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REMARKS ON NULL CONTROLLABILITY FOR SEMILINEAR HEAT EQUATION IN MOVING DOMAINS

by

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Abstract

We investigate in this article the null controllability for the semilinear heat operator $u' - \Delta u + f(u)$ in a domain which boundary is moving with the time t.

Key Words: heat operator, semilinear, moving boundary, null controllability.

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1. Introduction and Main Result

In this article we consider semilinear parabolic problems in domains which are moving with the time t. Given a time $T > 0$, the state equation is posed in an open set $\widehat{Q}$ of $\mathbb{R}^n \times (0,T)$ contained in $\mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$. The open set $\widehat{Q}$ is the union of open sets Ω_t of $\mathbb{R}^n$, for $0 < t < T$, which are images of a reference domain Ω_0 by a diffeomorphism $\tau_t \colon \Omega_0 \to \Omega_t$.

We identify Ω_0 to a bounded open set Ω of $\mathbb{R}^n$ and its points are represented by $y = (y_1, y_2, \ldots, y_n)$ and those of Ω_t by $x = (x_1, x_2, \ldots, x_n)$ are such that $x = \tau_t(y)$. We also employ the notation $\tau(y,t)$ instead of $\tau_t(y)$.

Thus, the noncylindrical domain $\widehat{Q}$ of $\mathbb{R}^{n+1}$ is defined by

$$\widehat{Q} = \bigcup_{0<t<T} \{\Omega_t \times \{t\}\}.$$

The boundary of Ω_t is represented by Γ_t and the lateral boundary of $\widehat{Q}$, denoted by $\widehat{\Sigma}$, is given by

$$\widehat{\Sigma} = \bigcup_{0<t<T} \{\Gamma_t \times \{t\}\}.$$

Let Q be the cylinder

$$Q = \Omega \times (0,T),$$

Ω the reference domain. We have the natural diffeomorphism between Q and $\widehat{Q}$ given by

$$(y,t) \in Q \to (x,t) \in \widehat{Q}, \qquad (x,t) = (\tau_t(y), t) = (\tau(y,t), t).$$

We will develope the article under the following assumptions.

(A1) For all $0 \leq t \leq T$, $\quad \tau_t$ is a C^2-diffeomorphism from Ω to Ω_t.

(A2) $\quad \tau(y,t) \in C^0([0,T]; C^2(\overline{\Omega}))$;

We assume $\Omega \subset \mathbb{R}^n$ bounded and of class C^2. It could be Lipschitz continuous and unbounded.

In this article we will work with the following state equation

$$
(1.1) \qquad \left| \begin{array}{l} u' - \Delta u + f(u) = h(x,t)\chi_{\hat{q}} \ \text{ in } \ \widehat{Q} \\ y = 0 \ \text{ on } \ \widehat{\Sigma} \\ u(x,0) = u_0(x) \ \text{ in } \ \Omega. \end{array} \right.
$$

In (1.1) we have $u = u(x,t)$, $\quad u' = \dfrac{\partial u}{\partial t}$; $\quad \Delta$ is the Laplace's operator in $\mathbb{R}^n$; $\hat{q}$ is an open, non-empty, subset of $\widehat{Q}$. We also denote by w_t the cross section of $\hat{q}$ at any $0 < t < T$; $\quad \chi_{\hat{q}}$ the characteristic function of $\hat{q}$. The function $h(x,t)$ is the control that acts on the state $u(x,t)$ localized in $\hat{q}$. The nonlinear function f is real and globally Lipschitz such that $f(0) = 0$. This means that there exists a constant K_0, called Lipschitz constant, such that

$$
(1.2) \qquad |f(\xi) - f(\eta)| \leq K_0 |\xi - \eta|
$$

for all $\xi, \eta \in \mathbb{R}$.

As we will see later, if $u_0 \in L^2(\Omega)$, $\quad h \in L^2(\widehat{Q})$ the system (1.1) has a unique solution

$$
u \in C^0([0,T]; L^2(\Omega_t)) \cap L^2(0,T; H_0^1(\Omega_t)).
$$

The null controllability problem for (1.1) can be formulated as follows: Give $T > 0$ and $u^0 \in L^2(\Omega)$, to find a controll $h \in L^2(\widehat{Q})$ such that the solution $u = u(x,t)$ of (1.1) satisfies the conditions:

$$
(1.3) \qquad \bullet \quad u(x,T) = 0 \quad \text{for all} \quad x \in \Omega_T \,,
$$

$$
(1.4) \qquad \bullet \quad |h|_{L^2(\widehat{Q})} \leq c|u^0|_{L^2(\Omega)} \,, \quad \text{for all} \quad u^0 \in L^2(\Omega).
$$

There is a large literature on the null controllability for heat equations in cylindrical domains. See for instance, and the bibliography therein, Lions [20,21,22], Fabre-Puel-Zuazua [12], Fernandez-Cara and Zuazua [14], Cabanillas-Menezes-Zuazua [4], Zuazua [38]. In the context of noncylindrical domain, Limaco-Medeiros-Zuazua [17], proved null controllability for linear heat equation.

The main result of the present paper is the following:

Theorem 1.1. Assume f is C^1 and satisfies (1.2) with $f(0) = 0$. Then, for all $T > 0$ and for every $u^0 \in L^2(\Omega)$, there exists $h \in L^2(\widehat{Q})$ such that the solution $u = u(x,t)$ of (1.1) satisfies (1.3). Moreover, (1.4) holds for a suitable constant $C > 0$ independent of u^0. In other words, system (1.1) is null controllable for $T > 0$. $\square$

The methodology of the proof of the Theorem 1.1 is based in the fixed point method, see Zuazua [39,40]. There is however a new difficulty related to the fact that $\widehat{Q}$ is noncylindrical. To set up this point we employ the idea contained in [27]. The first step on the fixed point method is to study the null controll for the linearized system. This problem is reduced, by duality, to obtain a observability inequality for the adjoint system. This is get as an application of Carlemann inequalities as in Imanuvilov-Yamamoto [17].

This work is organized as follows: Section 2 is devoted to prove the null controllability for the linearized system. In Section 3 we prove Theorem 1.1 by a fixed point method.

To close this section we mention some basic references on the analysis of Partial Differential Equations in noncylindrical domains. Among many references we mention the following: Lions [19]; Cooper and Bardos [9]; Medeiros [6]; Inoue [18]; Rabello [33]; Nakao and Narazaki [34], for wave equations. Bernardi, Bonfanti and Lutteroti [2], Miranda, Medeiros [9] for Schrödinger equations; Cheng-He and Ling-Hsiano [7] for Euler equation; Miranda and Limaco [33] for Navier-Stokes equations; Chen and Frid [6] for hyperbolic systems of conservation law. Note that in [29] and [30] they considered $\tau_t(y) = K(t)y$. In [2] the authors considered $\tau_t(y) = K(t)y + h(t), \quad h(t)$ a vector of $\mathbb{R}^n$. $\square$

2. Analysis of the Linear Problem

The main result of this article will be proved in Section 3 by means of a fixed point argument. As an step preliminary we need to analyse the null controllability of the following linearized system:

$$\left| \begin{array}{l} u' - \Delta u + a(x,t)u = h(x,t) \ \text{ in } \ \widehat{Q} \\ u = 0 \ \text{ on } \ \widehat{\Sigma} \\ u(x,0) = u^0 \ \text{ in } \ \Omega, \end{array} \right. \tag{2.1}$$

where the potential is assumed to be in $L^\infty(\widehat{Q})$.

First of all we study the existence and uniqueness of solution of the system (2.1).

2.1 Strong and Weak Solutions

We distinguish three classes of solutions for the system (2.1), as follows: strong, weak and ultra weak solutions defined by transposition.

Definition 2.1. a) A real function $u = u(x,t)$ defined in $\widehat{Q}$ is said to be a strong solution for the boundary value problem (2.1) if

$$u \in C^0([0,T]; H_0^1(\Omega_t)) \cap L^2(0,T; H^2(\Omega_t) \cap H_0^1(\Omega_t)) \cap H^1(0,T; L^2(\Omega_t)) \tag{2.2}$$

and the three conditions in (2.1) are satisfied almost everywhere in their corresponding domains.

b) We say that the function u is a weak solution of (2.1) if

$$u \in C^0([0,T]; L^2(\Omega_t)) \cap L^2(0,T; H_0^1(\Omega_t)) \tag{2.3}$$

and

$$\begin{aligned} &\int_0^T \int_{\Omega_t} u\varphi' \, dxdt + \int_0^T \int_{\Omega} u^0(x)\varphi(x,0) \, dxds + \\ &+ \int_0^T \int_{\Omega_t} \nabla_x u \, \nabla_x \varphi \, dxdt = \int_0^T \int_{\Omega_t} h\varphi \, dxdt \end{aligned} \tag{2.4}$$

for all $\varphi \in L^2(0,T; H_0^1(\Omega_t)) \cap C^1([0,T]; L^2(\Omega_t))$ such that $\varphi(T) = 0$.

Theorem 2.1. Assume that the noncylindrical domain $\widehat{Q}$ satisfies the conditions of the Section 1. Then, if $u^0 \in H_0^1(\Omega)$, $a(x,t) \in L^\infty(\widehat{Q})$ and $h \in L^1(0,T; L^2(\Omega_t))$, the problem (2.1) has a unique strong solution.

Moreover, there exists a positive constant C (depending on $\widehat{Q}$ but independent of u^0 and h) such that

$$||u||_{L^\infty(0,T;H_0^1(\Omega_t))} + |u'|_{L^2(\widehat{Q})} + |u|_{L^2(0,T;L^2(\Omega_t))} \leq C \left(||u^0||_{H_0^1(\Omega)} + |h|_{L^2(\widehat{Q})} \right) \tag{2.5}$$

and for $f \in L^2(0,T; H_0^1(\Omega_t))$,

$$||u||_{L^\infty(0,T;H_0^1(\Omega_t))} + |u|_{L^2(0,T;H^2(\Omega_t)))} \leq C \left(||u^0||_{H_0^1(\Omega)} + |h|_{L^2(0,T;H_0^1(\Omega_t))} \right) \tag{2.6}$$

Proof: As in [27] we employ the argument consisting in transforming the heat equation in the noncylindrical domain $\widehat{Q}$, into a variable coefficients parabolic equation in the reference cylinder Q by means of the diffeomorphism $(x,t) = (\tau_t(y),t) = (\tau(y,t),t)$ for $x \in \Omega_t$, $y \in \Omega$ and $0 \leq t \leq T$, i.e., for $(x,t) \in \widehat{Q}$ and $(y,t) \in Q$.

In fact we set

$$v(y,t) = u(\tau_t(y),t) = u(\tau(y,t),t), \text{ for } y \in \Omega, \, 0 \leq t \leq T, \tag{2.7}$$

or equivalently

$$u(x,t) = v(\tau_t^{-1}(x),t) = v(\rho(x,t),t),\ x \in \Omega_t\,,\ 0 \le t \le T. \tag{2.8}$$

Here and in the following τ_t^{-1} denotes the inverse of τ_t, which, according to assumption (A1) is a C^2 map from Ω_t to Ω, for all $0 \le t \le T$. This map will be denoted by ρ_t. We shall also employ the notation $\rho(x,t) = \rho_t(x)$, $\quad y_j = \rho_j(x,t)$, $1 \le j \le n$.

We obtain,

$$\frac{\partial}{\partial t}\, u(x,t) = u'(x,t) = \frac{\partial v}{\partial t}\,(\rho(x,t),t) + \nabla_y\, v(\rho(x,t),t) \cdot \frac{\partial}{\partial t}\,\rho(x,t),$$

where $\cdot$ is the scalar product in $\mathbb{R}^n$. In other words,

$$\frac{\partial}{\partial t}\, u(x,t) = u'(x,t) = \frac{\partial v}{\partial t}\,(y,t) + \nabla_y\, v(y,t) \cdot \tilde{b}(y,t)$$

where $\tilde{b}(y,t)$ denotes a vector field

$$\tilde{b}(y,t) = \frac{\partial}{\partial t}\,\rho(x,t). \tag{2.9}$$

Note that according to the assumption (A2),

$$\tilde{b} \in C^1(\overline{\Omega}). \tag{2.10}$$

On the other hand,

$$\frac{\partial}{\partial x_i}\, u(x,t) = \frac{\partial}{\partial y_j}\, v(y,t)\, \frac{\partial y_j}{\partial x_i} = \nabla_y\, v(y,t) \cdot \frac{\partial}{\partial x_i}\,\rho(x,t),$$

and

$$\frac{\partial^2}{\partial x_i^2}\, u(x,t) = \frac{\partial^2}{\partial y_j \partial y_k}\, v(y,t)\, \frac{\partial y_j}{\partial x_i}\, \frac{\partial y_k}{\partial x_i} + \frac{\partial}{\partial y_j}\, v(y,t)\, \frac{\partial^2 y_j}{\partial x_i^2}$$

or

$$\Delta u(x,t) = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial y_j \partial y_k}\, \frac{\partial \rho_j}{\partial x_i}\, \frac{\partial \rho_k}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial v}{\partial y_j}\, \frac{\partial^2 \rho_j}{\partial x_i^2}\,.$$

Thus by the mapping $x = \tau^{-1}(y)$ that takes $\widehat{Q}$ into the cylinder Q we transform (2.1) in an equivalent problem (2.11) given by

$$
(2.11) \qquad \left|\begin{array}{l}
\dfrac{\partial v}{\partial t} - \displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 v}{\partial y_j \partial y_k} \dfrac{\partial \rho_j}{\partial x_i} \dfrac{\partial \rho_k}{\partial x_i} - \sum_{i=1}^{n} \dfrac{\partial v}{\partial y_j} \Delta \rho_j + \\
\qquad + \tilde{b} \cdot \nabla_y v + a(y,t)v = h(y,t) \ \text{ in } Q \\
v = 0 \quad \text{ on } \Sigma, \text{ lateral boundary of } Q \\
v(y,0) = u^0(y) \ \text{ in } \ \Omega
\end{array}\right.
$$

Analysis of the operator $A(t)v = -\sum\limits_{i=1}^{n} \dfrac{\partial^2 v}{\partial y_j \partial y_k} \dfrac{\partial \rho_j}{\partial x_i} \dfrac{\partial \rho_k}{\partial x_i}$

For $v, \varphi \in L^2(0,T; H_0^1(\Omega))$ and Gauss' Lemma we obtain the bilinear form $a(t,v,\varphi)$ defined by

$$
a(t,v,\varphi) = (A(t)v, \varphi) = \sum_{i=1}^{n} \int_\Omega \frac{\partial v}{\partial y_j} \frac{\partial \varphi}{\partial y_k} \frac{\partial \rho_j}{\partial x_i} \frac{\partial \rho_k}{\partial x_i} \, dx.
$$

This bilinear form is bounded because $\rho \in C^2(-\overline{\Omega})$ by assumption (A2). Let us prove that it is $H_0^1(\Omega)$-coercive. In fact, set $\varphi = v \in H_0^1(\Omega)$. We have

$$
\begin{aligned}
a(t,v,v) &= \sum_{i=1}^{n} \int_\Omega \left(\frac{\partial v}{\partial y_k} \frac{\partial \rho_k}{\partial x_i} \right) \left(\frac{\partial v}{\partial y_j} \frac{\partial \rho_j}{\partial x_i} \right) dy \ = \\
&= \sum_{i=1}^{n} \int_\Omega \sum_{k=1}^{n} \left(\frac{\partial v}{\partial y_k} \frac{\partial \rho_k}{\partial x_i} \right)^2 dy.
\end{aligned}
$$

Note that $\left(\dfrac{\partial v}{\partial y_k} \dfrac{\partial \rho_k}{\partial x_i} \right)_{1 \le k \le n}$ is a vector of $\mathbb{R}^n$. Let us consider the $n \times n$ matrix M given by

$$
M = \left(\frac{\partial \rho_k}{\partial x_i} \right)_{1 \le i,k \le n}
$$

and the vector ξ of $\mathbb{R}^n$ defined by

$$
\xi = \left(\frac{\partial v}{\partial y_k} \right)_{1 \le k \le n}
$$

We observe that

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial v}{\partial y_k}\frac{\partial \rho_k}{\partial x_i}\right)^2 = ||M\xi||^2_{\mathbb{R}^n}\,.$$

But, by assumption, M is bounded and invertible what comes from assumptions on $x = \tau_t(y)$. Then,

$$||M^{-1}\xi||^2_{\mathbb{R}^n} \leq C_0||\xi||^2_{\mathbb{R}^n}\,, \quad C_0 > 0$$

or

$$||M\,\xi||^2_{\mathbb{R}^n} \geq \frac{1}{C^0}\,||\xi||_{\mathbb{R}^n}\,.$$

Thus, returning to the quadratic form, we obtain

$$a(t,v,v) = \int_\Omega ||M\,\xi||^2_{\mathbb{R}^n}\,dy \geq \frac{1}{C_0}\int_\Omega |\xi|^2_{\mathbb{R}^n}\,dy$$

or

$$a(t,v,v) \geq \frac{1}{C_0}\,||v||^2_{H^1_0(\Omega)}\,.$$

$\square$

In (2.11) set

$$b(y,t) = \tilde{b}(y,t) - \Delta\rho(y,t),\; b \in [L^\infty(Q)]^n.$$

Thus, from (2.11) we obtain for (2.1) in Q the following system

$$(2.12) \qquad \left|\begin{array}{l} v' + A(t)v + b\cdot\nabla_y v + a(y,t)v = h(y,t) \text{ in } Q \\ v = 0 \text{ on } \Sigma \\ v(y,0) = u^0(y) \text{ in } \Omega \end{array}\right.$$

Note that (2.12) is a linear parabolic system with variable coefficients in a cylinder $Q = \Omega \times (0,T)$, Ω a regular bounded open set of $\mathbb{R}^n$. Since $A(t)$ is coercive the boundary value problem (2.12) is a classical problem studies in Lions-Magenes [24]. If we take $u^0 \in H^1_0(\Omega)$ and $h \in L^2(0,T;L^2(\Omega))$ then (2.12) has strong solution $v \in C^0([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Otherwise, if $u^0 \in L^2(\Omega)$ and $h \in L^2(0,T;L^2(\Omega))$, then (2.14) has a weak solution $u \in C^0([0,T];L^2(\Omega) \cap L^2(0,T;H_0^1(\Omega))$. In both cases we have uniqueness.

From the assumption (A1) and (A2) the transformation $y \to x$ from Q in $\widehat{Q}$ maps the space $C^0([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ into the space $C^0([0,T];H_0^1(\Omega_t)) \cap L^2(0,T;L^2(\Omega_t)) \cap H^1(0,T;L^2(\Omega_t))$.

To prove the estimate (2.5) we first establish the classical energy estimate. In fact, multiplying (2.1) by u integrating for $x \in \Omega_t$ and $0 < t < T$, we get

$$
\begin{aligned}
&\int_0^t \int_{\Omega_t} u'u \, dxds + \int_0^t |\nabla u(s)|^2_{L^2(\Omega_s)} \, ds \; = \\
&= - \int_0^t \int_{\Omega_t} au^2 \, dxds + \int_0^t \int_{\Omega_t} hu \, dxds \; \leq \\
&\leq ||a||_{L^\infty(\widehat{Q})} \int_0^t \int_{\Omega} u^2 \, dxds + C(\varepsilon)|h|^2_{L^2(0,T;H^{-1}(\Omega_t))} + \\
&+ \varepsilon |u|^2_{L^2(0,T;H_0^1(\Omega_t))}
\end{aligned}
\tag{2.13}
$$

Note that since u vanishes on the lateral boundary $\widehat{\Sigma}$ of $\widehat{Q}$ we have (cf. Duvaut [11], p.26):

$$
\begin{aligned}
&\int_0^t \int_{\Omega_t} uu' \, dxdt = \frac{1}{2} \int_0^t \int_{\Omega_t} \frac{\partial}{\partial t} |u(x,t)|^2 \, dxdt \; = \\
&= \frac{1}{2} \left[\int_{\Omega_t} u^2(x,t) \, dx - \int_{\Omega} (u^0(x))^2 \, dx \right]
\end{aligned}
\tag{2.14}
$$

As a consequence of the assumptions (A1) and (A2) it follows that the Poincaré inequality is satisfied, uniformly, in the domain Ω_t for all $0 \leq t \leq T$. Thus, in view of (2.13) and (2.14) we have

$$
\begin{aligned}
&\frac{1}{2} \int_0^t \int_{\Omega_t} \frac{\partial}{\partial t} |u(x,t)|^2 \, dxdt + \frac{1}{2} \int_0^t |\nabla u(s)|^2_{L^2(\Omega_s)} \, ds \; \leq \\
&\leq c|h|^2_{L^2(0,T;H^{-1}(\Omega_t))} + ||a||_{L^\infty(\widehat{Q})} \int_0^t \int_{\Omega_t} u^2 \, dxdt
\end{aligned}
\tag{2.15}
$$

Then,

$$\int_{\Omega_t} u^2\,dx \le C\left(|u^0|^2_{L^2(\Omega)} + |h|^2_{L^2(0,T;H^{-1}(\Omega_t))}\right) \tag{2.16}$$

By (2.14), (2.15) and (2.16) we have

$$|u(t)|^2_{L^2(\Omega_t)} + \frac{1}{2}\int_0^t |\nabla u(s)|^2_{L^2(\Omega_s)}\,ds \le C\left[|u^0|^2_{L^2(\Omega)} + |h|^2_{L^2(0,T;H^{-1}(\Omega_t))}\right], \tag{2.17}$$

for a constant $C > 0$.

In particular, strong solutions satisfies the energy estimate

$$|u|^2_{L^2(0,T;L^2(\Omega_t))} + |u|^2_{L^2(0,T;H_0^1(\Omega_t))} \le C\left[|u^0|^2_{L^2(\Omega)} + |h|^2_{L^2(0,T;H^{-1}(\Omega_t))}\right], \tag{2.18}$$

with $C > 0$ constant independent of the solution.

Now we multiply (2.1) by $-\Delta u$ and integrate. We have

$$-\int_{\Omega_t} u'\,\Delta u\,dx + \int_{\Omega_t} |\Delta u|^2\,dx - \int_{\Omega_t} au\Delta u\,dx = -\int_{\Omega_t} h\Delta u\,dx. \tag{2.19}$$

Moreover

$$-\int_{\Omega_t} u'\Delta u\,dx = \int_{\Omega_t} \nabla u\cdot\nabla u'\,dx = \frac{d}{dt}\int_{\Omega_t} |\nabla u|^2\,dx - \int_{\Gamma_t} |\nabla u|^2\,w\cdot n_t\,d\sigma, \tag{2.20}$$

where n_t denote the unit outward normal vector to Ω_t and w is the velocity field $w = \left[\dfrac{\partial\tau}{\partial t}\right]\rho(x,t)$ [cf. Duvaut [11] p.26]. Note that according to the assumption (A1) and (A2), by uniform (with respect to t) elliptic regularity, classical trace results and interpolation, [cf. Lions-Magenes [24] p.49], we obtain

$$\begin{aligned} \left|\int_{\Gamma_t} |\nabla u|^2\,w\cdot n_t\,d\sigma\right| &\le c\int_{\Gamma_t} |\nabla u|^2\,d\sigma \le \\ &\le c_\alpha\left[\int_{\Omega_t} |\Delta u| dx\right]^\alpha \left[\int_{\Omega_t} |\nabla u|^2\,dx\right]^{1-\alpha} \end{aligned} \tag{2.21}$$

for all $\alpha \ge \dfrac{1}{2}\cdot$

Combining (2.19)-(2.21) and by Cauchy-Schwarz's inequality we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t}|\nabla u|^2\,dx \le -\frac{1}{2}\int_{\Omega_t}|\Delta u|^2\,dx + |\nabla h|_{L^2(\Omega_t)}\cdot|\nabla u|_{L^2(\Omega_t)} + \\
&+ c|\nabla u|^2_{L^2(\Omega_t)} + |a|_{L^\infty(\widehat{Q})}\,|\nabla u|^2_{L^2(\Omega_t)} \;= \\
&= -\frac{1}{2}\int_{\Omega_t}|\Delta u|^2\,dx + |\nabla h|_{L^2(\Omega_t)}\cdot|\nabla u|_{L^2(\Omega_t)} + C|\nabla u|^2_{L^2(\Omega_t)}\,.
\end{aligned}
\tag{2.22}
$$

Solving this differential inequality we deduce the existence of a constant C such that

$$
|u|^2_{L^2(0,T;H^1_0(\Omega_t))} + |u|^2_{L^2(0,T;H^2(\Omega_t))} \le C\left[|u^0|^2_{H^1_0(\Omega)} + ||h||_{L^1(0,1;L^2(\Omega_t))}\right]. \tag{2.23}
$$

A variation of this argument alows also to get

$$
||u||^2_{L^\infty(0,T;H^1_0(\Omega_t))} + |u|_{L^2(0,T;H^2(\Omega_t))} \le C\left[|u^0|^2_{H^1_0(\Omega_t)} + |h|^2_{L^2(\widehat{Q})}\right] \tag{2.24}
$$

In fact, to obtain (2.24) instead of (2.23) it is sufficient to estimate the term $\displaystyle\int_{\Omega_t} h\Delta u\,dx$ as follows

$$
\int_{\Omega_t} h\Delta u\,dx \le \frac{1}{2}\int_{\Omega_t}(|h|^2 + |\Delta u|^2)dx.
$$

This complete the proof of the Theorem 2.1. □

Remark 2.1 Note that we could also have obtained the above estimates using existence results for the variable coefficients parabolic equation satisfied by v and then doing the change of variables $x \to y$ of $\widehat{Q}$ into Q. □

Theorem 2.2 Given $u^0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega_t))$, there exists a unique weak solution of (2.1). Moreover, there exists a constant $C > 0$ (depending on $\widehat{Q}$ but independent of u^0 and h) such that

$$
||u||_{L^\infty(0,T;L^2(\Omega_t))} + |\nabla u|_{L^2(\widehat{Q})} \le C\left[|u^0|_{L^2(\Omega)} + |h|_{L^2(0,T;H^{-1}(\Omega_t))}\right] \tag{2.25}
$$

A similar argument allows to replace $h \in L^2(0,T;H^{-1}(\Omega_t))$ by the assumption $h \in L^1(0,T\ L^2(\Omega_t))$ and to obtain the estimate

$$
||u||_{L^\infty(0,T;L^2(\Omega_t))} + |\nabla u|_{L^2(\widehat{Q})} \leq C\left[|u^0|_{L^2(\Omega)} + ||h||_{L^1(0,T;L^2(\Omega_t))}\right] \tag{2.26}
$$

Proof: We follow the argument of reference [27]. We proceed by steps.

Step 1 (Existence). Let $u_m^0 \in H_0^1(\Omega)$ and $h_m \in L^2(\widehat{Q})$ be a sequence of regularized initial data and right hand side terms, respectively, such that $u_m^0 \to u^0$ strongly in $H_0^1(\Omega)$, $h_m \to h$ strongly in $L^2(0,T;H^{-1}(\Omega_t))$. Then, for each $m \in \mathbb{N}$, let us consider the unique strong solution u_m of (2.1) with initial data u_m^0 and right side h_m. Thus, for any $n,k \in \mathbb{N}$ we have

$$
(2.27)\qquad \left|\begin{array}{l}
(u_n-u_k)' - \Delta(u_n-u_k) + a(x,t)(u_n-u_k) = h_n - h_k \text{ a.e. in } \widehat{Q},\\
(u_n-u_k) = 0 \text{ on } \widehat{\Sigma}\\
(u_n-u_k)(0) = u_n^0 - u_k^0 \text{ in } \Omega
\end{array}\right.
$$

By the energy estimate (2.18) we obtain that (u_m) is a Cauchy sequence in the space

$$
C^0([0,T];L^2(\Omega_t)) \cap L^2(0,T;H_0^1(\Omega_t)).
$$

Thus it converges, as $m \to \infty$, to a limit $u \in C([0,T];L^2(\Omega_t)) \cap L^2(0,T;H_0^1(\Omega_t))$. The limit u is a weak solution of (2.1) satisfying (2.4) and the estimate (2.25). In fact, u_m is strong solution for every $m \in \mathbb{N}$. Then, multiplying the equation with u_m by a test function φ and integrating by parts we deduce that u_m satisfies the weak formulation (2.4).

The convergence of u_m to u in the space

$$
C^0([0,T];L^2(\Omega_t)) \cap L^2(0,T;H_0^1(\Omega_t))
$$

allows to pass to the limit in the weak formulation to conclude that u satisfy (2.4).

Step 2 (Uniqueness). Assume that the system (2.2) admits two weak solution u and $\hat{u}$ satisfying (2.4). Introduce $w = u - \hat{u}$. Then, w belongs to $C^0([0,T]; L^2(\Omega_t)) \cap L^2(0,T; H_0^1(\Omega_t))$ and satisfies

$$\int_0^T \int_{\Omega_t} w\varphi' \, dxdt + \int_0^T \int_{\Omega_t} \nabla_x w \cdot \nabla_x \varphi \, dxdt + \int_0^T \int_{\Omega_t} a(x,t) u\varphi \, dxdt = 0$$

for all test function φ. In order to conclude that $w = 0$, it is sufficient to consider $w = \varphi$ as a test function. Of course we cannot do it directly since w is not a test function. It is justified by regularization and cut-off argument.

In this way we complete the proof obtaining the energy estimate for w what guarantees that

$$\int_0^t \int_{\Omega_t} \frac{\partial}{\partial t} |w|^2 \, dxdt + \int_0^t |\nabla w(s)|^2 \, ds \leq ||a||_{L^\infty(\widehat{Q})} \int_0^t \int_{\Omega_t} |w|^2 \, dxdt.$$

Then $w = 0$ because $w(0) = 0$.

Step 3. To prove the estimate (2.26), it suffices to employ in (2.18) the estimate

$$\left| \int_0^T \int_{\Omega_t} \rho h \, dxdt \right| \leq |h|_{L^1(0,T;L^2(\Omega))} \cdot ||u||_{L^\infty(0,T;L^2(\Omega_t))} .$$

$\square$

2.2 Ultra Weak Solutions by Transposition Method

In this section we address the question of finding solutions u of the boundary value problem

$$\text{(2.28)} \qquad \left| \begin{array}{l} u' - \Delta u + a(x,t) u = 0 \text{ in } \Omega_t \,, \text{ for } 0 < t < T \\ u = 0 \text{ on } \Gamma_t \text{ for } 0 < t < T \\ u(0) = u^0 \text{ in } \Omega \end{array} \right.$$

when $u^0 \in H^{-1}(\Omega)$ and $a(x,t) \in L^\infty(\widehat{Q})$.

We employ the transposition method as in Lions-Magenes [23]. First of all we define what we understand by ultra weak solution by this method.

A function $u = u(x,t)$ is said to be ultra solution of (2.28) or solution by transposition if

$$u \in C^0([0,T]; H^{-1}(\Omega_t)) \cap L^2(0,T; L^2(\Omega_t)) \tag{2.29}$$

and

$$\int_0^T \int_{\Omega_t} u(x,t)h(x,t)\, dxdt = \langle u^0, \varphi(0)\rangle \text{ for all } h \in L^2(\widehat{Q}) \tag{2.30}$$

where φ is the unique strong solution of the adjoint system

$$\left| \begin{array}{l} -\varphi' - \Delta\varphi + a\varphi = h \text{ in } \Omega_t \text{ for } 0 < t < T \\ \varphi = 0 \text{ on } \Gamma_t \text{ for } 0 < t < T \\ \varphi(x,T) = 0 \text{ in } \Omega \end{array} \right. \tag{2.31}$$

Here, $\langle\ ,\ \rangle$ denotes the duality passing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

According to Theorem 2.1, the system (2.31) admits a unique strong solution φ. Thus the definition of ultra weak solution makes sense.

Note that the strong solution φ satisfies the following estimates:

$$||\varphi||_{L^\infty(0,T;H_0^1(\Omega_t))} \leq c|h|_{L^2(\widehat{Q})} \tag{2.32}$$

and

$$||\varphi||_{L^\infty(0,T;H_0^1(\Omega_t))} \leq c||h||_{L^1(0,T;H_0^1(\Omega_t))} \tag{2.33}$$

These estimates were proved in Theorem 2.1. Indeed, it is sufficient to make the change of variables $t \to T - t$ to reduce the system (2.31) to (2.1).

By Riesz-Fréchet theorem we deduce that there exists a unique ultra weak solution in the class (2.29). More precisely, in view of (2.32) we deduce the existence of unique solution $u \in L^2(\widehat{Q})$ and the second estimate (2.33) provides the aditional regularity $u \in L^\infty(0,T;H^{-1}(\Omega_t))$. Moreover, one deduces the existence of a constant, independent of u^0, such that

$$||u||_{L^\infty(0,T;H^{-1}(\Omega_t))} + |u|_{L^2(\widehat{Q})} \leq c|u^0|_{H^{-1}(\Omega)} \,. \tag{2.34}$$

In order to show that $u \in C^0([0,T];H^{-1}(\Omega_t))$ we use a classical density argument. When u^0 is smooth enough, u is a weak or strong solution, therefore u is continuous with respect to time with values in $H^{-1}(\Omega_t)$. According to (2.34), by density, we deduce that $u \in C^0([0,T];H^{-1}(\Omega_t))$ whenever $u^0 \in H^{-1}(\Omega)$. □

To complete this section, we observe that when u^0 is smooth so that exist weak or strong solution then they are also ultra weak solutions. It is sufficient to integrate by parts in the strong formulation of (2.1) or consider the weak formulation. □

2.3 Observability of the Linearized Adjoint System

As we said before we employ a fixed point argument in order to prove our results in the semilinear case. However, first we analyse the null controllability for the following linearized system:

$$\left|\begin{array}{l} u' - \Delta u + a(x,t)u = h(x,t)\chi_{\hat{q}} \text{ in } \widehat{Q} \\ u = 0 \text{ on } \widehat{\Sigma} \\ u(x,0) = u_0(x) \text{ in } \Omega, \end{array}\right. \tag{2.35}$$

where the potential $a = a(x,t)$ is assumed to be in $L^\infty(\widehat{Q})$. Remember we denote by $\hat{u}_t$ the cross section of $\hat{q}$ at any $0 < t < T$.

As we know, the null controllability of (2.35) is equivalent to a suitable observability property for the adjoint system of (2.35).

Thus, let us consider the adjoint system

$$(2.36) \qquad \left| \begin{array}{l} -\varphi' - \Delta\varphi + a\varphi = 0 \text{ in } \widehat{Q} \\ \varphi = 0 \text{ on } \widehat{\Sigma} \\ \varphi(T) = \varphi^0 \text{ in } \Omega_T \end{array} \right.$$

for which we have the following observability property.

Proposition 2.1. For all $T > 0$ and $R > 0$ there exists a positive constant $C > 0$ such that

$$(2.37) \qquad |\varphi(0)|^2_{L^2(\Omega)} \leq C \int_{\hat{q}} |\varphi(x,t)|^2 \, dxdt,$$

for every solution of (2.36) and for any $a \in L^\infty(\widehat{Q})$ such that $||a||_{L^\infty(\widehat{Q})} \leq R$.

Remark 2.2. The constant C in (2.37) will be referred to as the observability constant. It depends on $\widehat{Q}$, $\hat{q}$ the time T and the size R of the potential but does not depend of the solution φ of (2.36). $\square$

Proof of the Proposition 2.1: The inequality (2.37) is a consequence of the results in [17]. In fact, by the change of variables $x \to y$, from $\widehat{Q}$ into Q, the adjoint system (2.36) is transformed into a variable coefficient parabolic equation of the form

$$(2.38) \qquad \left| \begin{array}{l} -\psi' + A(t)\psi + b \cdot \nabla\psi + a\psi = 0 \text{ in } Q \\ \psi = 0 \text{ in } \Sigma \\ \psi(T) = \psi^0 \text{ in } \Omega_T \end{array} \right.$$

as in (2.12) with $\tilde{h} = 0$. Thus the coefficient of the principal part $A(t)$, according to the assumption (A1) and (A2) are of class C^1 and a and b are bounded. Then, the observability inequalities in [17] guarantee that for every $T > 0$ and every open subset q of Q, there exists a constant $C > 0$ such that

$$|\psi(0)|_{L^2(\Omega)} \leq C \int_q |\psi(y,t)|^2 \, dydt. \tag{2.39}$$

In particular it is true for $q \subset Q$ image of $\hat{q}$ by $x \to y$. Thus estimate (2.37) for φ is obtained from (2.39) for ψ by the change of variables $y \to x$. □

2.4 Approximate Controllability for the Linearized System

From the observability inequality (2.37) the null controllability result for the linearized system can be proved as the limit of an approximate controllability property.

In fact, given $u^0 \in L^2(\Omega)$ and $\delta > 0$ we introduce the quadratic functional

$$J_\delta(\varphi^0) = \frac{1}{2} \int_{\hat{q}} \varphi^2 \, dxdt + \delta |\varphi^0|_{L^2(\Omega_T)} + \int_\Omega u^0 \varphi(0) \, dx, \tag{3.40}$$

where φ is the solution of (2.36) with initial data φ^0. The functional J_δ is continuous and strictly convex in $L^2(\Omega_t)$. Moreover, J_δ is coercive. More precisely, in view of (2.37) we have

$$\liminf_{|\varphi^0|_{L^2(\Omega_T)} \to \infty} \frac{J_\delta(\varphi^0)}{|\varphi^0|_{L^2(\Omega_T)}} \geq \delta. \tag{2.41}$$

To prove (2.41) we follows the argument used in [27] which we will not repeat here.

Thus J_δ has a unique minimizer in $L^2(\Omega_T)$. Let us denote it by $\hat{\varphi}^{0,\delta}$. It is not difficult to prove that the control $h_\delta = \hat{\varphi}^\delta$, where $\hat{\varphi}^\delta$ is the solution of (2.36) associated to the minimizer $\hat{\varphi}^{0,\delta}$ is such that the solution u_δ of (2.1) satisfies

$$|u_\delta(T)|_{L^2(\Omega_T)} \leq \delta. \tag{2.42}$$

We refer to [12] for the details of the proof. □

2.5 Null Controllability of the Linearized System

The null controllability property may be obtained as the limit when δ tends to zero of the approximate controllability property above obtained. However, to pass to the limit we need a uniform bound of the control. To obtain this bound we observe that, by (2.37),

$$J_\delta(\varphi^0) \geq \frac{1}{2}\int_{\hat{q}} \varphi^2\, dxdt - C\left[\int_{\hat{q}} \varphi^2\, dxdt\right]^{1/2} |u^0|_{L^2(\Omega)}\,, \tag{2.43}$$

when $C > 0$ is independent of δ. On the order hand,

$$J_\delta(\hat{\varphi}^{0,\delta}) \leq J_\delta(0) = 0. \tag{2.44}$$

Writing (2.43) for $\hat{\varphi}^{0,\delta}$ instead of φ, with $\varphi^{0,\delta}$ the minimizer of J_δ in $L^2(\Omega_T)$ and combining it with (2.44), we deduce that

$$|h_\delta|_{L^2(0,T;L^2(\Omega_t))} \leq 2C|u^0|_{L^2(\Omega)}\,, \tag{2.45}$$

for all $\delta > 0$.

In other words, h_δ remains bounded in $L^2(0,T;L^2(\Omega_t))$ as $\delta \to 0$. Note also that the constant C in (2.45) is independent of $||a||_{L^\infty(\widehat{Q})} \leq R$.

Extracting a sub net h_δ deduce that

$$h_\delta \rightharpoonup h, \text{ as } \delta \to 0, \text{ weakly in } L^2(\widehat{Q}), \tag{2.46}$$

for some $h \in L^2(\widehat{Q})$.

We can prove that the limit h is such that the solution u of (2.1) satisfies (1.3). Moreover, by the lower semicontinuity of the norm with respect to the weak topology and by (2.46) we deduce that

$$|h|_{L^2(\widehat{Q})} \leq \liminf_{\delta\to 0} |h_\delta|_{L^2(\widehat{Q})} \leq 2C|u^0|_{L^2(\Omega)}\,. \tag{2.47}$$

By the process we complete the proof of the following result.

Theorem 2.3. Assume that the noncylindrical domain $\widehat{Q}$ satisfies the conditions fixed in Section 1 and that $a(x,t) \in L^\infty(\widehat{Q})$. Then, for every $T > 0$ and $u^0 \in L^2(\Omega)$.

there exists $h \in L^2(\widehat{Q})$ such that the solution of (2.1) satisfies (1.3). Moreover, there exists a constant $C > 0$, depending on $R > 0$, but independent of u^0, such that (1.4) holds for every potential $a = a(x,t)$ in $L^\infty(\widehat{Q})$ such that $||a||_{L^2(\widehat{Q})} \leq R$. $\square$

3. Proof of the Main Result

This section is devoted to prove Theorem 1.1. As we said, in the Introduction, it will be a consequence of Theorem 2.3 above and a fixed point argument.

In order to be self contained we will prove existence result.

Theorem 3.1. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is C^1 and globally Lipschitz function, such that $f(0) = 0$. Let $u^0 \in L^2(\Omega)$ and $h \in L^2(\widehat{Q})$. Then, there exists a unique solution $u \in C^0([0,T]; L^2(\Omega_t)) \cap L^2(0,T; H_0^1(\Omega_t))$ of the problem (1.1).

Proof: As in the proof of Theorem 2.1 we transform the problem in a noncylindrical domain $\widehat{Q}$ into a parabolic problem with variables coefficients in the cylinder Q. In fact, the change of variables (2.7), equivalently (2.8), transforms (1.1) in

$$
(3.1) \qquad \left| \begin{array}{l} v' + A(t)v + b \cdot \nabla_y v + f(v) = h(y,t) \text{ for } (y,t) \in Q \\ v = 0 \text{ for } (y,t) \in \Sigma \\ v(y,0) = v^0(y) \text{ for } y \in \Omega. \end{array} \right.
$$

Then we know that (3.1) admits a unique solution

$$
v \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).
$$

By the change of variable $y \to x$ we deduce the existence of a unique solution u of (1.1) in the class

$$
u \in C^0([0,T]; L^2(\Omega_t)) \cap L^2(0,T; H_0^1(\Omega_t)). \qquad \square
$$

As in [12] we introduce the nonlinearity

$$g(s) = \left| \begin{array}{ll} \dfrac{f(s)}{s} & \text{if} \quad s \neq 0 \\ f'(0) & \text{if} \quad s = 0. \end{array} \right. \tag{3.2}$$

Note that g is uniformly bounded with $||g||_\infty \leq ||f'||_\infty$.

Given any $z \in L^2(\widehat{Q})$ we consider the linearized system

$$\left| \begin{array}{l} u' - \Delta u + g(z)u = h\,\chi_{\hat{q}} \text{ in } \widehat{Q} \\ u = 0 \text{ on } \widehat{\Sigma} \\ u(x,0) = u^0(x) \text{ in } \Omega \end{array} \right. \tag{3.3}$$

Observe that (3.3) is a linear systsem in the state $u = u(x,t)$ with potential $a = g(z) \in L^\infty(\widehat{Q})$, satisfying the condition

$$||a||_{L^\infty(\widehat{Q})} \leq ||f'||_{L^\infty(\mathbb{R})}. \tag{3.4}$$

With this notation, the system (3.3) can be written as

$$\left| \begin{array}{l} u' - \Delta u + au = h\,\chi_{\hat{q}} \text{ in } \widehat{Q} \\ u = 0 \text{ on } \widehat{\Sigma} \\ u(x,0) = u^0(x) \text{ in } \Omega. \end{array} \right. \tag{3.5}$$

By the subsection 2.4, if $\delta > 0$ is fixed, for each $z \in L^2(\widehat{Q})$ we can define a control $h_\delta = h_\delta(x,t) \in L^2(\widehat{Q})$ such that the solution u_δ of (3.5) satisfies

$$|u_\delta(T)|_{L^2(\Omega_T)} \leq \delta, \tag{3.6}$$

see (2.42).

Moreover, for every $R > 0$ and all potential $a = a(x,t) \in L^\infty(\widehat{Q})$ such that $||a||_{L^\infty(\widehat{Q})} \leq R$, we have

$$|h_\delta|_{L^2(\widehat{Q})} \leq |u^0|_{L^2(\Omega)}, \tag{3.7}$$

for all $\delta > 0$. Therefore, the controls h_δ are uniformly bounded (with respect to z and δ) in $L^2(\widehat{Q})$.

This result allows to define a nonlinear mapping

$$N_\delta(z) = u, \text{ from } L^2(\widehat{Q}) \text{ into } L^2(\widehat{Q}) \tag{3.8}$$

where u satisfies (3.5) and (3.6).

In this way, the approximate controllability problem for (1.1) is reduced to find a fixed point for the map N_δ. Indeed, if $z \in L^2(\widehat{Q})$ is such that $N_\delta(z) = u = z$, u solution of (3.3) is solution of (1.1) Then the control $h_\delta = h_\delta(z)$ is the one we were looking for, since, by construction, $u_\delta = u_\delta(z)$ satisfies (3.6).

As we shall see, the nonlinear map N_δ satisfies the following properties:

$$N_\delta \quad \text{is continuous and compact,} \tag{3.9}$$

(3.10) The range of N_δ is bounded, i.e., exists $M > 0$ such that $|N_\delta(z)|_{L^2(\widehat{Q})} \leq M$ for all $z \in L^2(\widehat{Q})$.

Therefore, by (3.9), (3.10) and Schauder fixed point theorem, it follows that N_δ is a fixed point.

By the moment, assume that (3.9) and (3.10) are true which proof comes after. Then if (3.9) and (3.10) are true it follows, by Schauder's fixed point theorem, that we have a control h_δ in $L^2(\widehat{Q})$ such that the solution u_δ of

$$\left| \begin{array}{l} u_\delta' - \Delta u_\delta + f(u_\delta) = h_\delta(x,t)\chi_{\hat{q}} \text{ in } \widehat{Q} \\ u_\delta = 0 \text{ on } \widehat{\Sigma} \\ u_\delta(x,0) = u^0(x) \text{ in } \Omega. \end{array} \right. \tag{3.11}$$

satisfies

$$|u_\delta(T)|_{L^2(\Omega_T)} \leq \delta, \tag{3.12}$$

with an estimate of the form

$$|h_\delta|_{L^2(\widehat{Q})} \leq C|u^0|_{L^2(\Omega)}, \, , \tag{3.13}$$

with C independent of δ.

Passing to the limit as $\delta \to 0$, as in Section 2, we deduce the existence of a limit control $h \in L^2(\widehat{Q})$ such that the solution u of (1.1) satisfies (1.3) and (1.4).

To complete the argument we need to prove (3.9) and (3.10).

Continuity of N_δ. Assume that $z_j \to z$ in $L^2(\widehat{Q})$. Then the potential $a_j = g(z_j)$ is such that

$$a_j = g(z_j) \to a = g(z) \text{ in } L^2(\widehat{Q}). \tag{3.14}$$

In fact, we have

$$|g(z_j)| = \frac{|f(z_j)|}{|z_j|} \leq K_0 \, , \tag{3.15}$$

by hypothesis, K_0 the Lipschitz constant of f, then $|g(z_j)|^p \leq K_0^p \, , \;\; 1 \leq p < \infty$.

We also have $z_j \to z$ in $L^2(\widehat{Q})$ and consequently a subsequence $z_j \to z$ a.e. in $\widehat{Q}$. Then

$$|g(z_j)|^p = \frac{|f(z_j)|^p}{|z_j|^P} \to \frac{|f(z)|^p}{|z|^p} \quad \text{a.e. in } \widehat{Q}.$$

It follows by Lebesgue's bounded convergence theorem that

$$g(z_j) \to g(z) \quad \text{in} \quad L^p(\widehat{Q})$$

for all $1 \leq p < \infty$, that is $a_j \to a$ in $L^p(\widehat{Q})$. According to Theorem 2.3 the corresponding control are uniformly bounded, i.e.,

$$|h|_{L^2(\widehat{Q})} \leq C, \quad \text{for all} \quad y \geq 1, \tag{3.16}$$

and, more precisely,

$$h_j = \hat{\varphi}_j \quad \text{in} \quad \hat{q}, \tag{3.17}$$

where $\hat{\varphi}$ solves

$$\left| \begin{array}{l} -\varphi' - \Delta\varphi + g(z_j)\varphi = 0 \quad \text{in} \quad \widehat{Q} \\ \varphi = 0 \quad \text{on} \quad \widehat{\Sigma} \\ \varphi(x,T) = \hat{\varphi}_j^0 \quad \text{in} \quad \Omega_T\,, \end{array} \right. \tag{3.18}$$

with the initial data $\hat{\varphi}_j^0$ that minimizes the correspondent functional J_δ in $L^2(\widehat{Q})$.

We also have

$$|\hat{\varphi}_j^0|_{L^2(\Omega_T)} \leq C. \tag{3.19}$$

By extracting a subsequence $(\hat{\varphi}_j^0)$ we have

$$\hat{\varphi}_j^0 \rightharpoonup \hat{\varphi}^0 \quad \text{weakly in} \quad L^2(Q_T). \tag{3.20}$$

From (3.18), (3.14) and (3.15) we have a subsequence still represented by $(\hat{\varphi}_j)$ such that

$$\hat{\varphi}_j \rightharpoonup \chi \quad \text{weakly in} \quad L^2(0,T;H_0^1(\Omega_t)).$$

We will prove that $\chi = \hat{\varphi}$ and $\hat{\varphi}$ solves

$$\left| \begin{array}{l} -\varphi' - \Delta\varphi + g(z)\varphi = 0 \quad \text{in} \quad \widehat{Q} \\ \varphi = 0 \quad \text{on} \quad \widehat{\Sigma} \\ \varphi(x,T) = \hat{\varphi}^0 \quad \text{in} \quad \Omega_T\,. \end{array} \right.$$

It is sufficient to prove that

$$g(z_j)\hat{\varphi}_j \rightharpoonup g(z)\hat{\varphi} \quad \text{weakly in} \quad L^2(\Omega \times (0,T)).$$

In fact, with the change of variables $y \to x$ from $\widehat{Q}$ into Q the system (3.18) is transformed in one system in $\psi_j(x,t) = \varphi_j(y,t)$ with $y = \tau_t(x)$, as follows:

$$\left| \begin{array}{l} -\psi_j' + A(t)\psi_j + a_j\psi_j + b \cdot \nabla\psi_j = 0 \quad \text{in} \quad Q \\ \psi_j = 0 \quad \text{on} \quad \Sigma \\ \psi_j(T) = \hat{\psi}_j^0 \quad \text{in} \quad \Omega. \end{array} \right.$$

For the parabolic problem for ψ_j we obtain estimates in the cylinder which permits to employ compacteness argument of the type Lions-Aubin for ψ_j . When we change the variables $y \to x$ we obtain subsequence $(\hat{\varphi}_j)$ in $L^2(\widehat{Q})$ such that

$$\hat{\varphi}_j \to \hat{\varphi} \quad \text{strong} \quad L^2(\widehat{Q}).$$

This implies that

$$g(z_j)\hat{\varphi}_j \to g(z)\hat{\varphi} \quad \text{weakly in} \quad L^2(\widehat{Q}).$$

Therefore,

$$h_j \to h \quad \text{in} \quad L^2(\widehat{Q})$$

where

$$h = \hat{\varphi} \quad \text{in} \quad \hat{q}.$$

Note that u_j and u solve (3.11), what implies, by the estimates, that

$$u_j \to u \quad \text{in} \quad L^2(\widehat{Q}),$$

where u solves

$$\left| \begin{array}{l} u' - \Delta u + g(z)u = h\,\chi_{\hat{q}} \quad \text{in} \quad \widehat{Q} \\ u = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{array} \right. \tag{3.21}$$

and

$$|u(T)|_{L^2(\Omega)} \leq \delta. \tag{3.22}$$

To complete the proof of the continuity of N_δ it is sufficient to check that the limit $\hat{\varphi}^0$ obtained in (3.20) is the minimizer of the functional J_δ associated to the limit control problem (3.21) and (3.22).

To do this, given $\psi^0 \in L^2(\Omega_T)$ we have to show that

$$J_\delta(\hat{\varphi}^0) \leq J_\delta(\psi^0). \tag{3.23}$$

In fact, by weak lower continuity, we have

$$J_\delta(\hat{\varphi}^0) \leq \liminf_{j\to\infty} J_{\delta,j}(\hat{\varphi}^0_j). \tag{3.24}$$

We also have

$$J_\delta(\psi^0) = \liminf_{j\to\infty} J_{\delta,j}(\psi^0), \quad \text{for all} \quad \psi^0 \in L^2(\Omega_T).$$

But,

$$J_{\delta,j}(\hat{\varphi}^0_j) \leq J_{\delta,j}(\psi^0) \quad \text{for all} \quad \psi^0 \in L^2(\Omega_T), \tag{3.25}$$

because $\hat{\varphi}^0_j$ is the minimizer of $J_{\delta,j}$. Thus, by (3.25) and (3.24) we obtain (3.23). □

Compactness of $\boldsymbol{N_\delta}$. The above argument says that when z varies in a bounded set B of $L^2(\widehat{Q})$ implies that $u = N_\delta(z)$ lies in a bounded set of $L^2(\widehat{Q})$ where u solves

$$\left| \begin{array}{l} u' - \Delta u = h\,\chi_{\hat{q}} - g(z)u \quad \text{in} \quad \widehat{Q} \\ u = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u(x,0) = u^0(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{3.34}$$

Set $\beta = h\,\chi_{\hat{q}} - g(z)u$ and (3.34) can be write for $u = w + v$, with w fixe solution of

$$\left|\begin{array}{l} w' - \Delta w = 0 \quad \text{in} \quad \widehat{Q} \\ w = 0 \quad \text{on} \quad \widehat{\Sigma} \\ w(x,0) = u^0(x) \quad \text{in} \quad \Omega \end{array}\right.$$

and v solution of

$$\left|\begin{array}{l} v' - \Delta v = \beta \quad \text{in} \quad \widehat{Q} \\ v = 0 \quad \text{on} \quad \widehat{\Sigma} \\ v(x,0) = 0 \quad \text{in} \quad \Omega \end{array}\right.$$

It follows that w is fixe and belongs to

$$L^\infty(0,T;L^2(\Omega_t)) \cap L^2(0,T;H_0^1(\Omega_t)).$$

As β is uniformly bounded in $L^2(\widehat{Q})$ we have v varies in a bounded set of $L^2(0,T;$ $H_0^1(\Omega_t) \cap H^2(\Omega_t))$ and v' varies in a bounded set of $L^2(0,T;L^2(\Omega_t))$. Thus by Aubin-Lions compactness result, v varies in a relatively compact set of $L^2(0,T;L^2(\Omega_t))$. It then follows that $u = w + v$, with w in $L^2(0,T;L^2(\Omega_t))$ varies in a relatively compact set of $L^2(0,T;L^2(\Omega_t))$. $\square$

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