Electronic Journal of Qualitative Theory of Differential Equations
2004, No. **2**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

EIGENVALUE PROBLEMS FOR A THREE-POINT BOUNDARY-VALUE PROBLEM ON A TIME SCALE

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Abstract. Let $\mathbb{T}$ be a time scale such that $0, T \in \mathbb{T}$. We us a cone theoretic fixed point theorem to obtain intervals for λ for which the second order dynamic equation on a time scale,

$$u^{\Delta\nabla}(t) + \lambda a(t) f(u(t)) = 0, \quad t \in (0, T) \cap \mathbb{T},$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T),$$

where $\eta \in (0, \rho(T)) \cap \mathbb{T}$, and $0 < \alpha < T/\eta$, has a positive solution.

1. Introduction

Stefan Hilger [4] introduced the concept of time scales as a means of unifying differential and difference calculus. In this paper we obtain eigenvalue intervals for which a second order multi-point boundary value problem on a time scale has positive solutions. This work not only carries the works [11], (difference equation), and [12], (differential equation), to the case of time scales, but also generalizes some of the results in [1] and [6] as well as presents new results. For a thorough treatment of the theory of dynamical systems on time scales see the books by Bohner and Peterson [2] and by Kaymakcalan et. al. [5]. We begin by presenting some basic definitions, which can be found in [2], concerning time scales.

A *time scale* $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, we define the *forward jump operator*, σ, and the *backward jump operator*, ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T},$$

$$\rho(r) = \sup\{\tau \in \mathbb{T} \mid \tau < r\} \in \mathbb{T},$$

2000 *Mathematics Subject Classification.* 3410, 34B15, 34G20.

Key words and phrases. Time scale, cone, boundary-value problem, positive solutions.

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be *right scattered*, and if $\sigma(t) = t$, t is said to be *right dense*, (rd). If $\rho(t) < t$, t is said to be *left scattered*, and if $\rho(t) = t$, t is said to be *left dense*, (ld).

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (assume t is not left scattered if $t = \sup \mathbb{T}$), we define the *delta derivative* of $x(t)$, $x^{\Delta}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, U, of t such that

$$\big|x(\sigma(t)) - x(s) - x^{\Delta}(t)(\sigma(t) - s)\big| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (assume t is not right scattered if $t = \inf \mathbb{T}$), we define the *nabla derivative* of $x(t)$, $x^{\nabla}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, U, of t such that

$$\big|x(\rho(t)) - x(s) - x^{\nabla}(t)(\rho(t) - s)\big| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in U$.

Remarks: If $\mathbb{T} = \mathbb{R}$ then $x^{\Delta}(t) = x^{\nabla}(t) = x'(t)$. If $\mathbb{T} = \mathbb{Z}$ then $x^{\Delta}(t) = x(t+1) - x(t)$ is the forward difference operator while $x^{\nabla}(t) = x(t) - x(t-1)$ is the backward difference operator.

We consider the three-point dynamic equation on a time scale

$$u^{\Delta\nabla}(t) + \lambda a(t) f(u(t)) = 0, \quad t \in (0, T) \cap \mathbb{T}, \tag{1.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T), \tag{1.2}$$

where $\eta \in (0, \rho(T)) \cap \mathbb{T}, 0 < \alpha < T/\eta$.

Ma [8] used cone theoretic techniques to show the existence of positive solutions to the second order three-point boundary value problem $u'' + a(t)f(u(t)) = 0, u(0) = 0, \alpha u(\eta) = u(1)$ where $\eta \in (0,1), 0 < \alpha\eta < 1$. Subsequent works by Ma [9], Ma [10], and Cao and Ma [3] generalized these results in the case of differential equations. Recently, Raffoul [12] generalized [8] by considering $u'' + \lambda a(t)f(u(t)) = 0, u(0) = 0, \alpha u(\eta) = u(1)$ where $\eta \in (0,1), 0 < \alpha\eta < 1$. In [12], the author considered combinations (two at a time) of $(L1) \lim_{x\to 0} \frac{f(x)}{x} = \infty, (L2) \lim_{x\to\infty} \frac{f(x)}{x} = \infty, (L3) \lim_{x\to 0} \frac{f(x)}{x} = 0, (L4) \lim_{x\to\infty} \frac{f(x)}{x} = 0, (L5) \lim_{x\to 0} \frac{f(x)}{x} = l, 0 < l <$

$\infty, (L6) \lim_{x\to\infty} \frac{f(x)}{x} = L, 0 < L < \infty$, and found intervals filled with eigenvalues λ and showed the existence of positive solutions. Later on, in [11], the above mentioned second order three-point value boundary in the discrete case was considered but only under the assumption that $\lambda = 1$ and f is either linear or super-linear. Anderson [1], and later Kaufmann [6], showed the existence of multiple positive solutions for the time scale equation (1.1), (1.2) in the case when $\lambda = 1$.

Throughout the paper we assume that

(1) $f : [0, \infty) \to [0, \infty)$ is continuous, and
(2) $a : (0, T) \cap \mathbb{T} \to [0, \infty)$ is ld-continuous and there exists $t_0 \in (\eta, T) \cap \mathbb{T}$ such that $a(t_0) > 0$.

In addition we will assume that one of the following conditions holds.

(H1) There are $x_n \to 0$ such that $f(x_n) > 0$ for $n = 1, 2, \ldots$.
(H2) $f(x) > 0$ for $x > 0$.

In section 2 we present some important lemmas and a fixed point theorem. We also define an operator whose fixed points are solutions to (1.1), (1.2). In section 3, we state several theorems giving eigenvalue intervals for the existence of a positive solution to (1.1), (1.2). In the final section, section 4, we present eigenvalue intervals and conditions under which there exists two positive solutions of (1.1), (1.2).

2. Preliminaries

We begin this section by stating five preliminary lemmas concerning the boundary value problem,

$$u^{\Delta\nabla}(t) + \lambda y(t) = 0, \quad t \in (0, T) \cap \mathbb{T}, \tag{2.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T). \tag{2.2}$$

The proofs of Lemmas 2.1 through 2.5 follow along the lines of the proofs given in [8] for the continuous case, [11] for the discrete case and [1] for time scales in the case $\lambda = 1$, and will be omitted.

Lemma 2.1. *If $\alpha\eta \neq T$ then for $y \in C_{ld}(\mathbb{T},\mathbb{R})$ the boundary-value problem (2.1), (2.2) has the unique solution*

$$\begin{aligned} u(t) &= \lambda\Big[-\int_0^t (t-s)y(s)\,\nabla s - \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)y(s)\,\nabla s \\ &\quad + \frac{t}{T-\alpha\eta}\int_0^T (T-s)y(s)\,\nabla s\Big]. \end{aligned}$$

Lemma 2.2. *If $u(0)=0$ and $u^{\Delta\nabla} \leq 0$, then $\frac{u(s)}{s} \leq \frac{u(t)}{t}$ for all $s,t \in (0,T]\cap\mathbb{T}$ with $t \leq s$.*

Lemma 2.3. *Let $0 < \alpha < T/\eta$. If $y \in C_{ld}(\mathbb{T},\mathbb{R})$ and $y \geq 0$ then the solution u of boundary-value problem (2.1), (2.2) satisfies $u(t) \geq 0$ for all $t \in [0,T]\cap\mathbb{T}$.*

Lemma 2.4. *Let $\alpha\eta > T$. If $y \in C_{ld}(\mathbb{T},\mathbb{R})$ and $y \geq 0$ then the boundary-value problem (2.1), (2.2) has no nonnegative solution.*

We use the Banach space $\mathcal{B} = C_{ld}(\mathbb{T},\mathbb{R})$ with norm $\|u\| = \sup_{t\in[0,T]\cap\mathbb{T}} |u(t)|$. Define the operator $I\colon \mathcal{B} \to \mathcal{B}$ by

$$\begin{aligned} Iu(t) &= \lambda\Big[-\int_0^t (t-s)a(s)f(u(s))\,\nabla s - \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(u(s))\,\nabla s \\ &\quad + \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\,\nabla s\Big]. \end{aligned}$$

The first two terms on the right hand side are non-positive and so

$$Iu(t) \leq \frac{\lambda T}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\nabla s.$$

Also, as in [1], [6], and [12] we can show that

$$Iu(\eta) \geq \frac{\lambda\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)f(u(s))\nabla s.$$

To simplify some expression we define the quantities

$$A \equiv \frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s \tag{2.3}$$

and

$$B \equiv \frac{\eta}{T-\alpha\eta}\int_0^T (T-s)a(s)\,\nabla s \tag{2.4}$$

Lemma 2.5. *Let $0 < \alpha\eta < T$. If $y \in C_{ld}(\mathbb{T}, [0, \infty))$, then the unique solution u of (2.1), (2.2) satisfies*

$$\min_{t \in [\eta, T] \cap \mathbb{T}} u(t) \geq \gamma \|u\| \tag{2.5}$$

where

$$\gamma = \min\left\{\frac{\alpha\eta}{T}, \frac{\alpha(T-\eta)}{T-\alpha\eta}, \frac{\eta}{T}\right\}. \tag{2.6}$$

In view of Lemma 2.5, we define the cone $\mathcal{P} \subset \mathcal{B}$, by

$$\mathcal{P} = \{u \in \mathcal{B} \colon u(t) \geq 0, t \in \mathbb{T} \text{ and } \min_{t \in [\eta, T] \cap \mathbb{T}} u(t) \geq \gamma \|u\|\}.$$

From Lemma 2.5 we have $I \colon \mathcal{P} \to \mathcal{P}$. Standard arguments show that the operator I is completely continuous.

We use Theorem 2.6 below, due to Krasnosel'skiĭ [7], to obtain fixed points for the operator $I : \mathcal{P} \to \mathcal{P}$.

Theorem 2.6. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Assume Ω_1, Ω_2 are bounded open balls of $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that*

$$I \colon \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

is a completely continuous operator such that, either

(1) $\|Iu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Iu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(2) $\|Iu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Iu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then I has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We will use the following lemma in the proofs of our main theorems.

Lemma 2.7. *Assume that there exist two positive numbers a and b such that $a \neq b$,*

$$\max_{0 \leq x \leq a} f(x) \leq \frac{a}{\lambda A}, \tag{2.7}$$

and

$$\min_{\gamma b \leq x \leq b} f(x) \geq \frac{b}{\lambda B}. \tag{2.8}$$

Then there exists $\overline{y} \in \Omega$ which is a fixed point of I and satisfies $\min\{a, b\} \leq \|\overline{y}\| \leq \max\{a, b\}$.

Proof. Let $\Omega_\xi = \{w \in \mathcal{B} : \|w\| < \xi\}$. Assume that $a < b$. Then, for any $y \in \mathcal{B}$ which satisfies $\|y\| = a$, in view of (2.7), we have

$$(Iu)(t) \leq \lambda \frac{T \int_0^T (T-s)a(s)\nabla s}{T - \alpha\eta} \cdot \frac{a}{\lambda A} \leq \lambda A \cdot \frac{a}{\lambda A} = a. \tag{2.9}$$

That is, $\|Iy\| \leq \|y\|$ for $y \in \partial\Omega_a$. For any $y \in \mathcal{B}$ which satisfies $\|y\| = b$, we have

$$(Iy)(\eta) \geq \lambda \frac{\eta \int_\eta^T (T-s)a(s)\nabla s}{T - \alpha\eta} \cdot \frac{b}{\lambda B} \geq \lambda B \cdot \frac{b}{\lambda B}. \tag{2.10}$$

That is, we have $\|Iy\| \geq \|y\|$ for $y \in \partial\Omega_b$. In view of Theorem 2.6, there exists $\overline{y} \in \mathcal{B}$ which satisfies $a \leq \|\overline{y}\| \leq b$ such that $I\overline{y} = \overline{y}$. If $a > b$, (2.9) is replaced by $(Ty)(t) \geq b$ in view of (2.8), and (2.10) is replaced by $(Iy)(t) \leq a$ in view of (2.7). The same conclusion then follows. The proof is complete. $\square$

3. Positive Solutions

Before we state and prove our main theorems we define two functions q and p by

$$q(r) = \frac{r}{A \max_{0 \leq x \leq r} f(x)}, \tag{3.1}$$

where A is defined in (2.3), and

$$p(r) = \frac{r}{B \min_{\gamma r \leq x \leq r} f(x)}, \tag{3.2}$$

where B is defined in (2.4). The function q is well defined if (H1) holds. Furthermore, in this case, $q : [0,\infty) \to [0,\infty)$ is continuous. The function p is well defined if (H2) holds and, in this case, $p : [0,\infty) \to [0,\infty)$ is continuous.

Theorem 3.1. *Suppose either*

I. *(H1),* $\lim_{x\to 0} \frac{f(x)}{x} = \infty$*, and* $\lim_{r\to+\infty} \max_{0\leq x\leq r} \frac{f(x)}{r} = 0$ *hold, or*

II. *(H2),* $\lim_{x\to 0} \frac{f(x)}{x} = 0$*, and* $\lim_{r\to+\infty} \min_{\gamma r\leq x\leq r} \frac{f(x)}{r} = \infty$ *hold.*

Then for any $\lambda \in (0, +\infty)$*, there exists at least one positive solution of (1.1), (1.2).*

Proof. Suppose (H1), $\lim_{x\to 0} \frac{f(x)}{x} = \infty$, and $\lim_{r\to+\infty} \max_{0\leq x\leq r} \frac{f(x)}{r} = 0$ hold. Consider $q : [0,+\infty) \to [0,+\infty)$ as defined in (3.1). Then $\lim_{r\to 0} q(r) = 0$ and $\lim_{r\to\infty} q(r) = +\infty$. Let $\lambda \in (0,\infty)$. By the intermediate value theorem, there exists an a such that $q(a) = \lambda$. That is, $a/(A \max_{0 \le x \le a} f(x)) = \lambda$. Consequently, $\max_{0 \le x \le a} f(x) = a/(\lambda A)$. Thus we have $f(x) \le a/(\lambda A)$ for all $x \in [0, a]$.

Since $\lim_{x \to 0} \frac{f(x)}{x} = \infty$, there exists a $b \in (0, a)$ such that $\frac{f(x)}{x} \ge 1/(\lambda \gamma B)$ for all $x \in (0, b)$. In particular, $f(x) \ge b/(\lambda B)$ for all $x \in [\gamma b, b]$. An application of Lemma 2.7 leads to a positive solution of (1.1), (1.2).

Now suppose that (H2), $\lim_{x \to 0} \frac{f(x)}{x} = 0$, and $\lim_{x \to +\infty} \min_{\gamma r \le x \le r} \frac{f(x)}{r} = \infty$ hold. Consider $p : (0, +\infty) \to (0, +\infty)$ as defined in (3.2). Since $\lim_{x \to 0} \frac{f(x)}{x} = 0$ and $\lim_{r \to +\infty} \min_{\gamma r \le x \le r} \frac{f(x)}{r} = \infty$, we have $\lim_{r \to 0} p(r) = +\infty$ and $\lim_{r \to +\infty} p(r) = 0$. Let $\lambda \in (0, +\infty)$. By the intermediate value theorem, there exists a b such that $p(b) = \lambda$. That is, $b/(B \min_{\gamma b \le x \le b} f(x)) = \lambda$. And so, $f(x) \ge b/(\lambda B)$ for all $x \in [\gamma b, b]$. Since $\lim_{x \to 0} \frac{f(x)}{x} = 0$, there exists an $a < b$ such that $\frac{f(x)}{x} \le 1/(\lambda A)$ for all $x \in (0, a]$. Thus, $f(x) \le a/(\lambda A)$ for all $x \in (0, a]$. Again, an application of Lemma 2.7 yields a positive solution of (1.1), (1.2). $\square$

The proofs of the remaining theorems in this section are similar to the proof of Theorem 3.1. We will present only sketches of their proofs.

Theorem 3.2. *Suppose (H1), $\lim_{x \to 0} \frac{f(x)}{x} = \infty$, and $\lim_{x \to +\infty} \max_{0 \le x \le r} \frac{f(x)}{r} = L$ hold. Then for any $\lambda \in (0, \frac{1}{AL})$, there exists at least one positive solution of (1.1), (1.2).*

Proof. Consider $q : [0, +\infty) \to [0, +\infty)$ as defined in (3.1). Since $\lim_{x \to 0} \frac{f(x)}{x} = \infty$ and $\lim_{x \to +\infty} \max_{0 \le x \le r} \frac{f(x)}{r} = L$ we have $\lim_{r \to 0} q(r) = 0$ and $\lim_{r \to \infty} q(r) = 1/(AL)$. Let $\lambda \in (0, \frac{1}{AL})$ and let a be such that $q(a) = \lambda$. Then, $f(x) \le a/(\lambda A)$.

Since $\lim_{x \to 0} \frac{f(x)}{x} = \infty$, there exists a $b \in (0, a)$ such that $\frac{f(x)}{x} \ge 1/(\lambda \gamma B)$. And so for all $x \in (\gamma b, b)$ we have $f(x) \ge b/(\lambda B)$. Apply Lemma 2.7 to get the result. $\square$

Theorem 3.3. *Suppose (H2), $\lim_{r \to 0} \min_{\gamma r \le x \le r} \frac{f(x)}{r} = \ell$, and $\lim_{x \to +\infty} \frac{f(x)}{x} = \infty$ hold. Then for any $\lambda \in (0, \frac{1}{B\ell})$, there exists at least one positive solution of (1.1), (1.2).*

Proof. Consider $p : [0, +\infty) \to [0, +\infty)$ as defined in (3.2). By $\lim_{x \to +\infty} \frac{f(x)}{x} = \infty$ we have $\lim_{r \to +\infty} p(r) = 0$. By $\lim_{r \to 0} \min_{\gamma r \le x \le r} \frac{f(x)}{r} = \ell$ we have $\lim_{r \to 0} p(r) =$

$\frac{1}{B\ell}$. Let $\lambda \in (0, 1/(B\ell))$ and let b be such that $p(b) = \lambda$. Then, $f(x) \geq a/(\lambda B)$ for all $x \in [\gamma b, b]$.

Since $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$ there exists a $a > b$ such that $\frac{f(x)}{x} \leq 1/(\lambda A)$ for all $x > a$. Let $\delta = \max_{0\leq x\leq a} f(x)$. Then $f(x) \leq a_1/(\lambda A)$ for all $x \in [0, a_1]$ where $a_1 > a$ and $a_1 \geq \lambda\delta A$. Apply Lemma 2.7 to get the result. □

Theorem 3.4. *Suppose (H1),* $\lim_{x\to 0} \frac{f(x)}{x} = 0$*, and* $\lim_{x\to+\infty} \max_{0\leq x\leq r} \frac{f(x)}{r} = L$ *hold. Then for any* $\lambda \in (\frac{1}{AL}, +\infty)$*, there exists at least one positive solution of (1.1), (1.2).*

Proof. Again we use q as given in (3.1). Since $\lim_{x\to 0} \frac{f(x)}{x} = 0$, we have $q(r) \to \infty$ as $r \to 0$ and by $\lim_{x\to+\infty} \max_{0\leq x\leq r} \frac{f(x)}{r} = L$ we have $q(r) \to 1/(AL)$ as $r \to \infty$. Pick $\lambda \in (\frac{1}{AL}, +\infty)$. As in preceding proofs there exists a and b such that $0 < b < a$, $f(x) \leq a/(\lambda A)$ for all $x \in [0, a]$, and $f(x) \geq b/(\lambda B)$ for all $x \in [\gamma b, b]$. The result follows by Lemma 2.7. □

Theorem 3.5. *Suppose (H2),* $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$*, and* $\lim_{r\to 0} \min_{\gamma r\leq x\leq r} \frac{f(x)}{r} = \ell$ *hold. Then for any* $\lambda \in (\frac{1}{B\ell}, +\infty)$*, there exists at least one positive solution of (1.1), (1.2).*

Proof. Again we use p as given in (3.2). By $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$, we have $p(r) \to \infty$ as $r \to \infty$ and by $\lim_{r\to 0} \min_{\gamma r\leq x\leq r} \frac{f(x)}{r} = \ell$ we have $p(r) \to 1/(B\ell)$ as $r \to 0$. Pick $\lambda \in (\frac{1}{B\ell}, +\infty)$. As in preceding proofs there exists a and b such that $0 < a < b$, $f(x) \geq b/(\lambda B)$ for all $x \in [\gamma b, b]$, and $f(x) \leq a/(\lambda A)$ for all $x \in [0, a]$. The result follows by Lemma 2.7. □

4. Multiple Positive Solutions

In this section we determine intervals over which the eigenvalue problem (1.1), (1.2) has at least two positive solutions. To the authors' knowledge this is the first such criterion for the existence of at least two positive solutions.

Theorem 4.1. *Suppose (H1),* $\lim_{x\to 0} \frac{f(x)}{x} = \infty$ *and* $\lim_{x\to+\infty} \frac{f(x)}{x} = \infty$ *hold. Then for any* $\lambda \in (0, \lambda^*)$*, the boundary value problem (1.1), (1.2) has at least two* *positive solutions, where*

$$\lambda^* = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0 \le x \le r} f(x)},$$

and A is defined by (2.3).

Proof. Let $q(r) = r/(A \max_{0 \le x \le r} f(x))$. In view of (H1), we have that $q \in C((0,\infty),(0,\infty))$. Furthermore $\lim_{r\to 0} q(r) = \lim_{r\to\infty} q(r) = 0$. Thus, there exists $r_0 > 0$ such that $q(r_0) = \max_{r>0} q(r) = \lambda^*$. For any $\lambda \in (0, \lambda^*)$, by the intermediate value theorem, there exist $a_1 \in (0, r_0)$ and $a_2 \in (r_0, \infty)$ such that $q(a_1) = q(a_2) = \lambda$. Thus, we have $f(x) \le a_1/(\lambda A)$ for $x \in [0, a_1]$ and $f(x) \le a_2/(\lambda A)$ for $x \in [0, a_2]$.

Since $\lim_{x\to 0} \frac{f(x)}{x} = \infty$ and $\lim_{x\to+\infty} \frac{f(x)}{x} = \infty$, we see that there exist $b_1 \in (0, a_1)$ and $b_2 \in (a_2, \infty)$ such that $f(x)/x \ge 1/(\lambda\gamma B)$ for $x \in (0, b_1] \cup [b_2\gamma, \infty)$. That is, $f(x) \ge b_1/(\lambda B)$ for $x \in [b_1\gamma, b_1]$ and $f(x) \ge b_2/(\lambda B)$ for $x \in [b_2\gamma, b_2]$. An application of Lemma 2.7 leads to two distinct solutions of equation (1.1), (1.2). $\square$

Theorem 4.2. *Suppose (H2), $\lim_{x\to 0} \frac{f(x)}{x} = 0$ and $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$ hold. Then for any $\lambda > \lambda^{**}$, the boundary value problem (1.1), (1.2) has at least two positive solutions, where*

$$\lambda^{**} = \frac{1}{B} \inf_{r>0} \frac{r}{\min_{\gamma r \le x \le r} f(x)},$$

and B is defined by (2.4).

Proof. Let $p(r) = r/(B \min_{\gamma r \le x \le r} f(x))$. Clearly, $p \in C((0,\infty),(0,\infty))$. From $\lim_{x\to 0} \frac{f(x)}{x} = 0$ and $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$, we see that $\lim_{r\to 0} p(r) = \lim_{r\to\infty} p(r) = \infty$. Thus, there exists $r_0 > 0$ such that $p(r_0) = \min_{r>0} p(r) = \lambda^{**}$. For any $\lambda > \lambda^{**}$, there exist $b_1 \in (0, r_0)$ and $b_2 \in (r_0, \infty)$ such that $p(b_1) = p(b_2) = \lambda$. Thus, we have $f(x) \ge b_1/(\lambda B)$ for $x \in [\gamma b_1, b_1]$ and $f(x) \ge b_2/(\lambda B)$ for $x \in [\gamma b_2, b_2]$.

Since $\lim_{x\to 0} \frac{f(x)}{x} = 0$, we see that $f(0) = 0$ and that there exists $a_1 \in (0, b_1)$ such that $f(x)/x \le 1/(\lambda A)$ for $x \in (0, a_1]$. Thus, we have $f(x) \le a_1/(\lambda A)$. In view of $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$, we see that there exists $a \in (b_2, \infty)$ such that $f(x)/x \le 1/(\lambda A)$ for $x \in [a, \infty)$. Let $\delta = \max_{0 \le x \le a} f(x)$. Then we have $f(x) \le a_2/(\lambda A)$ for $x \in [0, a_2]$, where $a_2 > a$ and $a_2 \ge \lambda\delta A$. An application of Lemma 2.7 leads to two distinct solutions of (1.1), (1.2). $\square$

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