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Symmetric solutions to minimization of a p-energy functional with ellipsoid value

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Abstract The author proves the $W^{1,p}$ convergence of the symmetric minimizers $u_\varepsilon = (u_{\varepsilon1}, u_{\varepsilon2}, u_{\varepsilon3})$ of a p-energy functional as $\varepsilon \to 0$, and the zeros of $u_{\varepsilon1}^2 + u_{\varepsilon2}^2$ are located roughly. In addition, the estimates of the convergent rate of $u_{\varepsilon3}^2$ (to 0) are presented. At last, based on researching the Euler-Lagrange equation of symmetric solutions and establishing its $C^{1,\alpha}$ estimate, the author obtains the $C^{1,\alpha}$ convergence of some symmetric minimizer.
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1 Introduction

Denote $B = \{x \in R^2; x_1^2+x_2^2 < 1\}$. For $b > 0$, let $E(b) = \{x \in R^3; x_1^2+x_2^2+\frac{x_3^2}{b^2} = 1\}$ be a surface of an ellipsoid. Assume $g(x) = (e^{id\theta}, 0)$ where $x = (\cos\theta, \sin\theta)$ on ∂B, $d \in N$. We concern with the minimizer of the energy functional

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p dx + \frac{1}{2\varepsilon^p}\int_B u_3^2 dx \quad (p > 2)$$

in the function class

$$W = \{u(x) = (\sin f(r)e^{id\theta}, b\cos f(r)) \in W^{1,p}(B, E(b)); u|_{\partial B} = g\},$$

which is named the symmetric minimizer of $E_\varepsilon(u, B)$.

When $p = 2$, the functional $E_\varepsilon(u, B)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planar ferromagnets and antiferromagnets (see [5][8]). The asymptotic behavior of minimizers of $E_\varepsilon(u, B)$ has been considered in [3]. In particular, they discussed the asymptotic behavior of the symmetric minimizer with $E(1)$-value of $E_\varepsilon(u, B)$ in §5. When the term $\frac{u_3^2}{\varepsilon^2}$ is replaced by $\frac{(1-|u|^2)^2}{2\varepsilon^2}$, the functional is the Ginzburg-Landau functional, which was well studied in [1], [4] and [7]. The works in [1] and [3] enunciated that the study of minimizers of the functional $E_\varepsilon(u, B)$ is connected tightly with the study of harmonic map with $E(1)$-value. Due to this we may also research the asymptotic behavior of minimizers of $E_\varepsilon(u,B)$ by referring to the p-harmonic map with ellipsoid value (which was discussed in [2]).

In this paper, we always assume $p>2$. As in [1] and [3], we are interested in the behavior of minimizers of $E_\varepsilon(u,B)$ as $\varepsilon\to 0$. We will prove the $W^{1,p}_{loc}$ convergence of the symmetric minimizers. In addition, some estimates of the convergent rate of the symmetric minimizer will be presented and we will discuss the location of the points where $u_3^2=b^2$.

In polar coordinates, for $u(x)=(\sin f(r)e^{id\theta}, b\cos f(r))$, we have

$$|\nabla u|^2=(1+(b^2-1)\sin^2 f)f_r^2+d^2r^{-2}\sin^2 f,$$

$$\int_B |\nabla u|^p dx = 2\pi\int_0^1 r((1+(b^2-1)\sin^2 f)f_r^2+d^2r^{-2}\sin^2 f)^{p/2}dr.$$

If we denote

$$V=\{f\in W^{1,p}_{loc}(0,1]; r^{1/p}f_r, r^{(1-p)/p}\sin f\in L^p(0,1), f(r)\geq 0, f(1)=\frac{\pi}{2}\},$$

then $V=\{f(r); u(x)=(\sin f(r)e^{id\theta}, b\cos f(r))\in W\}$. It is not difficult to see $V\subset\{f\in C[0,1]; f(0)=0\}$. Substituting $u(x)=(\sin f(r)e^{id\theta}, b\cos f(r))\in W$ into $E_\varepsilon(u,B)$ we obtain

$$E_\varepsilon(u,B)=2\pi E_\varepsilon(f,(0,1)),$$

where

$$E_\varepsilon(f,(0,1))=\int_0^1[\frac{1}{p}(f_r^2(1+(b^2-1)\sin^2 f)+d^2r^{-2}\sin^2 f)^{p/2}+\frac{1}{2\varepsilon^p}b^2\cos^2 f]rdr.$$

This shows that $u=(\sin f(r)e^{id\theta}, b\cos f(r))\in W$ is the minimizer of $E_\varepsilon(u,B)$ if and only if $f(r)\in V$ is the minimizer of $E_\varepsilon(f,(0,1))$. Applying the direct method in the calculus of variations we can see that the functional $E_\varepsilon(u,B)$ achieves its minimum on W by a function $u_\varepsilon(x)=(\sin f_\varepsilon(r)e^{id\theta}, b\cos f_\varepsilon(r))$, hence $f_\varepsilon(r)$ is the minimizer of $E_\varepsilon(f,(0,1))$ in V. Observing the expression of the functional $E_\varepsilon(f,(0,1))$, we may assume that, without loss of generality, the function f satisfies $0\leq f\leq\frac{\pi}{2}$.

We will prove the following

Theorem 1.1 *Let u_ε be a symmetric minimizer of $E_\varepsilon(u,B)$ on W. Then for any small positive constant $\gamma\leq b$, there exists a constant $h=h(\gamma)$ which is independent of $\varepsilon\in(0,1)$ such that $Z_\varepsilon=\{x\in B; |u_{\varepsilon 3}|>\gamma\}\subset B(0,h\varepsilon)$.*

This theorem shows that all the points where $u_{\varepsilon 3}^2 = b^2$ are contained in $B(0,h\varepsilon)$. Hence as $\varepsilon \to 0$, these points converge to 0.

Theorem 1.2 *Let $u_\varepsilon(x) = (\sin f_\varepsilon(r)e^{id\theta}, b\cos f_\varepsilon(r))$ be a symmetric minimizer of $E_\varepsilon(u,B)$ on W. Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon = (e^{id\theta}, 0), \quad in \quad W^{1,p}(K, R^3) \tag{1.1}$$

for any compact subset $K \subset \overline{B} \setminus \{0\}$.

Theorem 1.3 *(convergent rate) Let $u_\varepsilon(x) = (\sin f_\varepsilon(r)e^{id\theta}, b\cos f_\varepsilon(r))$ be a symmetric minimizer of $E_\varepsilon(u,B)$ on W. Then for any $\eta \in (0,1)$ and $K = \overline{B} \setminus B(0,\eta)$, there exist $C, \varepsilon_0 > 0$ such that as $\varepsilon \in (0,\varepsilon_0)$,*

$$\int_\eta^1 r[(f_\varepsilon')^p + \frac{1}{\varepsilon^p}\cos^2 f_\varepsilon]dr \le C\varepsilon^p. \tag{1.2}$$

$$\sup_{x\in K} |u_{\varepsilon 3}(x)| \le C\varepsilon^{\frac{p-2}{2}}. \tag{1.3}$$

(1.2) gives the estimate of the convergent rate of f_ε to $\pi/2$ in $W^{1,p}(\eta,1]$ sense, and that of convergence of $|u_{\varepsilon 3}(x)|$ to 0 in $C(K)$ sense is showed by (1.3).

However, there may be several symmetric minimizers of the functional in W. We will prove that one of the symmetric minimizer $\tilde{u}_\varepsilon$ can be obtained as the limit of a subsequence $u_\varepsilon^{\tau_k}$ of the symmetric minimizer u_ε^τ of the regularized functionals

$$E_\varepsilon^\tau(u,B) = \frac{1}{p}\int_B (|\nabla u|^2 + \tau)^{p/2}dx + \frac{1}{2\varepsilon^p}\int_B u_3^2 dx, \quad (\tau \in (0,1))$$

on W as $\tau_k \to 0$. In fact, there exist a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ and $\tilde{u}_\varepsilon \in W$ such that

$$\lim_{\tau_k\to 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon, \quad in \quad W^{1,p}(B, E(b)). \tag{1.4}$$

Here $\tilde{u}_\varepsilon$ is a symmetric minimizer of $E_\varepsilon(u,B)$ in W. The symmetric minimizer $\tilde{u}_\varepsilon$ is called the regularized minimizer. Recall that the paper [3] studied the asymptotic behavior of minimizers $u_\varepsilon \in H_g^1(B, E(1))$ of the energy functional $E_\varepsilon(u,B)$ as $\varepsilon \to 0$. It turns out that

$$\lim_{\varepsilon\to 0} u_\varepsilon = (u_*, 0), \quad in \quad C_{loc}^{1,\alpha}(\overline{B} \setminus A) \tag{1.5}$$

for some $\alpha \in (0,1)$, where u_* is a harmonic map, A is the set of singularities of u_*. Theorem 1.2 has shown the $W_{loc}^{1,p}(\overline{B} \setminus \{0\})$ convergence (weaker than (1.5)) of the symmetric minimizer. We will prove that the convergence of (1.5) is still true for the regularized minimizer. The result holds only for the regularized minimizer, since the Euler-Lagrange equation for the symmetric minimizer u_ε is degenerate. To derive the $C^{1,\alpha}$ convergence of the regularized minimizer $\tilde{u}_\varepsilon$, we try to set up the uniform estimate of u_ε^τ by researching the classical Euler-Lagrange equation which u_ε^τ satisfies. By this and applying (1.4), one can see the $C^{1,\alpha}$ convergence of $\tilde{u}_\varepsilon$. So, the following theorem holds only for the regularized minimizer.

Theorem 1.4 *Let $\tilde{u}_\varepsilon$ be a regularized minimizer of $E_\varepsilon(u, B)$. Then for any compact subset $K \subset \overline{B} \setminus \{0\}$, we have*

$$\lim_{\varepsilon \to 0} \tilde{u}_\varepsilon = (e^{id\theta}, 0), \quad in \quad C^{1,\alpha}(K, E(b)), \quad \alpha \in (0, 1/2).$$

At the same time, the estimates of the convergent rate of the regularized minimizer, which is better than (1.3), will be presented as following

Theorem 1.5 *Let $\tilde{u}_\varepsilon(x)$ be the regularized minimizer of $E_\varepsilon(u, B)$. Then for any compact subset K of $(0, 1]$ there exist positive constants ε_0 and C (independent of ε), such that as $\varepsilon \in (0, \varepsilon_0)$,*

$$\sup_K |\tilde{u}_{\varepsilon 3}| \le C\varepsilon^{\lambda p}, \tag{1.6}$$

where $\lambda = \frac{1}{2}$. Furthermore, if K is any compact subset of $(0, 1)$, then (1.6) holds with $\lambda = 1$.

The proof of Theorem 1.1 will be given in §2. In §3, we will set up the uniform estimate of $E_\varepsilon(u_\varepsilon, K)$ which implies the conclusion of Theorem 1.2. By virtue of the uniform estimate we can also derive the proof of Theorem 1.3 in §4. For the regularized minimizer, we will give the proofs of Theorems 1.4 and 1.5 in §5 and §6, respectively.

2 Proof of Theorem 1.1

Proposition 2.1 *Let f_ε be a minimizer of $E_\varepsilon(f, (0, 1))$. Then*

$$E_\varepsilon(f_\varepsilon, (0, 1)) \le C\varepsilon^{2-p}$$

with a constant C independent of $\varepsilon \in (0, 1)$.

Proof. Denote

$$\begin{aligned} I(\varepsilon, R) &= Min\{\int_0^R [\tfrac{1}{p}(f_r^2(1+(b^2-1)\sin^2 f) + \tfrac{d^2}{r^2}\sin^2 f)^{\frac{p}{2}} \\ &+ \tfrac{1}{2\varepsilon^p} b^2 \cos^2 f] r dr; f \in V_R\}, \end{aligned}$$

where $V_R = \{f(r) \in W^{1,p}_{loc}(0, R]; f(R) = \frac{\pi}{2}, \sin f(r) r^{\frac{1}{p}-1}, f'(r) r^{\frac{1}{p}} \in L^p(0, R)\}$. Then

$$\begin{aligned} & I(\varepsilon, 1) = E_\varepsilon(f_\varepsilon, (0,1)) = \tfrac{1}{p}\int_0^1 r((f_\varepsilon)_r^2(1+(b^2-1)\sin^2 f) \\ & + d^2 r^{-2}(\sin f_\varepsilon)^2)^{p/2} dr + \tfrac{1}{2\varepsilon^p}\int_0^1 r b^2 \cos^2 f_\varepsilon dr \\ = & \tfrac{1}{p}\int_0^{1/\varepsilon} \varepsilon^{2-p} s((f_\varepsilon)_s^2(1+(b^2-1)\sin^2 f) + d^2 s^{-2}\sin^2 f_\varepsilon)^{p/2} ds \\ & + \tfrac{1}{2\varepsilon^p}\int_0^{\varepsilon^{-1}} \varepsilon^2 s b^2 \cos^2 f_\varepsilon ds = \varepsilon^{2-p} I(1, \varepsilon^{-1}). \end{aligned} \tag{2.1}$$

Let f_1 be the minimizer for $I(1,1)$ and define

$$f_2 = f_1, \quad as \quad 0 < s < 1; \quad f_2 = \frac{\pi}{2}, \quad as \quad 1 \le s \le \varepsilon^{-1}.$$

We have

$$\begin{aligned} & I(1, \varepsilon^{-1}) \\ \le & \tfrac{1}{p}\int_0^{\varepsilon^{-1}} s[(f_2')^2(1+(b^2-1)\sin^2 f) + d^2 s^{-2}\sin^2 f_2]^{p/2} ds \\ & + \tfrac{1}{2}\int_0^{\varepsilon^{-1}} s b^2 \cos^2 f_2 ds \\ \le & \tfrac{1}{p}\int_1^{\varepsilon^{-1}} s^{1-p} d^p ds + \tfrac{1}{p}\int_0^1 s((f_1')^2(1+(b^2-1)\sin^2 f) + d^2 s^{-2}\sin^2 f_1)^{p/2} ds \\ & + \tfrac{1}{2}\int_0^1 s b^2 \cos^2 f_1 ds \\ = & \tfrac{d^p}{p(p-2)}(1-\varepsilon^{p-2}) + I(1,1) \le \tfrac{d^p}{p(p-2)} + I(1,1) = C. \end{aligned}$$

Substituting into (2.1) follows the conclusion of Proposition 2.1.

By the embedding theorem we derive, from $|u_\varepsilon| = max\{1, b\}$ and proposition 2.1, the following

Proposition 2.2 *Let u_ε be a symmetric minimizer of $E_\varepsilon(u, B)$. Then there exists a constant C independent of $\varepsilon \in (0,1)$ such that*

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \le C\varepsilon^{(2-p)/p}|x - x_0|^{1-2/p}, \qquad \forall x, x_0 \in B.$$

As a corollary of Proposition 2.1 we have

Proposition 2.3 *Let u_ε be a symmetric minimizer of $E_\varepsilon(u,B)$. Then*

$$\frac{1}{\varepsilon^2}\int_B u_{\varepsilon 3}^2 dx \le C$$

with some constant $C>0$ independent of $\varepsilon \in (0,1)$.

Proposition 2.4 *Let u_ε be a symmetric minimizer of $E_\varepsilon(u,B)$. Then for any $\gamma \in (0,\gamma_0)$ with $\gamma_0 < b$ sufficiently small, there exist positive constants λ,μ independent of $\varepsilon \in (0,1)$ such that if*

$$\frac{1}{\varepsilon^2}\int_{B\cap B^{2l\varepsilon}} u_{\varepsilon 3}^2 dx \le \mu \tag{2.2}$$

where $B^{2l\varepsilon}$ is some disc of radius $2l\varepsilon$ with $l \ge \lambda$, then

$$|u_{\varepsilon 3}(x)| \le \gamma, \qquad \forall x \in B\cap B^{l\varepsilon}. \tag{2.3}$$

Proof. First we observe that there exists a constant $\beta > 0$ such that for any $x \in B$ and $0 < \rho \le 1$, $mes(B\cap B(x,\rho)) \ge \beta\rho^2$. To prove the proposition, we choose $\lambda = (\frac{\gamma}{2C})^{\frac{p}{p-2}}$, $\mu = \frac{\beta}{4}(\frac{1}{2C})^{\frac{2p}{p-2}}\gamma^{2+\frac{2p}{p-2}}$ where C is the constant in Proposition 2.2.

Suppose that there is a point $x_0 \in B\cap B^{l\varepsilon}$ such that (2.3) is not true, i.e.

$$|u_{\varepsilon 3}(x_0)| > \gamma. \tag{2.4}$$

Then applying Proposition 2.2 we have

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(x_0)| &\le C\varepsilon^{(2-p)/p}|x-x_0|^{1-2/p} \le C\varepsilon^{(2-p)/p}(\lambda\varepsilon)^{1-2/p} \\ &= C\lambda^{1-2/p} = \tfrac{\gamma}{2}, \qquad \forall x \in B(x_0,\lambda\varepsilon) \end{aligned}$$

which implies $|u_{\varepsilon 3}(x) - u_{\varepsilon 3}(x_0)| \le \frac{\gamma}{2}$. Noticing (2.4), we obtain $|u_{\varepsilon 3}(x)|^2 \ge [|u_{\varepsilon 3}(x_0)| - \frac{\gamma}{2}]^2 > \frac{\gamma^2}{4}$, $\forall x \in B(x_0,\lambda\varepsilon)$. Hence

$$\int_{B(x_0,\lambda\varepsilon)\cap B} u_{\varepsilon 3}^2 dx > \frac{\gamma^2}{4} mes(B\cap B(x_0,\lambda\varepsilon)) \ge \beta\frac{\gamma^2}{4}(\lambda\varepsilon)^2 = \mu\varepsilon^2. \tag{2.5}$$

Since $x_0 \in B^{l\varepsilon}\cap B$, and $(B(x_0,\lambda\varepsilon)\cap B) \subset (B^{2l\varepsilon}\cap B)$, (2.5) implies

$$\int_{B^{2l\varepsilon}\cap B} u_{\varepsilon 3}^2 dx > \mu\varepsilon^2,$$

which contradicts (2.2) and thus the proposition is proved.

To find the points where $u_{\varepsilon 3}^2=b^2$ based on Proposition 2.4, we may take (2.2) as the ruler to distinguish the discs of radius $\lambda\varepsilon$ which contain these points.

Let u_ε be a symmetric minimizer of $E_\varepsilon(u,B)$. Given $\gamma\in(0,1)$. Let λ,μ be constants in Proposition 2.4 corresponding to γ. If

$$\frac{1}{\varepsilon^2}\int_{B(x^\varepsilon,2\lambda\varepsilon)\cap B} u_{\varepsilon 3}^2 dx\le\mu,$$

then $B(x^\varepsilon,\lambda\varepsilon)$ is called $\gamma-$ good disc, or simply good disc. Otherwise $B(x^\varepsilon,\lambda\varepsilon)$ is called $\gamma-$ bad disc or simply bad disc.

Now suppose that $\{B(x_i^\varepsilon,\lambda\varepsilon), i\in I\}$ is a family of discs satisfying

$$(i): x_i^\varepsilon\in B, i\in I;\qquad (ii): B\subset\cup_{i\in I}B(x_i^\varepsilon,\lambda\varepsilon);$$

$$(iii): B(x_i^\varepsilon,\lambda\varepsilon/4)\cap B(x_j^\varepsilon,\lambda\varepsilon/4)=\emptyset, i\ne j. \tag{2.6}$$

Denote $J_\varepsilon=\{i\in I; B(x_i^\varepsilon,\lambda\varepsilon)\ \ is\ \ a\ \ bad\ \ disc\}$. Then, one has

Proposition 2.5 *There exists a positive integer N (independent of ε) such that the number of bad discs $Card\ J_\varepsilon\le N$.*

Proof. Since (2.6) implies that every point in B can be covered by finite, say m (independent of ε) discs, from Proposition 2.3 and the definition of bad discs,we have

$$\mu\varepsilon^2 Card J_\varepsilon\le\textstyle\sum_{i\in J_\varepsilon}\int_{B(x_i^\varepsilon,2\lambda\varepsilon)\cap B}u_{\varepsilon 3}^2 dx$$

$$\le m\textstyle\int_{\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,2\lambda\varepsilon)\cap B}u_{\varepsilon 3}^2 dx\le m\int_B u_{\varepsilon 3}^2 dx\le mC\varepsilon^2$$

and hence $Card\ \ J_\varepsilon\le\frac{mC}{\mu}\le N$.

Applying TheoremIV.1 in [1], we may modify the family of bad discs such that the new one, denoted by $\{B(x_i^\varepsilon,h\varepsilon); i\in J\}$, satisfies

$$\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_i^\varepsilon,h\varepsilon),\quad \lambda\le h;\quad Card\ J\le Card\ J_\varepsilon,$$

$$|x_i^\varepsilon-x_j^\varepsilon|>8h\varepsilon, i,j\in J, i\ne j.$$

The last condition implies that every two discs in the new family are not intersected. From Proposition 2.4 it is deduced that all the points where $|u_{\varepsilon 3}|=b$ are contained in these finite, disintersected bad discs.

Proof of Theorem 1.1. Suppose there exists a point $x_0 \in Z_\varepsilon$ such that $x_0 \overline{\in} B(0, h\varepsilon)$. Then all points on the circle $S_0 = \{x \in B;\ |x| = |x_0|\}$ satisfy

$$u_{\varepsilon 3}^2(x) = b^2 \cos^2 f_\varepsilon(|x|) = b^2 \cos^2 f_\varepsilon(|x_0|) = u_{\varepsilon 3}^2(x_0) > \gamma^2.$$

By virtue of Proposition 2.4 we can see that all points on S_0 are contained in bad discs. However, since $|x_0| \geq h\varepsilon, S_0$ can not be covered by a single bad disc. As a result, S_0 has to be covered by at least two bad disintersected discs. This is impossible.

3 Proof of Theorem 1.2

Let $u_\varepsilon(x) = (\sin f_\varepsilon(r) e^{id\theta}, b\cos f_\varepsilon(r))$ be a symmetric minimizer of $E_\varepsilon(u, B)$, namely f_ε be a minimizer of $E_\varepsilon(f, (0,1))$ in V. From Proposition 2.1, we have

$$E_\varepsilon(f_\varepsilon, (0,1)) \leq C\varepsilon^{2-p} \tag{3.1}$$

for some constant C independent of $\varepsilon \in (0,1)$. In this section we further prove that for any $\eta \in (0,1)$, there exists a constant $C(\eta)$ such that

$$E_\varepsilon(f_\varepsilon; \eta) := E_\varepsilon(f_\varepsilon, (\eta, 1)) \leq C(\eta) \tag{3.2}$$

for $\varepsilon \in (0, \varepsilon_0)$ with small $\varepsilon_0 > 0$. Based on the estimate (3.2) and Theorem 1.1, we may obtain the $W_{loc}^{1,p}$ convergence for minimizers.

To establish (3.2) we first prove

Proposition 3.1 *Given $\eta \in (0,1)$. There exist constants $\eta_j \in [\frac{(j-1)\eta}{N+1}, \frac{j\eta}{N+1}]$, $(N = [p])$ and C_j, such that*

$$E_\varepsilon(f_\varepsilon, \eta_j) \leq C_j \varepsilon^{j-p} \tag{3.3}$$

for $j = 2, ..., N$, where $\varepsilon \in (0, \varepsilon_0)$.

Proof. For $j = 2$, the inequality (3.3) is just the one in Proposition 2.1.

Suppose that (3.3) holds for all $j \leq n$. Then we have, in particular

$$E_\varepsilon(f_\varepsilon; \eta_n) \leq C_n \varepsilon^{n-p}. \tag{3.4}$$

If $n = N$ then we are done. Suppose $n < N$. We want to prove (3.3) for $j = n+1$.

Obviously (3.4) implies

$$\frac{1}{4\varepsilon^p} \int_{\frac{n\eta}{N+1}}^{\frac{(n+1)\eta}{N+1}} b^2 \cos^2 f_\varepsilon r dr \leq C_n \varepsilon^{n-p}$$

from which we see by integral mean value theorem that there exists $\eta_{n+1} \in [\frac{n\eta}{N+1}, \frac{(n+1)\eta}{N+1}]$ such that

$$[\frac{1}{\varepsilon^p} b^2 \cos^2 f_\varepsilon]_{r=\eta_{n+1}} \leq C_n \varepsilon^{n-p}. \tag{3.5}$$

Consider the functional

$$E(\rho, \eta_{n+1}) = \frac{1}{p} \int_{\eta_{n+1}}^1 (\rho_r^2 + 1)^{p/2} dr + \frac{1}{\varepsilon^p} \int_{\eta_{n+1}}^1 b^2 \cos^2 \rho dr.$$

It is easy to prove that the minimizer ρ_1 of $E(\rho, \eta_{n+1})$ in $W^{1,p}_{f_\varepsilon}((\eta_{n+1}, 1), R^+)$ exists and satisfies

$$-\varepsilon^p (v^{(p-2)/2} \rho_r)_r = \sin 2\rho, \quad \text{ in } \ (\eta_{n+1}, 1) \tag{3.6}$$

$$\rho|_{r=\eta_{n+1}} = f_\varepsilon, \quad \rho|_{r=1} = f_\varepsilon(1) = \frac{\pi}{2} \tag{3.7}$$

where $v = \rho_r^2 + 1$. It follows from the maximum principle that $\rho_1 \leq \pi/2$ and

$$\sin^2 \rho(r) \geq \sin^2 \rho(\eta_{n+1}) = \sin^2 f_\varepsilon(\eta_{n+1}) = 1 - \cos^2 f_\varepsilon(\eta_{n+1}) \geq 1 - \gamma^2, \tag{3.8}$$

the last inequality of which is implied by Theorem 1.1. Noting $\min\{1, b^2\} \leq 1 + (b^2 - 1)\sin^2 f \leq \max\{1, b^2\}$, applying (3.4) we see easily that

$$E(\rho_1; \eta_{n+1}) \leq E(f_\varepsilon; \eta_{n+1}) \leq C(b) E_\varepsilon(f_\varepsilon; \eta_{n+1}) \leq C_n \varepsilon^{n-p} \tag{3.9}$$

for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small.

Now, choosing a smooth function $\zeta(r)$ such that $\zeta = 1$ on $(0, \eta), \zeta = 0$ near $r = 1$, multiplying (3.6) by $\zeta \rho_r (\rho = \rho_1)$ and integrating over $(\eta_{n+1}, 1)$ we obtain

$$v^{(p-2)/2} \rho_r^2|_{r=\eta_{n+1}} + \int_{\eta_{n+1}}^1 v^{(p-2)/2} \rho_r (\zeta_r \rho_r + \zeta \rho_{rr}) dr = \frac{1}{\varepsilon^p} \int_{\eta_{n+1}}^1 \sin 2\rho \zeta \rho_r dr. \tag{3.10}$$

Using (3.9) we have

$$\begin{aligned} &|\textstyle\int_{\eta_{n+1}}^1 v^{(p-2)/2} \rho_r (\zeta_r \rho_r + \zeta \rho_{rr}) dr| \\ &\leq \textstyle\int_{\eta_{n+1}}^1 v^{(p-2)/2} |\zeta_r| \rho_r^2 dr + \frac{1}{p} |\int_{\eta_{n+1}}^1 (v^{p/2} \zeta)_r dr - \int_{\eta_{n+1}}^1 v^{p/2} \zeta_r dr| \\ &\leq C \textstyle\int_{\eta_{n+1}}^1 v^{p/2} dr + \frac{1}{p} v^{p/2}|_{r=\eta_{n+1}} + \frac{C}{p} \int_{\eta_{n+1}}^1 v^{p/2} dr \\ &\leq C_n \varepsilon^{n-p} + \frac{1}{p} v^{p/2}|_{r=\eta_{n+1}} \end{aligned} \tag{3.11}$$

and using (3.5)(3.9) we have

$$|\tfrac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1 \zeta\rho_r \sin 2\rho dr| = \tfrac{1}{\varepsilon^p}|\int_{\eta_{n+1}}^1 \zeta_r b^2\cos^2\rho dr - \int_{\eta_{n+1}}^1 (\zeta b^2\cos^2\rho)_r dr|$$
$$\leq \tfrac{1}{\varepsilon^p} b^2\cos^2\rho|_{r=\eta_{n+1}} + \tfrac{C}{\varepsilon^p}\int_{\eta_{n+1}}^1 \cos^2\rho dr \leq C_n\varepsilon^{n-p}. \tag{3.12}$$

Combining (3.10) with (3.11)(3.12) yields

$$v^{(p-2)/2}\rho_r^2|_{r=\eta_{n+1}} \leq C_n\varepsilon^{n-p} + \frac{1}{p}v^{p/2}|_{r=\eta_{n+1}}.$$

Hence

$$v^{p/2}|_{r=\eta_{n+1}} = v^{(p-2)/2}(\rho_r^2+1)|_{r=\eta_{n+1}}$$
$$\leq C_n\varepsilon^{n-p} + \tfrac{1}{p}v^{p/2}|_{r=\eta_{n+1}} + v^{(p-2)/2}|_{r=\eta_{n+1}}$$
$$\leq C_n\varepsilon^{n-p} + (\tfrac{1}{p}+\delta)v^{p/2}|_{r=\eta_{n+1}} + C(\delta)$$

from which it follows by choosing $\delta > 0$ small enough that

$$v^{p/2}|_{r=\eta_{n+1}} \leq C_n\varepsilon^{n-p}. \tag{3.13}$$

Noting (3.8), we can see $\sin\rho > 0$. Multiply both sides of (3.6) by $\cot\rho = \frac{\cos\rho}{\sin\rho}$ and integrate. Then

$$-\varepsilon^p v^{(p-2)/2}\rho_r\cot\rho|_{\eta_{n+1}}^1 = \varepsilon^p\int_{\eta_{n+1}}^1 v^{(p-2)/2}\rho_r^2\frac{1}{\sin^2\rho}dr + 2\int_{\eta_{n+1}}^1 \cos^2\rho dr.$$

Noting $\cot\rho(1) = 0$ (which is implied by (3.7)) and $\frac{1}{\sin^2\rho} \geq 1$, we have

$$E(\rho_1;\eta_{n+1}) = \tfrac{1}{p}\int_{\eta_{n+1}}^1 v^{p/2}dr + \tfrac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1 \cos^2\rho dr$$
$$\leq C[\int_{\eta_{n+1}}^1 v^{(p-2)/2}\rho_r^2 dr + \tfrac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1\cos^2\rho dr] \leq Cv^{(p-2)/2}\rho_r\cot\rho|_{r=\eta_{n+1}}.$$

From this, using(3.13)(3.5) and noticing that $n < p$, we obtain

$$E(\rho_1;\eta_{n+1}) \leq Cv^{(p-2)/2}\rho_r\cot\rho|_{r=\eta_{n+1}}$$
$$\leq Cv^{(p-1)/2}\cot\rho|_{r=\eta_{n+1}} \leq (C_n\varepsilon^{n-p})^{(p-1)/p}(\tfrac{C_n\varepsilon^n}{1-C_n\varepsilon^n})^{1/2} \tag{3.14}$$
$$\leq C_{n+1}\varepsilon^{n+1-p+(n/2-n/p)} \leq C_{n+1}\varepsilon^{n+1-p}.$$

Define $w_\varepsilon = f_\varepsilon,\ for\ r\in(0,\eta_{n+1});\ w_\varepsilon = \rho_1,\ for\ r\in[\eta_{n+1},1]$. Since f_ε is a minimizer of $E_\varepsilon(f)$, we have $E_\varepsilon(f_\varepsilon) \leq E_\varepsilon(w_\varepsilon)$, namely,

$$E_\varepsilon(f_\varepsilon;\eta_{n+1})$$
$$\leq \tfrac{1}{p}\int_{\eta_{n+1}}^1(\rho_r^2(1+(b^2-1)\sin^2\rho)+d^2r^{-2}\sin^2\rho)^{p/2}rdr + \tfrac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1\cos^2\rho rdr$$
$$\leq \tfrac{C}{p}\int_{\eta_{n+1}}^1(\rho_r^2+1)^{p/2}dr + \tfrac{C}{2\varepsilon^p}\int_{\eta_{n+1}}^1\cos^2\rho dr + C = CE(\rho_1;\eta_{n+1}) + C.$$

Thus, using (3.14) yields

$$E_\varepsilon(f_\varepsilon;\eta_{n+1}) \le C_{n+1}\varepsilon^{n-p+1}$$

for $\varepsilon \in (0,\varepsilon_0)$. This is just (3.3) for $j = n+1$.

Proposition 3.2 *Given $\eta \in (0,1)$. There exist constants $\eta_{N+1} \in [\frac{N\eta}{N+1}, \eta]$ and C_{N+1} such that*

$$E_\varepsilon(f_\varepsilon;\eta_{N+1}) \le C_{N+1}\varepsilon^{N-p+1} + \frac{1}{p}\int_{\eta_{N+1}}^1 \frac{d^p}{r^{p-1}}dr \tag{3.15}$$

where $N = [p]$.

Proof. Similar to the derivation of (3.5) we may obtain from Proposition 3.1 for $j = N$ that there exists $\eta_{N+1} \in [\frac{N\eta}{N+1}, \frac{(N+1)\eta}{N+1}]$, such that

$$\frac{1}{\varepsilon^p}\cos^2 f_\varepsilon|_{r=\eta_{N+1}} \le C_N\varepsilon^{N-p}. \tag{3.16}$$

Also similarly, consider the functional

$$E(\rho,\eta_{N+1}) = \frac{1}{p}\int_{\eta_{N+1}}^1 (\rho_r^2+1)^{p/2}dr + \frac{1}{\varepsilon^p}\int_{\eta_{N+1}}^1 \cos^2\rho dr$$

whose minimizer ρ_2 in $W_{f_\varepsilon}^{1,p}((\eta_{N+1},1),R^+)$ exists and satisfies

$$-\varepsilon^p(v^{(p-2)/2}\rho_r)_r = \sin 2\rho, \quad in \quad (\eta_{N+1},1)$$

$$\rho|_{r=\eta_{N+1}} = f_\varepsilon, \quad \rho|_{r=1} = f_\varepsilon(1) = \frac{\pi}{2}$$

where $v = \rho_r^2 + 1$. From (3.4) for $n = N$ it follows immediately that

$$E(\rho_2;\eta_{N+1}) \le E(f_\varepsilon;\eta_{N+1}) \le C_N E_\varepsilon(f_\varepsilon;\eta_{N+1}) \le C_N E_\varepsilon(f_\varepsilon;\eta_N) \le C_N\varepsilon^{N-p}.$$

Similar to the proof of (3.13) and (3.14), we get, from Proposition 3.1 and (3.16),

$$v^{p/2}|_{r=\eta_{N+1}} \le C_N\varepsilon^{N-p}, \quad and \quad E(\rho_2;\eta_{N+1}) \le C_{N+1}\varepsilon^{N+1-p}. \tag{3.17}$$

Now we define

$$w_\varepsilon = f_\varepsilon, \ for \ r \in (0,\eta_{N+1}); \quad w_\varepsilon = \rho_2, \ for \ r \in [\eta_{N+1},1]$$

and then we have $E_\varepsilon(f_\varepsilon) \le E_\varepsilon(w_\varepsilon)$. Notice that

$$\begin{aligned}
&\int_{\eta_{N+1}}^1 (\rho_r^2(1+(b^2-1)\sin^2\rho)+d^2r^{-2}\sin^2\rho)^{p/2}rdr \\
&-\int_{\eta_{N+1}}^1 (d^2r^{-2}\sin^2\rho)^{p/2}rdr \\
=&\tfrac{p}{2}\int_{\eta_{N+1}}^1\int_0^1[(\rho_r^2(1+(b^2-1)\sin^2\rho)+d^2r^{-2}\sin^2\rho)s \\
&+(d^2r^{-2}\sin^2\rho)(1-s)]^{(p-2)/2}]ds\rho_r^2rdr \\
\le& C\int_{\eta_{N+1}}^1(\rho_r^2+d^2r^{-2}\sin^2\rho)^{(p-2)/2}\rho_r^2rdr\int_0^1 s^{(p-2)/2}ds \\
&+C\int_{\eta_{N+1}}^1(d^2r^{-2}\sin^2\rho)^{(p-2)/2}\rho_r^2rdr\int_0^1(1-s)^{(p-2)/2}ds \\
\le& C(\int_{\eta_{N+1}}^1\rho_r^pdr+\int_{\eta_{N+1}}^1\rho_r^2dr)\le C\int_{\eta_{N+1}}^1(\rho_r^2+1)^{p/2}dr.
\end{aligned}$$

Hence

$$\begin{aligned}
&E_\varepsilon(f_\varepsilon;\eta_{N+1})\le\tfrac{1}{p}\int_{\eta_{N+1}}^1(d^2r^{-2}\sin^2\rho)^{p/2}rdr+\tfrac{C}{2\varepsilon^p}\int_{\eta_{N+1}}^1(\cos\rho_2)^2dr \\
&+C\int_{\eta_{N+1}}^1((\rho_2)_r^2+1)^{p/2}dr\le\tfrac{1}{p}\int_{\eta_{N+1}}^1 r(d^2r^{-2})^{p/2}dr+CE(\rho_2;\eta_{N+1}).
\end{aligned}$$

Using (3.17) we have

$$E_\varepsilon(f_\varepsilon;\eta_{N+1})\le\frac{1}{p}\int_{\eta_{N+1}}^1 r(d^2r^{-2})^{p/2}dr+C_{N+1}\varepsilon^{N-p+1}.$$

This is my conclusion.

Proof of Theorem 1.2. Without loss of generality, we may assume $K = B\setminus B(0,\eta_{N+1})$. From Proposition 3.2, We have $E_\varepsilon(u_\varepsilon,K)=2\pi E_\varepsilon(f_\varepsilon,\eta_{N+1})\le C$ where C is independent of ε, namely

$$\int_K|\nabla u_\varepsilon|^pdx\le C, \tag{3.18}$$

$$\int_K|u_{\varepsilon3}|^2dx\le C\varepsilon^p. \tag{3.19}$$

(3.18) and $|u_\varepsilon|\le\max\{1,b\}$ imply the existence of a subsequence u_{ε_k} of u_ε and a function $u_*\in W^{1,p}(K,R^3)$, such that

$$\lim_{\varepsilon_k\to0}u_{\varepsilon_k}=u_*,\quad weakly\ \ in\ \ W^{1,p}(K,R^3)$$

$$\lim_{\varepsilon_k\to0}u_{\varepsilon_k}=u_*,\quad in\ \ C^\alpha(K,R^3),\alpha\in(0,1-\frac{2}{p}). \tag{3.20}$$

(3.19) and (3.20) imply $u_* = (e^{id\theta}, 0)$. Noticing that any subsequence of u_ε has a convergence subsequence and the limit is always $(e^{id\theta}, 0)$, we can assert

$$\lim_{\varepsilon\to 0} u_\varepsilon = (e^{id\theta}, 0), \quad \textit{weakly in } W^{1,p}(K, R^3). \tag{3.21}$$

From this and the weakly lower semicontinuity of $\int_K |\nabla u|^p$, using Proposition 3.2, we have

$$\begin{aligned}\int_K |\nabla e^{id\theta}|^p dx &\leq \underline{\lim}_{\varepsilon_k\to 0} \int_K |\nabla u_\varepsilon|^p dx \leq \overline{\lim}_{\varepsilon_k\to 0} \int_K |\nabla u_\varepsilon|^p dx \\ &\leq C \lim_{\varepsilon\to 0} \varepsilon^{N+1-p} + 2\pi \int_{\eta_{N+1}}^1 (d^2 r^{-2})^{p/2} r dr\end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0} \int_K |\nabla u_\varepsilon|^p dx = \int_K |\nabla e^{id\theta}|^p dx$$

since

$$\int_K |\nabla e^{id\theta}|^p dx = 2\pi \int_{\eta_{N+1}}^1 (d^2 r^{-2})^{p/2} r dr.$$

Combining this with (3.21)(3.20) complete the proof.

4 Proof of Theorem 1.3

Firstly, it follows from Jensen's inequality that

$$\begin{aligned}E_\varepsilon(f_\varepsilon; \eta) &\geq \tfrac{1}{p} \int_\eta^1 (f'_\varepsilon)^p (1 + (b^2-1)\sin^2 f)^{p/2} r dr \\ &+ \tfrac{1}{2\varepsilon^p} \int_\eta^1 b^2 \cos^2 f_\varepsilon r dr + \tfrac{1}{p} \int_\eta^1 \tfrac{d^p}{r^p} \sin^p f_\varepsilon r dr.\end{aligned}$$

Combining this with (3.15) yields

$$\begin{aligned}&\tfrac{1}{p} \int_\eta^1 (f'_\varepsilon)^p (1 + (b^2-1)\sin^2 f)^{p/2} r dr + \tfrac{1}{2\varepsilon^p} \int_\eta^1 b^2 \cos^2 f_\varepsilon r dr \\ &\leq \tfrac{1}{p} \int_\eta^1 \tfrac{d^p}{r^p} (1 - \sin^p f_\varepsilon) r dr + C\varepsilon^{[p]+1-p}.\end{aligned}$$

Noticing that $1 - \sin^p f_\varepsilon \leq C(1 - \sin^2 f_\varepsilon) = C\cos^2 f_\varepsilon$ and (3.19), we obtain

$$\begin{aligned}&\int_\eta^1 (f'_\varepsilon)^p r dr + \tfrac{1}{\varepsilon^p} \int_\eta^1 b^2 \cos^2 f_\varepsilon r dr \\ &\leq C \int_\eta^1 \tfrac{d^p}{r^p} \cos^2 f_\varepsilon r dr + C\varepsilon^{[p]+1-p} \leq C\varepsilon^p + C\varepsilon^{[p]+1-p} \leq C\varepsilon^{[p]+1-p}.\end{aligned} \tag{4.1}$$

Using (4.1) and the integral mean value theorem we can see that there exists $\eta_1 \in [\eta, \eta(1 + 1/2)] \subset [R/2, R]$ such that

$$[\frac{1}{\varepsilon^p} \cos^2 f_\varepsilon]_{r=\eta_1} \leq C_1 \varepsilon^{[p]-p+1}. \tag{4.2}$$

Consider the functional

$$E(\rho,\eta_1)=\frac{1}{p}\int_{\eta_1}^1(\rho_r^2+1)^{p/2}dr+\frac{1}{2\varepsilon^p}\int_{\eta_1}^1\cos^2\rho dr.$$

It is easy to prove that the minimizer ρ_3 of $E(\rho,\eta_1)$ in $W^{1,p}_{f_\varepsilon}((\eta_1,1),R^+)$ exists.

By the same way to proof of (3.14), using (3.2) and (4.2) we have

$$E(\rho_3,\eta_1)\le v^{\frac{p-2}{2}}\rho_{3r}\cot\rho_3|_{r=\eta_1}\le C_1\cot\rho_3(\eta_1)\le C\varepsilon^{\frac{[p]+1-p}{2}+\frac{p}{2}}.$$

Hence, similar to the derivation of (3.15), we obtain

$$E_\varepsilon(f_\varepsilon;\eta_1)\le C\varepsilon^{\frac{[p]-p+1}{2}+\frac{p}{2}}+\frac{1}{p}\int_{\eta_1}^1\frac{d^p}{r^{p-1}}dr.$$

Thus (4.1) may be rewritten as

$$\int_{\eta_1}^1(f_\varepsilon')^p rdr+\frac{1}{\varepsilon^p}\int_{\eta_1}^1 b^2\cos^2 f_\varepsilon rdr\le C\varepsilon^{\frac{[p]+1-p}{2}+\frac{p}{2}}+C\varepsilon^p\le C_2\varepsilon^{\frac{[p]+1-p}{2}+\frac{p}{2}}.$$

Let $\eta_m=R(1-\frac{1}{2^m})$ where $R<1$. Proceeding in the way above (whose idea is improving the exponent of ε from $\frac{[p]+1-p}{2^k}+\frac{(2^k-1)p}{2^k}$ to $\frac{[p]+1-p}{2^{k+1}}+\frac{(2^{k+1}-1)p}{2^{k+1}}$ step by step), we can get that for any $m\in N$,

$$\int_{\eta_m}^1(f_\varepsilon')^p rdr+\frac{1}{\varepsilon^p}\int_{\eta_m}^1 b^2\cos^2 f_\varepsilon rdr\le C\varepsilon^{\frac{[p]+1-p}{2^m}+\frac{(2^m-1)p}{2^m}}+C\varepsilon^p.$$

Letting $m\to\infty$, we derive (1.2).

From (1.2) we can see that

$$\int_K u_{\varepsilon 3}^2dx\le C\varepsilon^{2p}. \tag{4.3}$$

On the other hand, for any $x_0\in K$, we have

$$|u_{\varepsilon 3}(x)-u_{\varepsilon 3}(x_0)|\le C\varepsilon^{(2-p)/p}|x-x_0|^{1-2/p},\qquad \forall x\in B(x_0,\alpha\varepsilon),$$

by applying Proposition 2.2, where $\alpha=(\frac{|u_{\varepsilon 3}(x_0)|}{2C})^{\frac{p}{p-2}}$. Thus

$$|u_{\varepsilon 3}(x)|\ge|u_{\varepsilon 3}(x_0)|-C\alpha^{1-2/p}\ge\frac{1}{2}|u_{\varepsilon 3}(x_0)|.$$

Substituting this into (4.3) we obtain

$$C\varepsilon^{2p}\ge\int_K u_{\varepsilon 3}^2dx\ge\int_{B(x_0,\alpha\varepsilon)}u_{\varepsilon 3}^2dx\ge\frac{\pi}{4}|u_{\varepsilon 3}(x_0)|^2(\alpha\varepsilon)^2,$$

which implies $|u_{\varepsilon 3}(x_0)|\le C\varepsilon^{\frac{p-2}{2}}$. Noting x_0 is an arbitrary point in K, we have

$$\sup_{x\in K}|u_{\varepsilon 3}(x)|\le C\varepsilon^{\frac{p-2}{2}}.$$

Thus (1.3) is derived and the proof of Theorem is complete.

5 Proof of Theorem 1.4

By the method in the calculus of variations we can see the following

Proposition 5.1 *The minimizer $f_\varepsilon \in V$ of the functional $E_\varepsilon(f,(0,1))$ satisfies the following equality*

$$\int_0^1 v^{(p-2)/2}[f_r\phi_r + \frac{b^2-1}{2}f_r^2(\sin 2f)\phi + \frac{d^2}{2r^2}(\sin 2f)\phi]rdr = \frac{1}{2\varepsilon^p}\int_0^1(\sin 2f)\phi rdr$$

for any function $\phi \in C_0^\infty[0,1]$, where $v = f_r^2(1+(b^2-1)sin^2 f) + \frac{d^2\sin^2 f}{r^2}$.

Assume $u_\varepsilon^\tau = (e^{id\theta}\sin f_\varepsilon^\tau, \cos f_\varepsilon^\tau)$ is the minimizer of the regularized functional $E_\varepsilon^\tau(u,B)$. It is easy to prove that the minimizer f_ε^τ is a classical solution of the equation

$$-(rA^{(p-2)/2}f_r)_r + \frac{r(b^2-1)}{2}A^{(p-2)/2}f_r^2\sin 2f + d^2A^{(p-2)/2}\frac{\sin 2f}{2r} = \frac{r\sin 2f}{2\varepsilon^p}, \tag{5.1}$$

where $A = v+\tau$. By the same argument of Theorem 1.1 and Proposition 3.2, we can also see that for any compact subset $K \in (0,1]$, there exist constants $\eta \in (0,1/2)$ and $C>0$ which are independent of ε and τ, such that

$$\eta \le f_\varepsilon^\tau(r) \le \frac{\pi}{2}, \quad r \in K, \tag{5.2}$$

$$E_\varepsilon^\tau(f_\varepsilon^\tau, K) \le C, \tag{5.3}$$

where

$$E_\varepsilon^\tau(f,K) = \int_K[\frac{1}{p}(f_r^2(1+(b^2-1)sin^2 f)+d^2r^{-2}\sin^2 f+\tau)^{p/2}+\frac{1}{2\varepsilon^p}b^2\cos^2 f]rdr.$$

Proposition 5.2 *Denote $f_\varepsilon^\tau = f$. Then for any closed subset $K \subset (0,1)$, there exists $C>0$ which is independent of ε, τ such that*

$$\|f\|_{C^{1,\alpha}(K,R)} \le C, \quad \forall \alpha \le 1/2.$$

Proof. Without loss of the generality, we assume $d=1$. Take $R>0$ sufficiently small such that $K \subset\subset (2R, 1-2R)$. Let $\zeta \in C_0^\infty([0,1],[0,1])$ be a function satisfying $\zeta = 0$ on $[0,R]\cup[1-R,1]$, $\zeta = 1$ on $[2R,1-2R]$ and $|\zeta_r| \le C(R)$ on $(0,1)$. Differentiating (5.1), multiplying with $f_r\zeta^2$ and integrating, we have

$$\begin{aligned}&-\int_0^1(A^{(p-2)/2}f_r)_{rr}(f_r\zeta^2)dr - \int_0^1(r^{-1}A^{(p-2)/2}f_r)_r(f_r\zeta^2)dr\\&+\tfrac{1}{2}\int_0^1[(r^{-2}+(b^2-1)f_r^2)A^{(p-2)/2}\sin 2f]_r(f_r\zeta^2)dr = \tfrac{b^2}{\varepsilon^p}\int_0^1(\cos 2f)f_r^2\zeta^2dr.\end{aligned}$$

Integrating by parts and noting $\cos 2f = 2\cos^2 f - 1$, we obtain

$$\begin{aligned}&\int_0^1 (A^{(p-2)/2} f_r)_r (f_r\zeta^2)_r dr\\ &= \int_0^1 A^{(p-2)/2}(f_r\zeta^2)_r[(\tfrac{1}{2r^2} + \tfrac{b^2-1}{2} f_r^2)\sin 2f - r^{-1} f_r]dr\\ &\quad + \tfrac{2}{\varepsilon^p}\int_0^1 (b^2\cos^2 f) f_r^2 \zeta^2 dr - \tfrac{1}{\varepsilon^p}\int_0^1 f_r^2\zeta^2 dr.\end{aligned}$$

Denote $I = \int_R^{1-R} \zeta^2 (A^{(p-2)/2} f_{rr}^2 + (p-2)A^{(p-4)/2} f_r^2 f_{rr}^2) dr$. Then for any $\delta \in (0,1)$, there holds

$$I \le \delta I + C(\delta)\int_R^{1-R} A^{p/2}\zeta_r^2 dr + \frac{2}{\varepsilon^p}\int_R^{1-R} (b^2\cos^2 f) f_r^2\zeta^2 dr \tag{5.4}$$

by using Young inequality. Noticing that (5.2) implies $\sin f > 0$ as $r \in [R, 1-R]$, from (5.1) we can see that

$$\begin{aligned}\tfrac{2}{\varepsilon^p}(\cos f)^2 \quad &= 4r^{-1}\cot f[-(A^{(p-2)/2} f_r)_r - r^{-1} A^{(p-2)/2} f_r\\ &\quad + A^{(p-2)/2}(\tfrac{1}{2r} + \tfrac{r(b^2-1)}{2} f_r^2)\sin 2f].\end{aligned}$$

Substituting it into the last term of the right hand side of (5.4) and applying Young inequality again we obtain that for any $\delta \in (0,1)$,

$$\frac{2}{\varepsilon^p}\int_R^{1-R} (\cos^2 f) f_r^2 \zeta^2 dr \le \delta I + C(\delta)\int_R^{1-R} A^{(p+2)/2}\zeta^2 dr.$$

Combining this with (5.4) and choosing δ sufficiently small, we have

$$I \le C\int_R^{1-R} A^{p/2}\zeta_r^2 dr + C\int_R^{1-R} A^{(p+2)/2}\zeta^2 dr. \tag{5.5}$$

To estimate the second term of the right hand side of (5.5), we take $\phi = \zeta^{2/q}|f_r|^{(p+2)/q}$ in the interpolation inequality (Ch II, Theorem 2.1 in [6])

$$\|\phi\|_{L^q} \le C\|\phi_r\|_{L^1}^{1-1/q}\|\phi\|_{L^1}^{1/q}, \quad q \in (1 + \frac{2}{p}, 2). \tag{5.6}$$

We derive by applying Young inequality that for any $\delta \in (0,1)$,

$$\begin{aligned}&\int_R^{1-R} |f_r|^{p+2}\zeta^2 dr \le C(\int_R^{1-R} \zeta^{2/q}|f_r|^{(p+2)/q} dr)\\ &\quad \cdot(\int_R^{1-R} \zeta^{2/q-1}|\zeta_r||f_r|^{(p+2)/q} + \zeta^{2/q}|f_r|^{(p+2)/q-1}|f_{rr}|dr)^{q-1}\\ &\le C(\int_R^{1-R} \zeta^{2/q}|f_r|^{(p+2)/q} dr)(\int_R^{1-R} \zeta^{2/q-1}|\zeta_r||f_r|^{(p+2)/q}\\ &\quad + \delta I + C(\delta)\int_R^{1-R} A^{\frac{p+2}{q}-\frac{p}{2}}\zeta^{4/q-2} dr)^{q-1}.\end{aligned} \tag{5.7}$$

Noting $q \in (1+\frac{2}{p}, 2)$, we may using Holder inequality to the right hand side of (5.7). Thus, by virtue of (5.3),

$$\int_R^{1-R} |f_r|^{p+2}\zeta^2 dr \le \delta I + C(\delta).$$

Substituting this into (5.5) and choosing δ sufficiently small, we obtain

$$\int_R^{1-R} A^{(p-2)/2} f_{rr}^2 \zeta^2 dr \le C,$$

which, together with (5.3), implies that $\|A^{p/4}\zeta\|_{H^1(R,1-R)} \le C$. Noticing $\zeta = 1$ on K, we have $\|A^{p/4}\|_{H^1(K)} \le C$. Using embedding theorem we can see that for any $\alpha \le 1/2$, there holds $\|A^{p/4}\|_{C^\alpha(K)} \le C$. From this it is not difficult to prove our proposition.

Applying the idea above, we also have the estimate near the boundary point $r = 1$.

Proposition 5.3 *Denote $f_\varepsilon^\tau = f(r)$. Then for any closed subset $K \subset (0,1]$, there exists $C > 0$ which is independent of ε, τ such that*

$$\|f\|_{C^{1,\alpha}(K,R)} \le C, \quad \forall \alpha \le 1/2.$$

Proof. Without loss of the generality, we assume $d = 1$. Let $g(r) = f(r+1) - 1$. Define

$$\tilde{g}(r) = g(r), \quad as \quad -1 < r \le 0;$$

$$\tilde{g}(r) = -g(-r) \quad as \quad 0 < r \le \tfrac{1}{2}.$$

If still denote $f(r) = \tilde{g}(r-1)+1$ on $(0, \frac{3}{2})$, then $f(r)$ solves (5.1) on $(0, \frac{3}{2})$. Take $R < \frac{1}{4}$ sufficiently small, and set $\zeta \in C^\infty[0,1]$, $\zeta = 1$ as $r \ge 1-R$, $\zeta = 0$ as $r \le 2R$. Differentiating (5.1), multiplying with $f_r\zeta^2$ and integrating over $[R,1]$, we have

$$\begin{aligned} &- \textstyle\int_R^1 (A^{(p-2)/2} f_r)_{rr} (f_r\zeta^2) dr - \int_R^1 (r^{-1}A^{(p-2)/2} f_r)_r (f_r\zeta^2) dr \\ &+ \textstyle\int_R^1 [(\frac{1}{2r^2} + \frac{b^2-1}{2} f_r^2) A^{(p-2)/2} \sin 2f]_r (f_r\zeta^2) dr = \frac{1}{\varepsilon^p} \int_R^1 (b^2 \cos 2f) f_r^2 \zeta^2 dr. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} &\textstyle\int_R^1 (A^{(p-2)/2} f_r)_r (f_r\zeta^2)_r dr \\ &\le |\textstyle\int_R^1 [A^{(p-2)/2} ((\frac{1}{2r^2} + \frac{b^2-1}{2} f_r^2) \sin 2f - r^{-1} f_r]_r (f_r\zeta^2) dr| \\ &\quad + \tfrac{2}{\varepsilon^p} \textstyle\int_R^1 (b^2 \cos^2 f) f_r^2 \zeta^2 dr + |I(1) - I(R)|, \end{aligned}$$

where $I(r) = -[(A^{(p-2)/2}f_r)_r + \frac{1}{r}A^{(p-2)/2}f_r - \frac{1}{2r^2}A^{(p-2)/2}\sin 2f]f_r\zeta^2$. The second term of the right hand side of the inequality above can be handled similar to the proof of Proposition 5.2. Computing the first term of the right hand side yields

$$
\begin{aligned}
&|\int_R^1 [A^{(p-2)/2}((\tfrac{1}{2r^2} + \tfrac{b^2-1}{2}f_r^2)\sin 2f - r^{-1}f_r]_r(f_r\zeta^2)dr| \\
&\le \delta \int_R^1 A^{(p-2)/2}f_{rr}^2\zeta^2 dr + C(\delta)\int_R^1 A^{(p+2)/2}dr
\end{aligned}
$$

with any $\delta \in (0,1)$ by using Young inequality. In view of (5.1), we have $I(r) = \frac{1}{2\varepsilon^p}(\sin 2f)f_r\zeta^2$. Hence, $I(1) = I(R) = 0$ since $\sin 2f(1) = 0$ and $\zeta(R) = 0$. Hence, we may also obtain the result as (5.5)

$$\int_R^1 A^{(p-2)/2}f_{rr}^2\zeta^2 dr \le C\int_R^1 (A^{p/2} + A^{(p+2)/2}\zeta^2)dr.$$

Now, if we take $\phi = \zeta^{2/q}|f_r|^{(p+2)/q}$, then the interpolation inequality (5.6) is invalid since $\phi \neq 0$ near $r = 1$. Thus, we apply a new interpolation inequality [6, (2.19) in Chapter 2]

$$\|\phi\|_{L^q} \le C(\|\phi_r\|_{L^1} + \|\phi\|_{L^1})^{1-1/q}\|\phi\|_{L^1}^{1/q}, \quad q \in (1 + \frac{2}{p}, 2).$$

Then it still follows the same result as (5.7). The rest of the proof is similar to the proof of Proposition 5.2.

Proof of Theorem 1.4. For every compact subset $K \subset B \setminus \{0\}$, applying Propositions 5.2 and 5.3 yields that for $\alpha \in (0, 1/2]$ one has

$$\|u_\varepsilon^\tau\|_{C^{1,\alpha}(K)} \le C = C(K), \tag{5.8}$$

where the constant does not depend on ε, τ.

Applying (5.8) and the embedding theorem we know that for any ε and $\beta_1 < \alpha$, there exist $w_\varepsilon^* \in C^{1,\beta_1}(K, E(b))$ and a subsequence of τ_k of τ such that as $k \to \infty$,

$$u_\varepsilon^{\tau_k} \to w_\varepsilon^*, \quad in \quad C^{1,\beta_1}(K, E(b)). \tag{5.9}$$

Combining this with (1.4) we know that $w_\varepsilon^* = \tilde{u}_\varepsilon$.

Applying (5.8) and the embedding theorem again we can see that for any $\beta_2 < \alpha$, there exist $w^* \in C^{1,\beta_2}(K, E(b))$ and a subsequence of τ_k which can be denoted by τ_m such that as $m \to \infty$,

$$u_{\varepsilon_m}^{\tau_m} \to w^*, \quad in \quad C^{1,\beta_2}(K, E(b)). \tag{5.10}$$

Noticing (1.2), we know that $w^* = (e^{id\theta}, 0)$. Denote $\gamma = min(\beta_1, \beta_2)$. Then as $m \to \infty$, we have

$$\begin{aligned} \|\tilde{u}_{\varepsilon_m} - (e^{id\theta}, 0)\|_{C^{1,\gamma}(K,E(b))} &\leq \|\tilde{u}_{\varepsilon_m} - u_{\varepsilon_m}^{\tau_m}\|_{C^{1,\gamma}(K,E(b))} \\ &\quad + \|u_{\varepsilon_m}^{\tau_m} - (e^{id\theta}, 0)\|_{C^{1,\gamma}(K,E(b))} \leq o(1) \end{aligned} \tag{5.11}$$

by applying (5.9) and (5.10).

Noting the limit $(e^{id\theta}, 0)$ is unique, we can see that the convergence (5.11) holds not only for some subsequence but for all $\tilde{u}_\varepsilon$. Theorem is proved.

6 Proof of Theorem 1.5

Without loss of the generality, we assume $d = 1$. Denote $f = f_\varepsilon^\tau$. Set $\psi = \frac{\cos f}{\varepsilon^p}$. Multiplying (5.1) by $\sin f$ we obtain

$$-(rA^{(p-2)/2}(\sin f)f_r)_r + r\cos f A^{(p-2)/2}(A-\tau) = r(\sin f)^2\psi. \tag{6.1}$$

Substituting $\psi_r = \frac{-\sin f}{\varepsilon^p} f_r$ into (6.1) we have

$$\varepsilon^p(rA^{(p-2)/2}\psi_r)_r + r\cos f A^{(p-2)/2}(A-\tau) = r(\sin f)^2\psi.$$

Suppose $\psi(r)$ achieves its maximum at the point r_0 in K, where K is an arbitrary open interval in any compact subset of $(0,1)$. Then $\psi_r(r_0) = 0$, $\psi_{rr}(r_0) \leq 0$. And $(\sin f)^2 \geq C_1 > 0$ with the constant C_1 independent of ε and τ which is implied by (5.2). Thus, it is deduced that, from Proposition 5.2,

$$\psi(r) \leq \psi(r_0) \leq \frac{1}{C_1} A^{(p-2)/2}(A-\tau)|_{r=r_0} \leq C,$$

which implies $\sup_K |\cos f| \leq C\varepsilon^p$ with the constant $C > 0$ independent of ε and τ, where K is any compact subset of $(0,1)$. Letting $\tau \to 0$ and using (5.9) we may see the conclusion

$$\sup_K |\cos f_\varepsilon| \leq C\varepsilon^p.$$

To derive estimate near the boundary $r = 1$, we use the idea of Pohozaev's equality. Choose $R \in (0, \frac{1}{4})$. Set $\zeta(r) \in C^\infty[0,1]$, $\zeta = 0$ as $r \in [0, 2R]$, $\zeta = 1$ as $r \in [1-R, 1]$. Then $\zeta_r \leq C(R)$. Multiplying (5.1) with $f_r\zeta$ and integrating over $[R, T]$ with T being an arbitrary constant in $(1-R, 1)$, we have

$$\begin{aligned} &-\int_R^T (rA^{(p-2)/2}f_r)_r f_r\zeta dr + \int_R^T A^{(p-2)/2}(\sin 2f)[\tfrac{1}{2r} + \tfrac{r(b^2-1)}{2}f_r^2]f_r\zeta dr \\ &= \tfrac{1}{2\varepsilon^p}\int_R^T rf_r(\sin 2f)\zeta dr. \end{aligned} \tag{6.2}$$

Integrating the right hand side of (6.2) by parts yields

$$\begin{aligned}&\tfrac{1}{2\varepsilon^p}\int_R^T rf_r(\sin 2f)\zeta dr = -\tfrac{1}{2\varepsilon^p}\int_R^T r(\cos^2 f)_r\zeta dr\\ =& -\tfrac{1}{2\varepsilon^p}r(\cos f)^2|_{r=T} + \tfrac{1}{2\varepsilon^p}\int_R^T(\cos^2 f)(r\zeta)_r dr.\end{aligned} \tag{6.3}$$

Similarly, the first term of the left hand side of (6.2) may be written as

$$\begin{aligned}&-\int_R^T(rA^{(p-2)/2}f_r)_r f_r\zeta dr = -rA^{(p-2)/2}f_r^2|_{r=T}\\ &+\int_R^T rA^{(p-2)/2}f_r f_{rr}\zeta dr + \int_R^T rA^{(p-2)/2}f_r^2\zeta_r dr = \Sigma_{i=1}^3.\end{aligned} \tag{6.4}$$

Combining I_2 with the second term of the left hand side of (6.2) we have

$$\begin{aligned}&I_2 + \int_R^T A^{(p-2)/2}\tfrac{\sin 2f}{2r}f_r\zeta dr\\ =& \int_R^T rA^{(p-2)/2}[f_r f_{rr} + (\tfrac{1}{2r^2} + \tfrac{b^2-1}{2}f_r^2)f_r\sin 2f)\zeta dr\\ =& \tfrac{1}{2}\int_R^T rA^{(p-2)/2}\zeta(A-\tau)_r dr + \tfrac{1}{2}\int_R^T r^{-2}A^{(p-2)/2}\zeta\sin^2 f dr\\ =& \tfrac{1}{p}rA^{p/2}|_{r=T} - \tfrac{1}{p}\int_R^T A^{p/2}(r\zeta)_r dr + \tfrac{1}{2}\int_R^T r^{-2}A^{(p-2)/2}\zeta\sin^2 f dr.\end{aligned}$$

Substituting this and (6.3),(6.4) into (6.2) yields

$$\begin{aligned}&\tfrac{1}{2\varepsilon^p}r(\cos f)^2|_{r=T} + \tfrac{1}{p}rA^{p/2}|_{r=T} + \int_R^T rA^{(p-2)/2}f_r^2\zeta_r dr\\ &+\tfrac{1}{2}\int_R^T A^{(p-2)/2}r^{-2}(\sin f)^2\zeta dr\\ =& \tfrac{1}{2\varepsilon^p}\int_R^T(\cos f)^2(r\zeta)_r dr + \tfrac{1}{p}\int_R^T A^{p/2}(r\zeta)_r dr + rA^{(p-2)/2}f_r^2|_{r=T}.\end{aligned}$$

Applying Proposition 5.3 and (5.3) we obtain $\frac{1}{2\varepsilon^p}T\cos^2 f(T) \le C$, with $C > 0$ independent of ε and τ. Letting $\tau \to 0$ and using (5.9) we derive

$$\frac{1}{2\varepsilon^p}\cos^2 f_\varepsilon(T) \le C.$$

By virtue of the arbitrary of the point T, it is not difficult to get our Theorem.

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