

Lemma 1 *[4]There exists a unique mild (resp. classical) solution to (2) (resp.(4)) and $y_n \to y$ in mean square uniformly on any bounded subset of $[s,\infty]$.*

Now we consider the following Lyapunov equation:

$$\frac{dQ(s)}{ds} + A^*(s)Q(s) + Q(s)A(s) + \sum_{i=1}^{m} G_i^*(s)Q(s)G_i(s) = 0, s \geq 0 \quad (5)$$

According with [4], we say that Q is a mild solution on an interval $J \subset \mathbf{R}_+$ of (5), if $Q \in C_s(J, L^+(H))$ and if for all $s \leq t$, $s, t \in J$ and $x \in H$ it satisfies

$$Q(s)x = U^*(t,s)Q(t)U(t,s)x + \int_s^t U^*(r,s)[\sum_{i=1}^{m} G_i^*(r)Q(r)G_i(r)]U(r,s)xdr. \quad (6)$$

If $A_n(t), n \in \mathbf{N}$ are the Yosida approximations of $A(t)$ then we introduce the approximating equation:

$$\frac{dQ_n(s)}{ds} + A_n^*(s)Q_n(s) + Q_n(s)A_n(s) + \sum_{i=1}^{m} G_i^*(s)Q_n(s)G_i(s) = 0, s \geq 0. \quad (7)$$

Lemma 2 *[4] Let $0 < T < \infty$ and let $R \in L^+(H)$. Then there exists a unique mild (resp. classical) solution Q (resp. Q_n) of (5) (resp. (7)) on $[0,T]$ such that $Q(T) = R$ (resp. $Q_n(T) = R$). They are given by*

$$\begin{aligned} Q(s)x &= U^*(T,s)RU(T,s)x \quad (8)\\ &+ \int_s^T U^*(r,s)[\sum_{i=1}^{m} G_i^*(r)Q(r)G_i(r)]U(r,s)xdr \end{aligned}$$

$$\begin{aligned} Q_n(s)x &= U_n^*(T,s)RU_n(T,s)x\\ &+ \int_s^T U_n^*(r,s)[\sum_{i=1}^{m} G_i^*(r)Q_n(r)G_i(r)]U_n(r,s)xdr \quad (9) \end{aligned}$$

and for each $x \in H$, $Q_n(s)x \to Q(s)x$ uniformly on any bounded subset of $[0,T]$. Moreover, if we denote these solutions by $Q(T,s;R)$ and respectively $Q_n(T,s;R)$ then they are monotone in the sense that $Q(T,s;R_1) \leq Q(T,s;R_2)$ if $R_1 \leq R_2$.

2 Differential equations on $\mathcal{H}_2$

For all $n \in \mathbf{N}$ and $t \geq 0$ we consider the mapping $L_n(t) : \mathcal{H}_2 \to \mathcal{H}_2$,

$$L_n(t)(P) = A_n(t)P + PA_n^*(t) + \sum_{i=1}^{m} G_i(t)PG_i^*(t), P \in \mathcal{H}_2. \tag{10}$$

It is easy to verify that $L_n(t) \in L(\mathcal{H}_2)$ and the adjoint operator $L_n^*(t)$ is the linear and bounded operator on $\mathcal{H}_2$ given by

$$L_n^*(t)(R) = RA_n(t) + A_n^*(t)R + \sum_{i=1}^{m} G_i^*(t)RG_i(t), \tag{11}$$

for all $t \geq 0, P \in \mathcal{H}_2$.

Lemma 3 *([7])If P1, P2 hold then*
a) $A_n \in C([0,\infty), L(H))$ for all $n \in N$ and
b) $L_n \in C_s([0,\infty), L(\mathcal{H}_2))$ for all $n \in N$.

Proof. a) If $t, s \geq 0$ we have $\|A_n(t) - A_n(s)\|$

$$\begin{aligned}
&= n^2 \|R(n, A(t)) - R(n, A(s))\| \\
&= n^2 \|R(n, A(t))(nI - A(t))(R(n, A(t)) - R(n, A(s)))\| \\
&\leq n^2 \|R(n, A(t))\| \, \|I - [(nI - A(s)) + A(s) - A(t)]R(n, A(s))\| \\
&\leq n^2 \|R(n, A(t))\| \, \|I - I + [A(s) - A(t)]R(n, A(s))\| \\
&\leq n \|R(n, A(t))\| \, \left\|[A(s) - A(t)]A(s)^{-1}\right\| \, \|nA(s)R(n, A(s))\| .
\end{aligned}$$

Now we use P1 (the statements b) and c)) and we deduce that there exist $\delta < 0$, $\alpha \in (0,1)$, $M > 0$ and $\widetilde{N} > 0$ such that we have

$$\|nA(s)R(n, A(s))\| = \left\|n^2 R(n, A(s)) - nI\right\| \leq n^2 \frac{M}{n - \delta} + n, \text{ for any } s \geq 0$$

and $\|A_n(t) - A_n(s)\| \leq n\frac{M}{n-\delta}(n^2\frac{M}{n-\delta}+n)\widetilde{N}\,|t - s|^{\alpha}$. The proof of a) is finished.

b) We deduce from a) that if $\mathcal{A}_n(t) : \mathcal{H}_2 \to \mathcal{H}_2$,

$$\mathcal{A}_n(t)(P) = A_n(t)P + PA_n^*(t), t \geq 0, n \in \mathbf{N}$$

then $\mathcal{A}_n \in C([0,\infty), L(\mathcal{H}_2))$. We only have to prove that $\mathcal{G}_i \in C_s([0,\infty), L(\mathcal{H}_2))$, where $\mathcal{G}_i(t) : \mathcal{H}_2 \to \mathcal{H}_2$,

$$\mathcal{G}_i(t)(P) = G_i(t)PG_i^*(t), i = 1, ..., m.$$

From Lemma 1 [7] and since $G_i \in C_s([0,\infty), L(H))$, $i = 1, ..., m$ it follows $G_iP \in C([0,\infty), \mathcal{H}_2)$ and $PG_i^* \in C([0,\infty), \mathcal{H}_2)$ for all $P \in \mathcal{H}_2$ and $i = 1, ..., m$. For $s \geq 0, P \in \mathcal{H}_2$ fixed and for every $i \in \{1, ..., m\}$ we have

$$\begin{aligned}\|\mathcal{G}_i(t)(P) - \mathcal{G}_i(s)(P)\|_2 &= \|G_i(t)PG_i^*(t) - G_i(s)PG_i^*(s)\|_2 \\ &\leq \|G_i(t)PG_i^*(t) - G_i(t)PG_i^*(s)\|_2 \\ &+ \|G_i(t)PG_i^*(s) - G_i(s)PG_i^*(s)\|_2 .\end{aligned}$$

If $\widetilde{G}_{i,s} = \sup\limits_{t\in[0,s+1]} \|G_i(t)\|$, then, for all $t \in [0, s+1]$, we have

$$\begin{aligned}\|\mathcal{G}_i(t)(P) - \mathcal{G}_i(s)(P)\|_2 &\leq \widetilde{G}_{i,s} \|PG_i^*(t) - PG_i^*(s)\|_2 \\ &+ \|G_i(t)PG_i^*(s) - G_i(s)PG_i^*(s)\|_2 .\end{aligned}$$

As $t \to s$, we obtain $\lim\limits_{t\to s} \|\mathcal{G}_i(t)(P) - \mathcal{G}_i(s)(P)\|_2 = 0$. If $s = 0$ we only have the limit from the right. ■

If E is a Banach space and $\mathcal{L} \in C_s([0,\infty), L(E))$, we consider the initial value problem

$$\frac{\partial v(t)}{\partial t} = \mathcal{L}(t)v(t),\ v(s) = x \in E, t \geq s \geq 0. \tag{12}$$

Let $T \geq s$. An E valued function $v : [s,T] \to E$ is a classical solution of (12) if v is continuous on $[s,T]$, continuously differentiable on $[s,T]$ and satisfies (12). The following results have a standard proof (see [11]).

Lemma 4 *For every $x \in E$ the initial value problem (12) has a unique classical solution v.*

We define the "solution operator" of the initial value problem (12) by $V(t,s)x = v(t), x \in E$ for $0 \leq s \leq t \leq T$, where v is the solution of (12).

Let us denote by I the identity operator on E.

Proposition 5 *For all $0 \leq s \leq t \leq T$, $V(t,s)$ is a bounded linear operator and*

1. $\|V(t,s)\| \leq e^{\lambda(t-s)}$, *where* $\lambda = \sup\limits_{t\in[0,T]} \|\mathcal{L}(t)\|$.

2. $V(s,s) = I$ *and* $V(t,s) = V(t,r)V(r,s)$ *for all* $0 \leq s \leq r \leq t \leq T$.

3. $V(t,s) \underset{t-s\to 0}{\to} I$ *in the uniform operator topology for all* $0 \leq s \leq t \leq T$.

4. $(t,s) \to V(t,s)$ *is continuous in the uniform operator topology on* $\{(t,s)/0 \leq s \leq t \leq T\}$.

5. $\frac{\partial V(t,s)x}{\partial t} = L(t)V(t,s)x$ *for all* $x \in E$ *and* $0 \leq s \leq t \leq T$.

6. $\frac{\partial V(t,s)x}{\partial s} = -V(t,s)L(s)x$ *for all* $x \in E$ *and* $0 \leq s \leq t \leq T$.

The operator $V(t,s)$ is called the evolution operator generated by the family $\mathcal{L}$. Let us consider the equation

$$\frac{dP_n(t)}{dt} = L_n(t)P_n(t), \quad P_n(s) = S \in \mathcal{H}_2, t \geq s \geq 0 \tag{13}$$

on $\mathcal{H}_2$, where L_n is given by (10). From Lemma 3, Lemma 4 and the above proposition it follows that the unique classical solution of (13) is

$$P_n(t) = \mathcal{U}_n(t,s)(S),$$

where $\mathcal{U}_n(t,s) \in L(\mathcal{H}_2)$ is the evolution operator generated by L_n and

$$\frac{\partial \mathcal{U}_n(t,s)S}{\partial s} = -\mathcal{U}_n(t,s)L_n(s)S$$

for all $t \geq s \geq 0$, $S \in \mathcal{H}_2$. Now it is clear that

$$\frac{\partial}{\partial \sigma} \left\langle \mathcal{U}_n^*(t,\sigma)R, S\right\rangle_2 = \left\langle -L_n^*(\sigma)\mathcal{U}_n^*(t,\sigma)R, S\right\rangle_2, S, R \in \mathcal{H}_2 \text{ for all } t \geq \sigma \geq 0.$$

We take $S = x \otimes x, x \in H$. It is easy to see that $\langle Fx, x\rangle = TrFS$ for all $F \in L(H)$. If $F \in \mathcal{H}_2$ then $\langle F, S\rangle_2 = \langle Fx, x\rangle$. Integrating from s to t, we have

$$\langle \mathcal{U}_n^*(t,s)Rx, x\rangle - \langle Rx, x\rangle = \int_s^t \langle L_n^*(\sigma)\mathcal{U}_n^*(t,\sigma)Rx, x\rangle \, d\sigma, R \in \mathcal{H}_2. \tag{14}$$

Let $Q_n(t, s; R)$ be the unique classical solution of (7) such as $Q_n(t) = R, R \geq 0$. We have

$$\langle Q_n(t, s; R)x, x\rangle - \langle Rx, x\rangle = \int_s^t \langle L_n^*(\sigma)Q_n(t, \sigma; R)x, x\rangle \, d\sigma, R \geq 0 \qquad (15)$$

If $R \in \mathcal{H}_2, R \geq 0$ it follows from (14) and (15)

$$\langle [\mathcal{U}_n^*(t, s)R - Q_n(t, s; R)] \, x, x\rangle = \int_s^t \langle L_n^*(\sigma) \, [\mathcal{U}_n^*(t, \sigma)R - Q_n(t, \sigma; R)] \, x, x\rangle \, d\sigma.$$

By the Uniform Boundedness Principle there exists $l_T > 0$ such that $\|L_n^*(t)P\| \leq l_T \|P\|$ for all $t \in [0, T]$, $P \in L(H)$ and we obtain

$$\|\mathcal{U}_n^*(t, s)R - Q_n(t, s; R)\| \leq \int_s^t l_T \, \|\mathcal{U}_n^*(t, \sigma)R - Q_n(t, \sigma; R)\| \, d\sigma.$$

Now we use Gronwall's inequality and we get

$$\mathcal{U}_n^*(t, s)R = Q_n(t, s; R), \text{ for all } R \in \mathcal{H}_2, R \geq 0, t \geq s \qquad (16)$$

From Proposition 5 and (1) we deduce that for all $R \in \mathcal{H}_2$ the map

$$(t, s) \rightarrow Q_n(t, s; R) \text{ is } \|.\| - \text{continuous on } \{(s, t)/0 \leq s \leq t\} \text{ and} \qquad (17)$$

$$Q_n(t, s; \alpha R + \beta S) = \alpha Q_n(t, s; R) + \beta Q_n(t, s; S) \qquad (18)$$

for all $\alpha, \beta \in \mathbf{R}_+$ and $R, S \in \mathcal{H}_2, R, S \geq 0$.

3 The covariance operator of the mild solutions of linear stochastic differential equations and the Lyapunov equations

Let $\xi \in L^2(\Omega, H)$. We denote by $E(\xi \otimes \xi)$ the bounded and linear operator which act on H given by $E(\xi \otimes \xi)(x) = E(\langle x, \xi\rangle \, \xi)$.

The operator $E(\xi \otimes \xi)$ is called the covariance operator of ξ.

Lemma 6 *[10] Let V be another real, separable Hilbert space and $A \in L(H,V)$. If $\xi \in L^2(\Omega, H)$ then*

$$E\,\|A(\xi)\|^2 = \|AE(\xi \otimes \xi)A^*\|_1 < \infty.$$

Particularly $E\,\|\xi\|^2 = \|E(\xi \otimes \xi)\|_1$.

Proposition 7 *If $y_n(t,s;\xi)$, $\xi \in L_s^2(H)$ is the classical solution of (4) then $E[y_n(t,s;\xi)\otimes y_n(t,s;\xi)]$ is the unique classical solution of the following initial value problem*

$$\frac{dP_n(t)}{dt} = A_n(t)P_n(t) + P_n(t)A_n^*(t) + \sum_{i=1}^{m} G_i(t)P_n(t)G_i^*(t) \tag{19}$$
$$P_n(s) = E(\xi \otimes \xi).$$

Proof. Let $u \in H$ and $T \geq 0$, fixed. We consider the function

$$F_u \overset{not}{=} F : \mathbf{R}_+ \times H \to \mathbf{R}, F(t,x) = \langle (x \otimes x)\,u, u\rangle\,.$$

Using Ito's formula for F and $y_n(t,s;\xi)$ we obtain for all $0 \leq s \leq t \leq T$

$$\begin{aligned}
&\langle [y_n(t,s;\xi) \otimes y_n(t,s;\xi)]\,u,u\rangle - \langle (\xi \otimes \xi)\,u,u\rangle \\
&= \int_s^t \langle [A_n(r)y_n(r,s;\xi) \otimes y_n(r,s;\xi)]\,u,u\rangle \\
&+ \langle [y_n(r,s;\xi) \otimes A_n(r)y_n(r,s;\xi)]\,u,u\rangle \\
&+ \sum_{i=1}^{m} \langle [G_i(r)y_n(r,s;\xi) \otimes G_i(r)y_n(r,s;\xi)]\,u,u\rangle\,dr \\
&+ \sum_{i=1}^{m} \int_s^t \langle [y_n(r,s;\xi) \otimes G_i(r)y_n(r,s;\xi)]\,u,u\rangle \\
&+ \langle [G_i(r)y_n(r,s;\xi) \otimes y_n(r,s;\xi)]\,u,u\rangle\,dw_i(r)
\end{aligned}$$

Taking expectations, we have

$$\begin{aligned}
&\langle E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]u,u\rangle - \langle E[\xi \otimes \xi]u,u\rangle \\
&= \int_s^t \langle E[y_n(r,s;\xi) \otimes y_n(r,s;\xi)]u, A_n^*(r)u\rangle
\end{aligned}$$

$$+ \langle E[y_n(r,s;\xi) \otimes y_n(r,s;\xi)] A_n^*(r)u, u\rangle$$
$$+ \sum_{i=1}^{m} \langle E[y_n(r,s;\xi) \otimes y_n(r,s;\xi)] G_i^*(r)u, G_i^*(r)u\rangle \, dr.$$

If $P_n(t) = E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]$ then

$$\langle P_n(t)u, u\rangle - \langle E[\xi \otimes \xi]u, u\rangle = \int_s^t \langle A_n(r)P_n(r)u, u\rangle \tag{20}$$
$$+ \langle P_n(r)A_n^*(r)u, u\rangle + \sum_{i=1}^{m} \langle G_i(r)P_n(r)G_i^*(r)u, u\rangle \, dr.$$

According with lemmas L.3, L.4 and the statements of the last section, the equation (19) has a unique classical solution $\mathcal{U}_n(t,s)E\,(\xi \otimes \xi)$ in $\mathcal{H}_2$ and we have

$$\mathcal{U}_n(t,s)E\,(\xi \otimes \xi) = E\,(\xi \otimes \xi) + \int_s^t L_n(r)\mathcal{U}_n(r,s)E\,(\xi \otimes \xi) \, dr.$$

We note that $\mathcal{U}_n(t,s)$ is the evolution operator generated by L_n. Then

$$\langle \mathcal{U}_n(t,s)E\,(\xi \otimes \xi)\,, u \otimes u\rangle_2$$
$$= \langle E\,(\xi \otimes \xi)\,, u \otimes u\rangle_2 + \int_s^t \langle L_n(r)\mathcal{U}_n(r,s)E\,(\xi \otimes \xi)\,, u \otimes u\rangle_2 \, dr$$

or equivalently $\langle \mathcal{U}_n(t,s)E\,(\xi \otimes \xi)\,u, u\rangle = \langle E\,(\xi \otimes \xi)\,u, u\rangle +$

$$\int_s^t \langle L_n(r)\mathcal{U}_n(r,s)E\,(\xi \otimes \xi)\,u, u\rangle \, dr.$$

From (20) and the last equality we obtain

$$\langle [\mathcal{U}_n(t,s)E\,(\xi \otimes \xi) - P_n(t)]\,u, u\rangle = \int_s^t \langle L_n(r)\,[\mathcal{U}_n(r,s)E\,(\xi \otimes \xi) - P_n(r)]\,u, u\rangle \, dr.$$

Since there exists $l_T > 0$ such that $\|L_n(t)\| \leq l_T$ for all $t \in [0,T]$ and $\mathcal{U}_n(t,s)E(\xi \otimes \xi), P_n(t) \in \mathcal{E}$ we can use the Gronwall's inequality to deduce that

$$E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)] = \mathcal{U}_n(t,s)E(\xi \otimes \xi) \tag{21}$$

for all $t \in [s,T]$. Since T is arbitrary we obtain the conclusion. ■

The following theorem gives a representation of the covariance operator associated to the mild solution of (2), by using the mild solution of the Lyapunov equation (5).

Theorem 8 *Let V be another real separable Hilbert space and $B \in L(H,V)$. If $y(t,s;\xi)$, $\xi \in L_s^2(H)$ is the mild solution of (2) and $Q(t,s,R)$ is the unique mild solution of (5) with the final value $Q(t) = R \geq 0$ then*

a) $\langle E[y(t,s;\xi) \otimes y(t,s;\xi)]u, u\rangle = TrQ(t,s;u \otimes u)E(\xi \otimes \xi)$ for all $u \in H$

b)

$$E\|By(t,s;\xi)\|^2 = TrQ(t,s;B^*B)E(\xi \otimes \xi).$$

Proof. a) Let $u \in H, \xi \in L_s^2(H)$ and $y_n(t,s;\xi)$ be the classical solution of (4). By (21) we obtain successively

$$\begin{aligned}&< E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]u, u >=< u \otimes u, \mathcal{U}_n(t,s)E(\xi \otimes \xi) >_2 \\ &=< \mathcal{U}_n^*(t,s)(u \otimes u), E(\xi \otimes \xi) >_2 = Tr\mathcal{U}_n^*(t,s)(u \otimes u)E(\xi \otimes \xi).\end{aligned}$$

If $Q_n(t,s;u \otimes u)$ is the solution of (7) with $Q_n(t) = u \otimes u$ we obtain from (16)

$$\langle E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]u, u\rangle = TrQ_n(t,s;u \otimes u)E(\xi \otimes \xi) \tag{22}$$

As $n \to \infty$ we get the conclusion. Indeed, since $Q_n(t,s;u\otimes u) \underset{n\to\infty}{\to} Q(t,s;u\otimes u)$ in the strong operator topology (Lemma 2) then it is not difficult to deduce from Lemma 1 [7] that $Q_n(t,s;u\otimes u)E(\xi \otimes \xi) \underset{n\to\infty}{\to} Q(t,s;u\otimes u)E(\xi \otimes \xi)$ in $C_1(H)$.

It is known that the map $Tr : C_1(H) \to \mathbf{C}$ is continuous. So we obtain

$$TrQ_n(t,s;u \otimes u)E(\xi \otimes \xi) \underset{n\to\infty}{\to} TrQ(t,s;u \otimes u)E(\xi \otimes \xi).$$

On the other hand, for all $u \in H$ we have

$$
\begin{aligned}
&|\langle \{E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)] - E[y(t,s;\xi) \otimes y(t,s;\xi)]\}u,u\rangle| \\
&= \left|E(\langle y_n(t,s;\xi),u\rangle^2 - \langle y(t,s;\xi),u\rangle^2)\right| \\
&\le E(\|y_n(t,s;\xi) - y(t,s;\xi)\|^2 + 2\,\|y_n(t,s;\xi) - y(t,s;\xi)\|\,\|y(t,s;\xi)\|)\,\|u\|^2 \\
&\le \{E\,\|y_n(t,s;\xi) - y(t,s;\xi)\|^2 \\
&+ 2(E\,\|y_n(t,s;\xi) - y(t,s;\xi)\|^2\,E\,\|y(t,s;\xi)\|^2)^{1/2}\}\,\|u\|^2 .
\end{aligned}
$$

From Lemma 1 and the last inequality we get

$$\langle E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]u,u\rangle \underset{n\to\infty}{\to} \langle E[y(t,s;\xi) \otimes y(t,s;\xi)]u,u\rangle$$

and the proof is finished.

b) Let $\xi \in L_s^2(H)$ and $n \in \mathbf{N}$. It is sufficient to prove that

$$E\,\|By_n(t,s;\xi)\|^2 = TrQ_n(t,s;B^*B)E\,(\xi \otimes \xi)\,. \tag{23}$$

By Lemma 6 we have

$$E\,\|By_n(t,s;\xi)\|^2 = \|BE[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]B^*\|_1\,. \tag{24}$$

If $\{e_i\}_{i\in\mathbf{N}^*}$ is an orthonormal basis in V then we deduce from (a)

$$
\begin{aligned}
\|BE[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]B^*\|_1 &= \sum_{i=1}^{\infty} \langle E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]B^*e_i, B^*e_i\rangle \\
&= \sum_{i=1}^{\infty} TrQ_n(t,s;B^*e_i \otimes B^*e_i)E\,(\xi \otimes \xi)\,.
\end{aligned}
$$

Since $B^*e_i \otimes B^*e_i \in \mathcal{H}_2$ and $B^*e_i \otimes B^*e_i \ge 0, i \in \mathbf{N}^*$, we have by (18)

$$
\begin{aligned}
&\|BE[\ y_n(t,s;\xi) \otimes y_n(t,s;\xi)]B^*\|_1 \\
&= \lim_{p\to\infty} TrQ_n(t,s;\sum_{i=1}^{p} B^*e_i \otimes e_iB)E\,(\xi \otimes \xi)\,.
\end{aligned} \tag{25}
$$

The sequence $B_p = \sum\limits_{i=1}^{p} B^*e_i \otimes e_iB$ is increasing and bounded above:

$$\langle B_px,x\rangle = \sum_{i=1}^{p} \langle Bx,e_i\rangle^2 \le \|Bx\|^2 = \langle B^*Bx,x\rangle\,.$$

Then $\{B_p\}_{p\in\mathbf{N}}$ converges in the strong operator topology to the operator $B^*B \in L^+(H)$. By Lemma 2 we deduce that the sequence $\{Q_n(t,s;B_p\}_{p\in\mathbf{N}^*}$ is increasing ($Q_n(t,s;B_p) \le Q_n(t,s;B^*B)$ for all $p \in \mathbf{N}^*$) and consequently it converges in the strong operator topology to the operator $Q_n(t,s) \in L^+(H)$.

If $U_n(t,s)$ is the evolution operator relative to $A_n(t)$, we have for all $x \in H$

$$\langle Q_n(t,s;B_p)x,x\rangle = \langle B_p U_n(t,s)x, U_n(t,s)x\rangle \tag{26}$$
$$+\sum_{i=1}^{m}\int_s^t \langle Q_n(t,r;B_p)G_i(r)U_n(r,s)x, G_i(r)U_n(r,s)x\rangle\, dr.$$

Since $B_p \in \mathcal{H}_2$ and $B_p \ge 0$ we deduce from (17) and the hypothesis that

$$r \to \phi_{p,n,s,t}(r) = (\langle Q_n(t,r;B_p)G_i(r)U_n(r,s)x, G_i(r)U_n(r,s)x\rangle$$

is continuous. On the other hand we have for all $r \in [s,t]$

$$\lim_{p\to\infty}\phi_{p,n,s,t}(r) = \phi_{n,s,t}(r) = \langle Q_n(t,r)G_i(r)U_n(r,s)x, G_i(r)U_n(r,s)x\rangle\,.$$

Thus it follows that $r \to \phi_{n,s,t}(r)$ is a Borel measurable and nonnegative function defined on $[s,t]$ and bounded above by a continuous function, namely $r \to \langle Q_n(t,r;B^*B)G_i(r)U_n(r,s)x, G_i(r)U_n(r,s)x\rangle$.

From the Monotone Convergence Theorem we can pass to limit $p \to \infty$ in (26) and we have

$$\langle Q_n(t,s)x,x\rangle = \langle B^*BU_n(t,s)x, U_n(t,s)x\rangle$$
$$+\sum_{i=1}^{m}\int_s^t \langle Q_n(t,r)G_i(r)U_n(r,s)x, G_i(r)U_n(r,s)x\rangle\, dr,$$

where the integral is in Lebesgue sense. From (26) it follows

$$\langle [Q_n(t,s;B^*B) - Q_n(t,s)]x,x\rangle$$
$$= \sum_{i=1}^{m}\int_s^t \langle [Q_n(t,r;B^*B) - Q_n(t,r)]G_i(r)U_n(r,s)x, G_i(r)U_n(r,s)x\rangle\, dr. \tag{27}$$

The map $x \to \langle [Q_n(t,r;B^*B) - Q_n(t,r)]x,x\rangle$, $x \in H$ is continuous and $r \to \langle [Q_n(t,r;B^*B) - Q_n(t,r)]x,x\rangle$, $r \in [s,t]$ is a Borel measurable function.

Since $B_1 = \{x \in H, \|x\| = 1\}$ is separable [1], then there exists a net $\{y_n\}_{n \in \mathbf{N}} \subset B_1$ which is dense in B_1 and

$$\|Q_n(t,r;B^*B) - Q_n(t,r)\| = \sup_{y_n \in B_1} \langle [Q_n(t,r;B^*B) - Q_n(t,r)]y_n, y_n \rangle .$$

Thus $r \to \|Q_n(t,r;B^*B) - Q_n(t,r)\|$, $r \in [s,t]$ is a Borel measurable function. Since $0 \le Q_n(t,r;B^*B) - Q_n(t,r) \le Q_n(t,r;B^*B)$ it is clear that

$$r \to \|Q_n(t,r;B^*B) - Q_n(t,r)\| \, \|U_n(r,s)\|^2$$

is Lebesgue integrable. By (27) we have

$$\begin{aligned} &\|Q_n(t,s;B^*B) - Q_n(t,s)\| \\ &\le \sum_{i=1}^{m} \widetilde{G}_i \int_s^t \|Q_n(t,r;B^*B) - Q_n(t,r)\| \, \|U_n(r,s)\|^2 \, dr. \end{aligned}$$

Using the Gronwall's inequality, we get $\|Q_n(t,s;B^*B) - Q_n(t,s)\| = 0$. Thus $Q_n(t,s;B_p)x \underset{p\to\infty}{\to} Q_n(t,s;B^*B)x$ for all $x \in H$ and, from Lemma 1 in [7], we deduce that $Q_n(t,s;B_p)E(\xi \otimes \xi) \underset{p\to\infty}{\to} Q_n(t,s;B^*B)E(\xi \otimes \xi)$ in $\|.\|_1$.

By (24), (25) and since Tr is continuous on $C_1(H)$ we obtain (23). As $n \to \infty$ we obtain the conclusion. ■

We note that if A is time invariant ($A(t) = A$, for all $t \ge 0$), then the condition P1 can be replaced with the hypothesis

H0 : A is the infinitesimal generator of a C_0-semigroup

and arguing as above we can prove the following result.

Proposition 9 *If P2 and H0 hold, then the conclusions of the above theorem stay true. Particularly, if we replace P2 with the condition $G_i \in L(H)$, $i = 1, ..., m$ the statement b) becomes:*

$$E\|By(t,s;\xi)\|^2 = TrQ(t,s,0;B^*B)E(\xi \otimes \xi) = TrQ(t-s;B^*B)E(\xi \otimes \xi)$$

It is not difficult to see that if the coefficients of the stochastic equation (2) verify the condition

H1 : $A, G_i \in C(R_+, L(H)), i = 1, ..., m,$

then we don't need to work with the approximating systems and all the main results of the last two sections (including this) can be reformulated (and proved) adequately. So, we have the following proposition:

Proposition 10 *If the assumption H1 holds then the statements a) and b) of the Theorem 8 are true.*

4 The solution operators associated to the Lyapunov equations

Let $Q(T,s;R), R \in L^+(H), T \geq s \geq 0$ be the unique mild solution of the Lyapunov equation (5), which satisfies the condition $Q(T) = R$.

Lemma 11 *a) If $R_1, R_2 \in L^+(H)$ and $\alpha, \beta > 0$ then*

$$Q(T,s;\alpha R_1 + \beta R_2) = \alpha Q(T,s;R_1) + \beta Q(T,s;R_2).$$

b) $Q(p,s;Q(t,p;R)) = Q(t,s;R)$ for all $R \in L^+(H), t \geq p \geq s \geq 0$.

Proof. a) Let us denote $K(s) = Q(T,s;\alpha R_1 + \beta R_2) - \alpha Q(T,s;R_1) - \beta Q(T,s;R_2)$, $K(s) \in \mathcal{E}$, $T \geq s \geq 0$. By Lemma 2 we get

$$K(s)x = \int_s^T U^*(r,s)[\sum_{i=1}^m G_i^*(r)K(r)G_i(r)]U(r,s)xdr \text{ and}$$

$$\|K(s)\| = \sup_{x\in H, \|x\|=1} |\langle K(s)x, x\rangle| \leq \sum_{i=1}^m \int_s^T \|K(r)\| \, \|G_i(r)\| \, \|U(r,s)\| \, dr.$$

Using the Gronwall's inequality we deduce $\|K(s)\| = 0$ for all $s \in [0,T]$ and the conclusion follows. Similarly we can prove b). $\blacksquare$

The following lemma is known [13].

Lemma 12 *Let $T \in L(\mathcal{E})$. If $T(L^+(H)) \subset L^+(H)$ then $\|T\| = \|T(I)\|$, where I is the identity operator on H.*

If $R \in \mathcal{E}$ then there exist $R_1, R_2 \in L^+(H)$ such that $R = R_1 - R_2$ (we take for example $R_1 = \|R\| I$ and $R_2 = \|R\| I - R$).

We introduce the mapping $\mathcal{T}(t,s) : \mathcal{E} \to \mathcal{E}$,

$$\mathcal{T}(t,s)(R) = Q(t,s;R_1) - Q(t,s;R_2) \tag{28}$$

for all $t \geq s \geq 0$. The mapping $\mathcal{T}(t,s)$ has the following properties:

1. $\mathcal{T}(t,s)$ is well defined. Indeed if R_1', R_2' are another two nonnegative operators such as $R = R_1' - R_2'$ we have $R_1' + R_2 = R_1 + R_2'$. From lemmas L.2 and L.11 we have $Q(t,s;R_1'+R_2) = Q(t,s;R_1+R_2')$ and $Q(t,s;R_1') + Q(t,s;R_2) = Q(t,s;R_1) + Q(t,s;R_2')$. The conclusion follows.

2. $\mathcal{T}(t,s)(-R) = -\mathcal{T}(t,s)(R), R \in \mathcal{E}$.

3. $\mathcal{T}(t,s)(R) = Q(t,s;R)$ for all $R \in L^+(H)$ and $t \geq s \geq 0$.

4. $\mathcal{T}(t,s)(L^+(H)) \subset L^+(H)$.

5. For all $R \in \mathcal{E}$ and $x \in H$ we have

$$\langle \mathcal{T}(t,s)(R)x, x \rangle = E \langle Ry(t,s;x), y(t,s;x) \rangle . \tag{29}$$

   (It follows from the Theorem 8 and from the definition of $\mathcal{T}(t,s)(R)$.)

6. $\mathcal{T}(t,s)$ is a linear and bounded operator and $\|\mathcal{T}(t,s)\| = \|\mathcal{T}(t,s)(I)\|$.

   From 5. we deduce that $\mathcal{T}(t,s)$ is linear. If $R \in \mathcal{E}$, we use (29) and we get

$$\|\mathcal{T}(t,s)(R)\| \leq \|R\| \sup_{x \in H, \|x\|=1} E \|y(t,s;x)\|^2 = \|R\| \|Q(t,s;I)\| .$$

   Thus $\mathcal{T}(t,s)$ is bounded. Using 4. and Lemma 12 we obtain the conclusion.

7. $\mathcal{T}(p,s)\mathcal{T}(t,p)(R) = \mathcal{T}(t,s)(R)$ for all $t \geq p \geq s \geq 0$ and $R \in \mathcal{E}$.

   It follows from Lemma 11 and the definition of $\mathcal{T}(t,s)$.

If we change the definition of the mild solution of (5) by replacing the condition $Q \in C_s(J, L^+(H))$ with $Q \in C_s(J, \mathcal{E})$, then the statements of Lemma 2 stay true.

Proposition 13 *Let $R \in \mathcal{E}$ and $T > 0$. There exists a unique mild solution (resp. classical) Q (resp. Q_n) of (5) (resp. (7)) on $[0, T]$ such that $Q(T) = R$ (resp. $Q_n(T) = R$). They are given by (8) respectively (9). Moreover, $Q(T, s; R) = \mathcal{T}(T, s)(R)$.*

Proof. Let $R = R_1 - R_2 \in \mathcal{E}$, $R_1, R_2 \geq 0$. It is easy to see that $Q(T, s; R_1) - Q(T, s; R_2) \in C_s([0, T], \mathcal{E})$ satisfies the integral equation (8). If $Q' \in C_s([0, T], \mathcal{E})$ is another mild solution of (5) such that $Q'(T) = R$ then we denote $K(s) = Q(T, s; R_1) - Q(T, s; R_2) - Q'(s) \in C_s([0, T], \mathcal{E})$ and we have

$$\begin{aligned} \|K(s)\| &= \sup_{x \in H, \|x\|=1} \left| \sum_{i=1}^{m} \int_s^T \langle K(r)G_i(r)U(r,s)x, G_i(r)U(r,s)x \rangle \, dr \right| \\ &\leq \sum_{i=1}^{m} \int_s^T \|K(r)\| \, \|G_i(r)\| \, \|U(r,s)\|^2 \, dr. \end{aligned}$$

Now, we use the Gronwall's inequality and we obtain the conclusion. The proof for the approximating equation (7) goes on similarly. ■

5 The uniform exponential stability of linear stochastic system with periodic coefficients

We need the following hypothesis:

P3 *There exists $\tau > 0$ such that $A(t) = A(t + \tau), G_i(t) = G_i(t + \tau), i = 1, ..., m$ for all $t \geq 0$.*

It is known (see [12], [3]) that if P1, P3 hold then we have

$$U(t + \tau, s + \tau) = U(t, s) \text{ for all } t \geq s \geq 0. \tag{30}$$

Definition 14 *We say that (2) is uniformly exponentially stable if there exist the constants $M \geq 1$, $\omega > 0$ such that $E \|y(t, s; x)\|^2 \leq Me^{-\omega(t-s)} \|x\|^2$ for all $t \geq s \geq 0$ and $x \in H$.*

Proposition 15 *If P3 holds and $Q(t,s;R)$ is the unique mild solution of (5) such that $Q(t)=R, R\geq 0$, then for all $t\geq s\geq 0$ and $x\in H$ we have*

a) $Q(t+\tau,s+\tau;R)=Q(t,s;R)$.

b) $\mathcal{T}(t+\tau,s+\tau)=\mathcal{T}(t,s)$

c)$\mathcal{T}(n\tau,0)=\mathcal{T}(\tau,0)^n$

d) $E\left\|y(t+\tau,s+\tau;x)\right\|^2=E\left\|y(t,s;x)\right\|^2$

Proof. a) Since P3 holds we deduce from (30) and Lemma 2 that

$$Q(t+\tau,s+\tau;R)x=U^*(t+\tau,s+\tau)RU(t+\tau,s+\tau)x+\int_{s+\tau}^{t+\tau}U^*(r,s+\tau)$$

$$[\sum_{i=1}^{m}G_i^*(r)Q(t+\tau,r;R)G_i(r)]U(r,s+\tau)xdr \text{ and}$$

$$Q(t+\tau,s+\tau;R)x=U^*(t,s)RU(t,s)x+\int_{s}^{t}U^*(v,s)$$

$$[\sum_{i=1}^{m}G_i^*(v)Q(t+\tau,v+\tau;R)G_i(v)]U(v,s)xdv.$$

Now, we can use (8) and Gronwall's inequality to deduce the conclusion.

The statement b) follows from a) and from the definition of the operator $\mathcal{T}(t,s)$. Using b) and the property 7. of the operator $\mathcal{T}(t,s)$ we obtain c). d) follows from Theorem 8 and a). ■

Next remark is a consequence of the Theorem 8 and of the property 6. of the operator $\mathcal{T}(t,s)$.

Remark 16 *The following statements are equivalent:*

a) the equation (2) is uniformly exponentially stable

b) there exist the constants $M\geq 1$, $\omega>0$ such that

$$Q(t,s;I)\leq Me^{-\omega(t-s)}I \text{ for all } t\geq s\geq 0,$$

c) there exist the constants $M\geq 1$, $\omega>0$ such that $\|\mathcal{T}(t,s)\|\leq Me^{-\omega(t-s)}$.

Now we establish the main result of this section.

Theorem 17 *The following assertions are equivalent:*

a) the equation (2) is uniformly exponentially stable;

b) $\lim_{n\to\infty} E\left\|y(n\tau,0;x)\right\|^2 = 0$ *uniformly for* $x \in H$, $\|x\| = 1$*;*

c) $\rho(\mathcal{T}(\tau,0)) < 1$.

Proof. The implication "a) $\Rightarrow$ b)" is a consequence of the Definition 14.

We will prove "b) $\Rightarrow$ a)". Since b) holds we deduce that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbf{N}$ such that $E\left\|y(n\tau,0;x)\right\|^2 < \varepsilon$ for all $n \geq n(\varepsilon)$ and $x \in H$, $\|x\| = 1$. By (29) we get $E\left\|y(n\tau,0;x)\right\|^2 = \langle \mathcal{T}(n\tau,0)(I)x,x\rangle$.

Therefore $\langle \mathcal{T}(n\tau,0)(I)x,x\rangle < \varepsilon$ for all $n \geq n(\varepsilon)$ and $x \in H$, $\|x\| = 1$ or equivalently $\|\mathcal{T}(n\tau,0)(I)\| < \varepsilon$ for all $n \geq n(\varepsilon)$.

Let $\varepsilon = \frac{1}{2}$. We use the property 6. of the operator $\mathcal{T}(t,s)$ and we deduce that there exists $n(\frac{1}{2}) \in \mathbf{N}$ such as $\left\|\mathcal{T}(n(\frac{1}{2})\tau,0)\right\| < \frac{1}{2}$. We denote $\widehat{\tau} = n(\frac{1}{2})\tau$.

If $t \geq s \geq 0$, then there exist unique $\alpha,\gamma \in \mathbf{N}$ and $r_1, r_2 \in [0,\widehat{\tau})$ such as $t = \alpha\widehat{\tau} + r_1$, $s = \gamma\widehat{\tau} + r_2$.

For $\alpha \neq \gamma$ we deduce by Proposition 15 that

$$\mathcal{T}(t,s) = \mathcal{T}(\widehat{\tau},r_2)\mathcal{T}(\widehat{\tau},0)^{\alpha-\gamma-1}\mathcal{T}(r_1,0).$$

Hence

$$\|\mathcal{T}(t,s)\| \leq \|\mathcal{T}(\widehat{\tau},r_2)\| \, \|\mathcal{T}(\widehat{\tau},0)\|^{\alpha-\gamma-1} \|\mathcal{T}(r_1,0)\|.$$

Using Lemma 2 and Gronwall's inequality it is easy to deduce that there exists $M_{\widehat{\tau}} > 0$ such that $\|Q(t,s;I)\| \leq M_{\widehat{\tau}}$ for all $0 \leq s \leq t \leq \widehat{\tau}$.

Then $\|\mathcal{T}(t,s)\| = \|\mathcal{T}(t,s)(I)\| = \|Q(t,s;I)\| \leq M_{\widehat{\tau}}$ for all $0 \leq s \leq t \leq \widehat{\tau}$.

If we denote $\omega = -\frac{1}{\widehat{\tau}}\ln\left(\frac{1}{2}\right) > 0$, we obtain

$$\|\mathcal{T}(t,s)\| \leq M_{\widehat{\tau}}^2 e^{-\omega(t-s)} 2^{\frac{\widehat{\tau}+r_1-r_2}{\widehat{\tau}}} \leq 4M_{\widehat{\tau}}^2 e^{-\omega(t-s)}.$$

If $\alpha = \gamma$ we have $\|\mathcal{T}(t,s)\| \leq M_{\widehat{\tau}} e^{\omega\widehat{\tau}} e^{-\omega(t-s)} = 2M_{\widehat{\tau}} e^{-\omega(t-s)}$.

Now, we take $\beta = 4M_{\widehat{\tau}}^2 > 2M_{\widehat{\tau}}$ (as $M_{\widehat{\tau}} > 1$) and we deduce that

$$\|\mathcal{T}(t,s)\| \leq \beta e^{-\omega(t-s)}$$

for all $t \geq s \geq 0$. The conclusion follows from Remark 16.

"a) $\Rightarrow$ c)". From T.2.38 of [2] we have

$$\rho(\mathcal{T}(\tau,0)) = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{T}(\tau,0)^n\|} = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{T}(n\tau,0)\|}$$

Using Remark 16 and Definition 14 we deduce $\|\mathcal{T}(n\tau,0)\| \leq \beta e^{-\omega n\tau}$. Thus

$$\lim_{n\to\infty} \sqrt[n]{\|\mathcal{T}(n\tau,0)\|} \leq \lim_{n\to\infty} \sqrt[n]{\beta e^{-\omega n\tau}} \leq e^{-\omega\tau} < 1,$$

and the conclusion follows.

"c) $\Rightarrow$ b)" Let $\rho(\mathcal{T}(\tau,0)) = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{T}(\tau,0)^n\|} = s < 1$ and let $\varepsilon > 0$ be such that $s + \varepsilon = \alpha < 1$.

Then, there exists $k_0 \in \mathbf{N}$ such that for all $n \geq k_0$ we have $\|\mathcal{T}(\tau,0)^n\| \leq \alpha^n$ and $\|\mathcal{T}(n\tau,0)\| \leq \alpha^n$ (by Proposition 15). Thus $\lim_{n\to\infty} \|\mathcal{T}(n\tau,0)\| = 0$ or equivalently $\lim_{n\to\infty} \|\mathcal{T}(n\tau,0)(I)\| = 0$. Using (29) we get the conclusion. Since "b) $\Rightarrow$ a)" we get "c) $\Rightarrow$ a)".The proof is complete. ■

Remark 18 *The condition b) of the previous theorem is equivalent, according Theorem 8, with the following statement* $\lim_{n\to\infty} \|Q(n\tau,0;I)\| = 0$.

It is not difficult to see that under the hypothesis H1 the Lyapunov equation 5 with final condition has a unique classical solution. Consequently the operator $\mathcal{T}(t,s)$ is well defined and has the properties 1.-7. stated in the last section. From propositions P. 10 and P. 9 we obtain the following result:

Proposition 19 *Assume that P3 hold. If either H0 and P2 or H1 hold, then the statements of the above theorem stay true.*

We give here two simple examples to illustrate the theory.

Example 20 *Consider an example of equation (2)*

$$dy = e^{-\sin^2(t)} y dt + cos(t) y dw(t), t \geq 0 \tag{31}$$

where $w(t)$ is a real Wiener process. It is clear that H1 and P3 (with $\tau = 2\pi$) hold. The Lyapunov equation associated to (31) is

$$dQ + (2e^{-\sin^2(t)} + \cos^2(t))Q dt = 0 \text{ and}$$

$$Q(2\pi,0;I) = \exp(-\int_0^{2\pi} 2e^{-\sin^2(t)} + \cos^2(t) dt) I$$

$$\leq e^{-\pi} \exp(-\int_0^{2\pi} 2e^{-\sin^2(t)} dt) I < I.$$

Since

$$\rho(\mathcal{T}(2\pi,0)) \leq \|\mathcal{T}(2\pi,0)\| = \|\mathcal{T}(2\pi,0)(I)\| = \|Q(2\pi,0;I)\| < 1$$

we can deduce from the Proposition 19 that the solution of the stochastic equation (31) is uniformly exponentially stable.

Example 21 *Consider a parabolic equation*

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + \cos(t) y dw(t), \tag{32}$$
$$y(0,t) = y(1,t) = 0$$

where $w(t)$ is a real Wiener process, $A = \frac{\partial^2}{\partial x^2}$, $D(A) = H_0^1(0,1) \cap H^2(0,1) \subset H = L_2(0,1)$. The coefficient of the stochastic part is periodic with $\tau = 2\pi$.

It is known that the operator A is self adjoint, $\langle Ay, y\rangle \leq -\pi^2 \|y\|^2$ for all $y \in D(A)$ and A is the infinitesimal generator of an analytic semigroup $S(t)$ [11], which satisfies the following inequality:

$$\|S(t)\| \leq e^{-\pi^2 t}, t \geq 0 \tag{33}$$

The Lyapunov equation associated to (32) is

$$dQ(s) + \left[AQ(s) + Q(s)A + \cos^2(s)Q(s)\right] ds = 0 \text{ and}$$

$$\langle Q(t,s;I)x, x\rangle \leq \|S(t-s)x\|^2 + \int_s^t \|S(r-s)x\|^2 \cos^2(r) \|Q(t,r;I)\| dr.$$

By (33) and Gronwall's inequality we get

$$e^{-2\pi^2 s} \|Q(t,s;I)\| \leq e^{-2\pi^2 t} \exp\left(\int_s^t \cos^2(r) dr\right).$$

Thus $\|Q(2n\pi, 0; I)\| \leq e^{-4n\pi^3} \exp\left(\int_0^{2n\pi} \cos^2(r) dr\right)$ *and*

$$\lim_{n\to\infty} \|Q(2n\pi,0;I)\| \leq \lim_{n\to\infty} e^{-4n\pi^3 + n\pi} = 0.$$

We use Proposition 19 and Remark (18) to deduce that the solution of (32) is uniformly exponentially stable.

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