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ON THE UNIFORMLY CONTINUITY OF THE SOLUTION MAP FOR TWO DIMENSIONAL WAVE MAPS

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Abstract

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Abstract

Abstract. The aim of this paper is to analyze the properties of the solution map to the Cauchy problem for the wave map equation with a source term, when the target is the hyperboloid $\mathcal{H}^2$ that is embedded in $\mathcal{R}^3$. The initial data are in $\dot{H}^1 \times L^2$. We prove that the solution map is not uniformly continuous.

Abstract

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In this paper we study the properties of the solution map $(u_\circ, u_1, g) \longrightarrow u(t,x)$ to the Cauchy problem

$$u_{tt} - \triangle u - (|u_t|^2 - |\nabla_x u|^2)u = g(t,x), \tag{1}$$

$$u(0,x) = u_\circ(x) \in \dot{H}^1(\mathcal{R}^2), \quad u_t(0,x) = u_1(x) \in L^2(\mathcal{R}^2) \tag{2}$$

in the case when $x \in \mathcal{R}^2$ and the target is the hyperboloid $\mathcal{H}^2 : u_1^2 + u_2^2 - u_3^2 = -1$, $\mathcal{H}^2 \hookrightarrow \mathcal{R}^3$. Here

$$|u_t|^2 = u_{1t}^2 + u_{2t}^2 - u_{3t}^2,$$

$$|\nabla_x u|^2 = |\nabla_{x_1} u|^2 + |\nabla_{x_2} u|^2,$$
$$|\nabla_{x_i} u|^2 = u_{1x_i}^2 + u_{2x_i}^2 - u_{3x_i}^2, \quad i = 1,2.$$

More precisely, we prove that the solution map $(u_\circ, u_1, g) \longrightarrow u(t,x)$ to the Cauchy problem (1), (2) is not uniformly continuous.

In [1] is proved that the solution map isn't uniformly continuous in the case when $g \equiv 0$.

When we say that the solution map $(u_\circ, u_1, g) \longrightarrow u(t,x)$ is uniformly continuous we understand: *for every positive constant ϵ there exist positive constants δ and R such that for any two solutions $u, v : \mathcal{R} \times \mathcal{R}^2 \longrightarrow \mathcal{H}^2$ of (1), (2), with right hands $g = g_1$, $g = g_2$ of (1), so that*

(3)
$$E(0, u-v) \le \delta, \quad ||g_1||_{L^1([0,1]L^2(\mathcal{R}^2))} \le R, \quad ||g_2||_{L^1([0,1]L^2(\mathcal{R}^2))} \le R, \quad ||g_1 - g_2||_{L^1([0,1]L^2(\mathcal{R}^2))} \le R,$$

the following inequality holds

$$E(t,u-v)\leq\epsilon \quad for \quad \forall t\in[0,1], \tag{4}$$

where

$$E(t,u):=||\partial_t u(t,\cdot)||^2_{L^2(\mathcal{R}^2)}+||\nabla_x u(t,\cdot)||^2_{L^2(\mathcal{R}^2)}.$$

Here we prove

Theorem 1. *There exist constant $\epsilon>0$ such that for every pair of positive constants δ and R there exists smooth solutions u, v: $\mathcal{R}\times\mathcal{R}^2\longrightarrow\mathcal{H}^2$ of (1), (2), with right hands $g=g_1$, $g=g_2$ of (1), so that*

$$E(0,u-v)\leq\delta,\quad ||g_1||_{L^1([0,1]L^2(\mathcal{R}^2))}\leq R,\quad ||g_2||_{L^1([0,1]L^2(\mathcal{R}^2))}\leq R,\quad ||g_1-g_2||_{L^1([0,1]L^2(\mathcal{R}^2))}\leq\delta,$$

and

$$E(1,u-v)\geq\epsilon.$$

Proof. We suppose that the solution map $(u_\circ,u_1,g)\longrightarrow u(t,x)$ to the Cauchy problem (1), (2) is uniformly continuous. Then for every $\epsilon>0$ there exist positive constants δ and R such that for any solution u of (1), (2) with right hand g of (1) for which

$$E(0,u)\leq\delta,\quad ||g||_{L^1([0,1]L^2(\mathcal{R}^2))}\leq R \tag{5}$$

and the inequality

$$E(t,u)\leq\epsilon \tag{6}$$

holds for every $t\in[0,1]$ (in this case $v=0$, which is solution of (1) with right hand $g=0$). Let

$$\begin{aligned}&u=(u_1,u_2,u_3),\\ &u_1=\sinh\chi\cos\phi_1,\\ &u_2=\sinh\chi\sin\phi_1,\\ &u_3=\cosh\chi,\quad \chi\geq0,\quad \phi_1\in[0,2\pi],\end{aligned} \tag{$\star$}$$

$\chi=Y^2$, where Y is solution to the Cauchy problem

$$Y_{tt}-\triangle Y=0, \tag{7}$$

$$Y(0,x)=0,\quad Y_t(0,x)=q(x), \tag{8}$$

$$q(x)=\int_{\mathcal{R}^2}sin(x\xi)\phi(\xi)d\xi,$$

$$\phi(\xi)\equiv\phi_N(\xi)=H(A_N)\frac{1}{\sqrt{|\xi|}},$$

$H(\cdot)$ is the characteristic function of correspond set, $x\xi=x_1\xi_1+x_2\xi_2$,

$$A_N=\{\xi\in\mathcal{R}^2,\xi_1=rcos\phi,\xi_2=rsin\phi,N_\circ\leq|\xi|\leq N,\phi\in\left(\frac{\pi}{6},\frac{\pi}{4}\right)\},$$

$N > N_\circ > 0$ are fixed such that $N_\circ$ is close enough to N, $sin(\xi\eta) \geq a_1$, $cos(|\eta|) \geq a_2$, $sin(|\xi|) \geq a_4$ for $\xi \in A_N$, $\eta \in A_N$, where $0 < a_1 < 1$, $0 < a_2 < 1$, $0 < a_4 < 1$ (for instance $N_\circ = 1 - p$, $N = 1$, p is close enough to zero), $g = (g_1, g_2, g_3)$,

$$g_1 = cosh\chi cos\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2} sinh\chi cos\phi_1 - \frac{2x_2}{r^2} cosh\chi sin\phi_1 \chi_{x_1} +$$

$$+\frac{2x_1}{r^2} cosh\chi \chi_{x_2} sin\phi_1 + \frac{sinh^3\chi cos\phi_1}{r^2},$$

$$g_2 = cosh\chi sin\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2} sinh\chi sin\phi_1 + \frac{2x_2}{r^2} cosh\chi cos\phi_1 \chi_{x_1} -$$

$$\frac{2x_1}{r^2} cosh\chi \chi_{x_2} cos\phi_1 + \frac{sinh^3\chi sin\phi_1}{r^2},$$

$$g_3 = sinh\chi f,$$

$$f = 2Y_t^2 - 2Y_r^2 + \frac{sinh(2Y^2)}{2r^2},$$

$x_1 = rcos\phi_1$, $x_2 = rsin\phi_1$, $r > 0$. Then the function u which is defined with $(\star)$ is a solution to (1).

We can to write the solution of the problem (7), (8) in the form

$$Y(t, x) = \int_{\mathcal{R}^2} sin(t|\xi|) sin(x\xi) \frac{\phi_N(\xi)}{|\xi|} d\xi. \tag{9}$$

For the function Y, which is defined with (9), we have the following estimates

$$||Y||_{L^2(\mathcal{R}^2)} \leq \Big|\Big| sin(t|x|) \frac{\phi(x)}{|x|} \Big|\Big|_{L^2(\mathcal{R}^2)} = \tag{10}$$

$$= \Big(\int_{A_N} \Big(sin(t|x|) \frac{\phi_N(x)}{|x|} \Big)^2 dx \Big)^{\frac{1}{2}} \leq \sqrt{\frac{\pi}{12}} \Big(\int_{N_\circ}^N \frac{1}{\rho^2} d\rho \Big)^{\frac{1}{2}} = \sqrt{\frac{\pi(N - N_\circ)}{12 N N_\circ}},$$

$$|Y(t, x)| \leq \int_{\mathcal{R}^2} \Big| sin(t|\xi|) sin(x\xi) \frac{\phi_N(\xi)}{|\xi|} \Big| d\xi \leq \int_{A_N} \frac{1}{|\xi|^{\frac{3}{2}}} d\xi = \tag{11}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{N_\circ}^N \frac{\rho}{\rho^{\frac{3}{2}}} d\rho d\phi = \frac{\pi}{6} (\sqrt{N} - \sqrt{N_\circ}),$$

$$|Y| \leq |x| \frac{\pi}{18} (N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}). \tag{12}$$

$$||Y_t||_{L^2(\mathcal{R}^2)} \leq ||cos(t|x|)\phi(x)||_{L^2(\mathcal{R}^2)} \leq \Big(\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{N_\circ}^N \frac{1}{\rho} \rho d\rho d\phi \Big)^{\frac{1}{2}} = \sqrt{\frac{\pi}{12}(N - N_\circ)}. \tag{13}$$

$$|Y_t(t, x)| = \Big| \int_{\mathcal{R}^2} cos(t|\xi|) sin(x\xi) \phi_N(\xi) d\xi \Big| \leq \int_{\mathcal{R}^2} \phi_N(\xi) d\xi = \frac{\pi}{18} (N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}), \tag{14}$$

Similarly, we have

$$|Y_{x_i}| \leq \frac{\pi}{18}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}), \tag{15}$$

$$||Y_{x_i}||_{L^2(\mathcal{R}^2)} \leq \sqrt{\frac{\pi}{12}(N - N_\circ)}. \tag{16}$$

On the other hand

$$||f||_{L^2(\mathcal{R}^2)} \leq ||2Y_t^2||_{L^2(\mathcal{R}^2)} + ||2Y_r^2||_{L^2(\mathcal{R}^2)} + \Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)}. \tag{17}$$

Now we use (13), (14). Then

$$||2Y_t^2||_{L^2(\mathcal{R}^2)} \leq 2\frac{\pi}{18}\Big(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})||Y_t||_{L^2(\mathcal{R}^2)} \leq \tag{18}$$

$$\frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_\circ)}{12}}.$$

Similarly

$$||2Y_r^2||_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_\circ)}{12}}. \tag{18'}$$

Let $\Omega = \{x \in \mathcal{R}^2 : |x| \leq 1\}$. Then

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} \leq \Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\Omega)} + \Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2\backslash\Omega)} \tag{19}$$

Since (12) holds, we have that there exists constant c_1 such that $|sinh(2Y^2)| \leq c_1(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2|x|^2$ and

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\Omega)} = \Big(\int_\Omega\Big(\frac{sinh(2Y^2)}{2r^2}\Big)^2 dx\Big)^{\frac{1}{2}} \leq \tag{20}$$

$$\leq c_1(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2\Big(\int_\Omega\Big(\frac{|x|^2}{2|x|^2}\Big)^2 dx\Big)^{\frac{1}{2}} = \sqrt{2\pi}\frac{c_1}{2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2.$$

On the other hand (here we use (11) and the fact that $sinhx$ increases for every x)

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2\backslash\Omega)} \leq sinh(\frac{\pi^2}{18}(\sqrt{N} - \sqrt{N_\circ})^2)\frac{\sqrt{\pi}}{2}. \tag{21}$$

From (19), (20), (21) we get

$$\Big|\Big|\frac{sinh(2Y^2)}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} \leq sinh(\frac{\pi^2}{18}(\sqrt{N} - \sqrt{N_\circ})^2)\frac{\sqrt{\pi}}{2} + \sqrt{2\pi}\frac{c_1}{2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2 \tag{22}$$

and from (17), (18), (18'), (22)

$$||f||_{L^2(\mathcal{R}^2)} \leq \tag{22'}$$

$$\leq 2\frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N-N_\circ)}{12}} + sinh(\frac{\pi^2}{18}(\sqrt{N}-\sqrt{N_\circ})^2)\frac{\sqrt{\pi}}{2} + \sqrt{2\pi}\frac{c_1}{2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^2,$$

(23) $$||g_3||_{L^2(\mathcal{R}^2)} = ||sinh\chi f||_{L^2(\mathcal{R}^2)} \leq sinh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)||f||_{L^2(\mathcal{R}^2)}.$$

We note that when $N_\circ$ is close enough to N $||g_3||_{L^1([0,1]L^2(\mathcal{R}^2))}$ is close enough to zero .
From third equation of (1) we get that χ is solution to the equation

$$\chi_{tt} - \Delta\chi + \frac{sinh(2\chi)}{2r^2} = f,$$

i.e.

$$\chi_{tt} - \Delta\chi = -\frac{sinh(2\chi)}{2r^2} + f.$$

Then(here we use (11) and the fact that the functions $sinhx$, $coshx$ are increasing for every $x \geq 0$)

(24) $$||g_1||_{L^2(\mathcal{R}^2)} \leq cosh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|f - \frac{sinh2\chi}{2r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} +$$

$$cosh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|\frac{2x_2}{r^2}\chi_{x_1}\Big|\Big|_{L^2(\mathcal{R}^2)} +$$

$$cosh(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|\frac{2x_1}{r^2}\chi_{x_2}\Big|\Big|_{L^2(\mathcal{R}^2)} +$$

$$+\Big|\Big|\frac{sinh\chi}{r^2}\Big|\Big|_{L^2(\mathcal{R}^2)} + sinh^2(\frac{\pi^2}{36}(\sqrt{N}-\sqrt{N_\circ})^2)\Big|\Big|\frac{sinh\chi}{r^2}\Big|\Big|_{L^2(\mathcal{R}^2)}.$$

Since $\chi_{x_1} = 2YY_{x_1}$,

$$\Big|\frac{2x_2}{r^2}\chi_{x_1}\Big| \leq 2\frac{|x_2|}{r^2}|Y||Y_{x_1}| \leq$$

(from (12))

$$\leq \frac{\pi}{9}\frac{|x_2||x|}{r^2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})|Y_{x_i}|,$$

(25) $$\Big|\Big|\frac{2x_2}{r^2}\chi_{x_1}\Big|\Big|_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})||Y_{x_1}||_{L^2(\mathcal{R}^2)} \leq$$

(here we use (16))

$$\leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N-N_\circ)}{12}}.$$

Similarly

(26). $$\Big|\Big|\frac{2x_1}{r^2}\chi_{x_2}\Big|\Big|_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})\sqrt{\frac{\pi(N-N_\circ)}{12}}$$

From (17), (22'), (22), (24), (25), (26) we get

(27) $$||g_1||_{L^2(\mathcal{R}^2)} \leq C_1,$$

where C_1 is close enough to zero when $N_\circ$ is close enough to N. Similarly,

$$||g_2||_{L^2(\mathcal{R}^2)} \leq C_2, \tag{28}$$

where C_2 is close enough to zero when $N_\circ$ is close enough to N. From (23), (27), (28) we have

$$||g||_{L^2(\mathcal{R}^2)} \leq C,$$

where C is close enough to zero when $N_\circ$ is close enough to N. From here the second inequality of (5) is hold for every $R > 0$ when $N_\circ$ is close enough to N.
Sinse

$$Y(0,x) = 0, \quad \chi(0,x) = 0, \quad \chi_t(0,x) = 2Y(0,x)Y_t(0,x) = 0,$$

$$\chi_{x_i}(0,x) = 2Y(0,x)Y_{x_i}(0,x) = 0, \quad i = 1,2,$$

$$u_{1t}(0,x) = cosh\chi(0,x)cos\phi_1\chi_t(0,x) = 0,$$

$$u_{2t}(0,x) = cosh\chi(0,x)sin\phi_1\chi_t(0,x) = 0,$$

$$u_{3t}(0,x) = sinh\chi\chi_t(0,x) = 0,$$

$$u_{1x_1}(0,x) = cosh\chi(0,x)cos\phi_1\chi_{x_1}(0,x) + sinh\chi(0,x)sin\phi_1\frac{x_2}{r^2} = 0,$$

$$u_{2x_1}(0,x) = cosh\chi(0,x)sin\phi_1\chi_{x_1}(0,x) - sinh\chi(0,x)cos\phi_1\frac{x_2}{r^2} = 0,$$

$$u_{3x_1}(0,x) = sinh\chi(0,x)\chi_{x_1}(0,x) = 0,$$

$$u_{1x_2}(0,x) = cosh\chi(0,x)cos\phi_1\chi_{x_2}(0,x) - sinh\chi(0,x)sin\phi_1\frac{x_1}{r^2} = 0,$$

$$u_{2x_2}(0,x) = cosh\chi(0,x)sin\phi_1\chi_{x_2}(0,x) + sinh\chi(0,x)cos\phi_1\frac{x_1}{r^2} = 0,$$

$$u_{3x_2}(0,x) = sinh\chi(0,x)\chi_{x_2}(0,x) = 0,$$

we have

$$E(0,u) = 0,$$

i.e. the first inequality of (5) holds for every $\delta > 0$.
From (6) we get that

$$||\partial_t u||^2_{L^2(\mathcal{R}^2)} \leq \epsilon \quad \forall \quad t \in [0,1]. \tag{29}$$

On the other hand

$$||\partial_t u||^2_{L^2(\mathcal{R}^2)} = ||\partial_t u_1||^2_{L^2(\mathcal{R}^2)} + ||\partial_t u_2||^2_{L^2(\mathcal{R}^2)} - ||\partial_t u_3||^2_{L^2(\mathcal{R}^2)} =$$

$$= ||cosh\chi cos\phi_1\chi_t||^2_{L^2(\mathcal{R}^2)} + ||cosh\chi sin\phi_1\chi_t||^2_{L^2(\mathcal{R}^2)} - ||sinh\chi\chi_t||^2_{L^2(\mathcal{R}^2)} =$$

$$= \int_{\mathcal{R}^2} \chi_t^2(cosh^2\chi - sinh^2\chi)dx = \int_{\mathcal{R}^2} \chi_t^2 dx = ||\chi_t||^2_{L^2(\mathcal{R}^2)}.$$

Therefore, using (29), we get

$$||\chi_t||_{L^2(\mathcal{R}^2)} \leq \epsilon^{\frac{1}{2}}$$

or

$$2||YY_t||_{L^2(\mathcal{R}^2)} \leq \epsilon^{\frac{1}{2}}.$$

From here

$$2\int_{\mathcal{R}^2}\psi Y Y_t dx \le ||\psi||_{L^2(\mathcal{R}^2)}\epsilon^{\frac{1}{2}} \tag{30}$$

for any function $\psi \in L^2(\mathcal{R}^2)$. Let

$$B := a_1^4 a_2^2 a_4^2 \frac{\pi^5}{18.126^2}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})(N^{\frac{7}{4}} - N_\circ^{\frac{7}{4}})^2(\sqrt{N} - \sqrt{N_\circ})^2 \tag{31}$$

and $\epsilon = \frac{B}{2}$,

$$\psi \equiv \psi_N(\xi) = H(A_N)\frac{1}{|\xi|^{\frac{1}{4}}}.$$

For $t = 1$ and $x \in A_N$ we have

$$Y \ge a_1 a_4 \frac{\pi}{6}(\sqrt{N} - \sqrt{N_\circ}) > 0, \tag{32}$$

$$Y_t \ge a_1 a_2 \frac{\pi}{18}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}) > 0. \tag{33}$$

$$\int_{A_N}\psi dx = \frac{\pi}{21}(N^{\frac{7}{4}} - N_\circ^{\frac{7}{4}}), \tag{34}$$

$$||\psi||_{L^2(A_N)} = \sqrt{\frac{\pi}{18}}(\sqrt{N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}}}). \tag{35}$$

From (30), (32), (33), (34), (35) we have for $t = 1$

$$a_1^2 a_2 a_4 \frac{\pi^{\frac{5}{2}}}{126\sqrt{18}}(N^{\frac{3}{2}} - N_\circ^{\frac{3}{2}})^{\frac{1}{2}}(N^{\frac{7}{4}} - N_\circ^{\frac{7}{4}})(\sqrt{N} - \sqrt{N_\circ}) \le \epsilon^{\frac{1}{2}},$$

i. e. $\epsilon \ge B$ which is contradiction with $\epsilon = \frac{B}{2}$. Therefore the solution map is not uniformly continuous.

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