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On the structure of spectra of travelling waves

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Abstract

The linear stability of the travelling wave solutions of a general reaction-diffusion system is investigated. The spectrum of the corresponding second order differential operator L is studied. The problem is reduced to an asymptotically autonomous first order linear system. The relation between the spectrum of L and the corresponding first order system is dealt with in detail. The first order system is investigated using exponential dichotomies. A self-contained short presentation is shown for the study of the spectrum, with elementary proofs. An algorithm is given for the determination of the exact position of the essential spectrum. The Evans function method for determining the isolated eigenvalues of L is also presented. The theory is illustrated by three examples: a single travelling wave equation, a three variable combustion model and the generalized KdV equation.

Keywords: linear stability of travelling waves, exponential dichotomies, Evans function

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1 Introduction

We investigate the stability of the travelling wave solutions of the system

$$\partial_\tau u = D\partial_{xx}u + f(u), \tag{1}$$

where $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^m$, D is a diagonal matrix with positive diagonal elements and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a continuously differentiable function. The travelling wave solution of this equation has the form $u(\tau, x) = U(x - c\tau)$, where $U : \mathbb{R} \to \mathbb{R}^m$. For this function we have

$$DU''(z) + cU'(z) + f(U(z)) = 0, \qquad (z = x - c\tau) \tag{2}$$

The stability of U can be determined by linearization. Putting $u(\tau, x) = U(x - c\tau) + v(\tau, x - c\tau)$ in (1) the linearized equation for v takes the form

$$\partial_\tau v = D\partial_{zz}v + c\partial_z v + f'(U(z))v. \tag{3}$$

According to well known stability theorems (see e.g. [21]) the stability of the travelling wave solution U is determined by the spectrum of the second order differential operator

$$(LV)(t) = DV''(t) + cV'(t) + Q(t)V(t), \tag{4}$$

defined in $C_0(\mathbb{R}, \mathbb{C}^m) \cap C^2(\mathbb{R}, \mathbb{C}^m)$, where $Q(t) = f'(U(t))$,

$$C_0(\mathbb{R}, \mathbb{C}^m) = \{V : \mathbb{R} \to \mathbb{C}^m \mid V \text{ is continuous}, \lim_{t\to\pm\infty} V(t) = 0\},$$

endowed with the supremum norm $\|V\| = \max_{\mathbb{R}} |V(t)|$. It is assumed that $Q : \mathbb{R} \to \mathbb{C}^{m\times m}$ is continuous, and the limits $Q^{\pm} = \lim_{t\to\pm\infty} Q(t)$ exist.

The complex number $\lambda \in \mathbb{C}$ is called a regular value of L if the operator $L-\lambda I$ has bounded inverse that is defined in the whole space $C_0(\mathbb{R}, \mathbb{C}^m)$. That is for any $W \in C_0(\mathbb{R}, \mathbb{C}^m)$ there exists a unique solution of $LV - \lambda V = W$ in $C_0(\mathbb{R}, \mathbb{C}^m) \cap C^2(\mathbb{R}, \mathbb{C}^m)$, and there exists $M > 0$ such that for any $W \in C_0(\mathbb{R}, \mathbb{C}^m)$ we have $\|V\| \leq M\|W\|$. The spectrum $\sigma(L)$ of L consists of non-regular values. The number λ is called an eigenvalue if $L - \lambda I$ has no inverse. The essential spectrum of L consists of those points of the spectrum which are not isolated eigenvalues with finite multiplicity. It is useful to introduce the first order system corresponding to equation $LV - \lambda V = W$. Let $x = (V, V')^T$, $y = (0, W)^T$, then the first order system is

$$\dot{x}(t) = A_\lambda(t)x(t) + y(t), \tag{5}$$

where

$$A_\lambda(t) = \begin{pmatrix} 0 & I \\ D^{-1}(\lambda I - Q(t)) & -cD^{-1} \end{pmatrix}. \tag{6}$$

The spectrum of L has been widely investigated. The position of the essential spectrum has been estimated, see e.g. [14], and Weyl's lemma in [21]. This can be done using exponential dichotomies for the first order system (5), [5, 18]. This approach was also generalized to infinite dimensional systems, when A is a bounded operator on a Banach space [6] and also for unbounded operators [4]. Fredholm properties are also relevant when determining the spectrum of L [11, 18]. The relation between these two concepts is dealt with in [16, 18]. The determination of the isolated eigenvalues requires to solve a linear system with non-constant coefficients, which can be done in general only numerically. For the investigation of the isolated eigenvalues two concepts were introduced. The first was the Evans function [10], which is an analytic function on the complex plane, the zeros of which are the isolated eigenvalues of L. Later a topological invariant was introduced for the study of the spectrum [1]. These methods were applied for several systems in physics [3, 12, 17], chemistry [2, 7, 13, 19] and biology [10, 15].

The aim of the paper is to present a self-contained detailed study of the spectrum of L, and to fill the gap between the abstract results (on exponential dichotomies and on topological invariants) and the applications. The novelties of the paper are as follows.

- An algorithm is given for the determination of the exact position of the essential spectrum. The statements concerning the essential spectrum are proved by elementary methods. (Most of the known results give only sufficient conditions for the essential spectrum to lie in the left half plane.)
- All the theorems are proved in the finite dimensional case. The presentation does not need abstract techniques, hence for those applying the theory a self-contained method is shown. (According to the author's knowledge a self-contained explanation, including the proofs, is only given for the case of unbounded operators [4]. The proof of the finite dimensional case must be compiled from different sources, e.g. [5, 8, 14, 16].)
- The relation between the spectrum of L and the corresponding first order system is dealt with in detail. (The standard reference in this context is [14] but the relation is not proved in that book.)

2 Relation between the spectrum of L and the first order system

Lemma 1 connects the determination of the spectrum of L with the study of system (5). In order to prove the lemma we will need the following Proposition.

Proposition 1 *If $W \in C_0(\mathbb{R}, \mathbb{C}^m)$ and $V \in C_0(\mathbb{R}, \mathbb{C}^m)$ is the solution of $LV - \lambda V = W$, then $V' \in C_0(\mathbb{R}, \mathbb{C}^m)$.*

Proof We will show that $V' \to 0$ at $+\infty$, it can be verified similarly, that $V' \to 0$ at $-\infty$. Let us denote by z the real or imaginary part of the k-th coordinate of V, (k is arbitrary), and let $d = D_{kk}$. We will prove that $\lim_{t\to+\infty} \dot{z}(t) = 0$ that implies the desired statement. First we prove in the case $c \neq 0$. Since V and W tend to 0 at $+\infty$, for any $\varepsilon > 0$ there exists t_0 such that

$$-\varepsilon < d\ddot{z}(t) + c\dot{z}(t) < \varepsilon \qquad \text{for all } t > t_0.$$

Dividing by d and multiplying by $\exp(ct/d)$ we obtain

$$-\frac{\varepsilon}{d}\mathrm{e}^{\frac{ct}{d}} < (\dot z(t)\mathrm{e}^{\frac{ct}{d}})' < \frac{\varepsilon}{d}\mathrm{e}^{\frac{ct}{d}} \qquad \text{for all } t > t_0.$$

Integrating in $[t_0, t]$ we get

$$-\frac{\varepsilon}{c}(1-\mathrm{e}^{\frac{c(t_0-t)}{d}}) + \dot z(t_0)\mathrm{e}^{\frac{c(t_0-t)}{d}} < \dot z(t) < \frac{\varepsilon}{c}(1-\mathrm{e}^{\frac{c(t_0-t)}{d}}) + \dot z(t_0)\mathrm{e}^{\frac{c(t_0-t)}{d}} \qquad \text{for all } t > t_0. \tag{7}$$

In the case $c > 0$, there exists $t_1 > t_0$ for which

$$-2\frac{\varepsilon}{c} < \dot z(t) < 2\frac{\varepsilon}{c} \qquad \text{for all } t > t_1$$

yielding $\lim_{+\infty} \dot z = 0$, what we wanted to prove.

In the case $c < 0$ we prove by contradiction. Assume that there exists $\alpha > 0$ and a sequence $t_n \to \infty$, such that $|\dot z(t_n)| = \alpha$. Let $\varepsilon = -c\alpha/2$ and apply (7) for $t_0 = t_n$ when n is large enough. If $\dot z(t_n) = \alpha$, then the inequality in the left hand side of (7) yields for $t > t_n$ that

$$\dot z(t) > \frac{\alpha}{2}(1+\mathrm{e}^{\frac{c(t_n-t)}{d}}) \to +\infty$$

as $t \to +\infty$. This contradicts to the boundedness of z. If $\dot z(t_n) = -\alpha$, then the inequality in the right hand side of (7) yields for $t > t_n$ that

$$\dot z(t) < -\frac{\alpha}{2}(1+\mathrm{e}^{\frac{c(t_n-t)}{d}}) \to -\infty$$

as $t \to +\infty$. This contradicts to the boundedness of z.

Finally, let us consider the case $c = 0$. Then from the differential equation we get that $\ddot z$ tends to zero at infinity. According to the Landau-Kolmogorov inequality (see e.g. [9]) $\|\dot z\| \le 4\|\ddot z\|\|z\|$. Defining $\|\cdot\|$ as the supremum norm on $[T, +\infty)$ for T large enough, we get that $\dot z \to 0$ at infinity. $\square$

Lemma 1 *(i) If system (5) has a unique solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ for any $y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ and x depends continuously on y, (i.e. there exists $M > 0$, such that $\|x\| \le M(\|y\|$ for all $y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$), then λ is a regular value of L.*

(ii) If λ is a regular value of L, then for any differentiable function $y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$, for which $\dot y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$, there exists a unique solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ of system (5); and there exists $M > 0$ such that for any y satisfying the above conditions the corresponding unique solution x satisfies $\|x\| \le M(\|y\| + \|\dot y\|)$.

Proof

(i)

Let $W \in C_0(\mathbb{R}, \mathbb{C}^m)$, and $y = (0, W)^T \in C_0(\mathbb{R}, \mathbb{C}^{2m})$. Then there exists a unique solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ of (5). Let $V = (x_1, \dots, x_m)^T$, $U = (x_{m+1}, \dots, x_{2m})^T$, then $V \in C_0(\mathbb{R}, \mathbb{C}^m)$ and $U = V'$, hence V is twice differentiable, and $LV - \lambda V = W$. The continuity follows from $\|x\| \le M\|y\|$, namely

$$\|V\| \le \|x\| \le M\|y\| = M\|W\|.$$

(ii)

First we show that for any $y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ which is differentiable and $\dot y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ there exists a unique solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ of (5). Let $x = (V, U)^T$, $y = (y_1, y_2)^T$, where $V, U, y_1, y_2 : \mathbb{R} \to \mathbb{C}^m$, then system (5) takes the form

$$\begin{aligned} \dot V &= U + y_1 \\ \dot U &= D^{-1}(\lambda I - Q)V - cD^{-1}U + y_2 \end{aligned}$$

The differentiability of y_1 implies that V is twice differentiable and $\ddot V = \dot U + \dot y_1$, hence from the second equation

$$D\ddot V + c\dot V + QV - \lambda V = D\dot y_1 + cy_1 + y_2.$$

Since λ is a regular value of L, this equation has a unique solution $V \in C_0(\mathbb{R}, \mathbb{C}^m)$, and there exists $M_1 > 0$ for which

$$\|V\| \le M_1(\|\dot{y}_1\| + \|y_1\| + \|y_2\|). \tag{8}$$

Moreover, according to Proposition 1 $\dot{V} \in C_0(\mathbb{R}, \mathbb{C}^m)$, yielding $U = \dot{V} - y_1 \in C_0(\mathbb{R}, \mathbb{C}^m)$. Thus $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ what we wanted to prove.

Now we show the continuous dependence of x on y. Let us denote the maximum point of $\dot{V}_k$ (for some $k = 1, \dots, m$) by t_k. Since $\dot{V}_k \to 0$ at $\pm\infty$, the maximum point is not at infinity, hence $\ddot{V}_k(t_k) = 0$. Therefore using (8) we get that there exists $M_2 > 0$ for which

$$|\dot{V}_k(t_k)| \le M_2(\|\dot{y}_1\| + \|y_1\| + \|y_2\|).$$

Similar inequality can be proved for the minimum of $\dot{V}_k$, and for all coordinates of V, hence there exists $M_3 > 0$ for which $\|\dot{V}\| \le M_3(\|\dot{y}_1\| + \|y_1\| + \|y_2\|)$. Since $U = \dot{V} - y_1$, there exists $M_4 > 0$ for which $\|U\| \le M_4(\|\dot{y}_1\| + \|y_1\| + \|y_2\|)$. Thus with $M = 2(M_1 + M_4)$ we have $\|x\| \le M(\|y\| + \|\dot{y}\|)$. $\square$

Now we turn to the study of general first order systems, the special case of which is system (5).

3 First order systems

Now for short let $C_0 = C_0(\mathbb{R}, \mathbb{C}^n)$ endowed with the supremum norm $\|x\| = \max_{\mathbb{R}} |x(t)|$, and let $A : \mathbb{R} \to \mathbb{C}^{n \times n}$ be a continuous function for which the limits

$$A^{\pm} = \lim_{t \to \pm\infty} A(t)$$

exist. Let us consider the first order system

$$\dot{x}(t) = A(t)x(t) + y(t). \tag{9}$$

Our aim is to give necessary and sufficient condition for the existence and uniqueness of a solution $x \in C_0$ of (9) for any $y \in C_0$, and for the continuous dependence of x on y.

Since A is continuous, system (9) has solutions for any $y \in C_0$, that can be given by the variation of constants formula as

$$x(t) = \Psi(t)x_0 + \int_0^t \Psi(t)\Psi(s)^{-1}y(s)\mathrm{d}s, \tag{10}$$

where Ψ is the fundamental system of solutions of the homogeneous equation satisfying $\Psi(0) = I$, i.e. the n columns of the matrix $\Psi(t)$ are n independent solutions of the homogeneous system

$$\dot{x}(t) = A(t)x(t). \tag{11}$$

Hence the question is that for a given $y \in C_0$ does there exist a unique $x_0 \in \mathbb{C}^n$, such that $x \in C_0$ (x is given by (10)), and that does x depend continuously on y.

The dimension of the stable, unstable and central subspaces of the matrices $A^{\pm}$ play important role. Let us denote the number of eigenvalues (with multiplicity) of A^+ with positive, negative, zero real part by n_u^+, n_s^+, n_c^+, respectively. We define n_u^-, n_s^-, n_c^- similarly using A^-.

First we show that the continuous dependence is violated when $n_c^+ > 0$ or $n_c^- > 0$. The main point in these cases is to prove the existence of a bounded solution of the homogeneous equation, which does not tend to zero.

Theorem 1 *Let us assume that at least one of the following two conditions holds:*

(a) $n_c^+ > 0$ *and* $\int_0^{+\infty} |A(t) - A^+| < \infty$

(b) $n_c^- > 0$ *and* $\int_{-\infty}^0 |A(t) - A^-| < \infty$.

Then the solution of (9) does not depend continuously on y, in the sense that there is no $M > 0$ for which $\|x\| \le M(\|y\| + \|\dot{y}\|)$ holds for any differentiable function $y \in C_0$, for which $\dot{y} \in C_0$.

Proof We prove in case (a), the other case is similar. According to Theorem 1.10.1 of [8] there exists $z_0 \in \mathbb{C}^n$, such that the solution $t \mapsto \Psi(t)z_0$ of (11) is bounded in $[0, +\infty)$ but does not tend to zero as $t \to +\infty$. Then there exist $a > 0$ and a sequence $t_k \to +\infty$, such that $|\Psi(t_k)z_0| = a$ for all $k = 1, 2, \ldots$. Let $h_k : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions satisfying the following conditions

$$h_k|_{(-\infty,0]} = 0, \quad |h_k| \leq 1, \quad |\dot{h}_k| \leq 1, \quad \int_0^{t_k} h_k = 1, \quad \int_0^{+\infty} h_k = 0, \quad \lim_{+\infty} h_k = 0 \quad \lim_{+\infty} \dot{h}_k = 0.$$

Let $y_k(t) = h_k(t)\Psi(t)z_0$, then $y_k \in C_0$, y_k is differentiable and

$$\dot{y}_k(t) = \dot{h}_k(t)\Psi(t)z_0 + h_k(t)\dot{\Psi}(t)z_0 = \dot{h}_k(t)\Psi(t)z_0 + h_k(t)A(t)\Psi(t)z_0.$$

Hence $\dot{y}_k \to 0$ at $+\infty$, since $h_k \to 0$, $\dot{h}_k \to 0$ and $A(t)$ is bounded. On the other hand, there exists $M_1 > 0$ for which $\|\dot{y}_k\| \leq M_1$ for all $k = 1, 2, \ldots$. Let

$$x_k(t) = \int_0^t \Psi(t)\Psi(s)^{-1} y_k(s) \mathrm{d}s$$

Then x_k is a solution of (9) belonging to y_k, and $x_k(t) = \Psi(t)z_0 \int_0^t h_k(s)\mathrm{d}s$, hence $x_k \in C_0$. However,

$$\|x_k\| = \max_{t\in\mathbb{R}} |x_k(t)| \geq |x_k(t_k)| = t_k a.$$

Since $\|y_k\| \leq \|\Psi(\cdot)z_0\|$ for all $k = 1, 2, \ldots$, the inequality $\|x_k\| \leq M(\|y_k\| + \|\dot{y}_k\|)$ would mean $t_k a \leq M(\|\Psi(\cdot)z_0\| + M_1)$ for all $k = 1, 2, \ldots$, which contradicts to $t_k \to +\infty$. □

In the further considerations we will assume $n_c^+ = 0$ or $n_c^- = 0$. In this case we will use exponential dichotomies to answer the above question.

Definition 1 System (11) possesses an exponential dichotomy in the interval J if there exist a projection P and positive numbers K, L, α, β, such that

$$\|\Psi(t)P\Psi(s)^{-1}\| \leq K\mathrm{e}^{-\alpha(t-s)} \qquad \text{for } t \geq s, \ t, s \in J \tag{12}$$

$$\|\Psi(t)(I-P)\Psi(s)^{-1}\| \leq L\mathrm{e}^{-\beta(s-t)} \qquad \text{for } s \geq t, \ t, s \in J \tag{13}$$

We will show that the exponential dichotomy on $\mathbb{R}$ implies the existence, uniqueness and continuous dependence of the solution of (9). It can be shown that system (11) possesses an exponential dichotomy on $\mathbb{R}$ if $A(t)$ is constant and it has no eigenvalues on the imaginary axis, that is when $n_c^+ = n_c^- = 0$. For the time dependent case there is no exponential dichotomy on $\mathbb{R}$ in general when $n_c^+ = n_c^- = 0$. However, we will show that system (11) possesses exponential dichotomies on $\mathbb{R}_+$ and on $\mathbb{R}_-$. If the projections of these two dichotomies are the same, then system (11) possesses an exponential dichotomy on $\mathbb{R}$ as well, and the existence, uniqueness and continuous dependence follows.

We will use the following properties of exponential dichotomies.

Lemma 2 *(i) Let P_1 and P_2 be projections, for which $Im\, P_1 = Im\, P_2$. If system (11) possesses an exponential dichotomy in the interval J with the projection P_1, then it possesses an exponential dichotomy in the interval J with the projection P_2 as well.*

(ii) System (11) possesses an exponential dichotomy in any closed and bounded interval $[a, b]$, with any projection.

(iii) If system (11) possesses an exponential dichotomy in the intervals $(a, b]$ and $[b, c)$ with the same projection P, then it possesses an exponential dichotomy in the interval (a, c) (here a can be $-\infty$, and c can be $+\infty$).

Proof

(i) This statement is proved in [5] pp. 16.

(ii) The function $(t,s)\mapsto \|\Psi(t)P\Psi(s)^{-1}\|\mathrm{e}^{\alpha(t-s)}$ is continuous on the square $[a,b]\times[a,b]$, hence it has a maximum K, implying (12). Inequality (13) follows similarly.
(iii) We only have to prove that (12) holds for $t\in[b,c)$, $s\in(a,b]$, and that (13) holds for $s\in[b,c)$, $t\in(a,b]$. We will show only the first one, the proof of the second one is similar. Thus let $a<s\le b\le t<c$. Then

$$\|\Psi(t)P\Psi(s)^{-1}\| = \|\Psi(t)P\Psi(b)^{-1}\Psi(b)P\Psi(s)^{-1}\| \le K\mathrm{e}^{-\alpha(t-b)}K\mathrm{e}^{-\alpha(b-s)} = K^2\mathrm{e}^{-\alpha(t-s)}.$$

The following Lemma can be proved using the results in [5].

Lemma 3 *(i) If $n_c^+=0$, then system (11) possesses an exponential dichotomy in $[0,+\infty)$, the projection of which is denoted by P^+. Moreover, $\dim(\ker P^+)=n_u^+$, $\dim(Im\, P^+)=n_s^+$.*

(ii) If $n_c^-=0$, then system (11) possesses an exponential dichotomy in $(-\infty,0]$, the projection of which is denoted by P^-. Moreover, $\dim(\ker P^-)=n_u^-$, $\dim(Im\, P^-)=n_s^-$.

Proof We prove only (i), the second statement can be verified similarly. Since $n_c^+=0$, system $\dot x = A_+x$ with constant coefficients possesses an exponential dichotomy in $[0,+\infty)$ the projection of which, denoted by P^+, has an n_s^+ dimensional range and n_u^+ dimensional kernel, see [5] pp.10. For any $\delta>0$ there exists $t_0>0$, such that $\|A(t)-A_+\|<\delta$ for $t>t_0$. Hence according to proposition 1 of [5] on pp. 34 system (11) possesses an exponential dichotomy in $[t_0,+\infty)$ with projection P^+. Applying statements (ii) and (iii) of Lemma 2 we can complete the proof. □

Let us introduce the following subspaces.

$$E_s^+=\mathrm{Im}\,P^+,\quad E_u^+=\ker P^+=\mathrm{Im}\,(I-P^+),\quad E_s^-=\mathrm{Im}\,P^-,\quad E_u^-=\ker P^-=\mathrm{Im}\,(I-P^-).$$

According to the next Proposition the subspace E_s^+ consists of those initial conditions x_0, for which the solution of the homogeneous equation tend to 0 as $t\to+\infty$. Similarly, the subspace E_u^- consists of those initial conditions x_0, for which the solution of the homogeneous equation tend to 0 as $t\to-\infty$.

Proposition 2 *(i) Assume $n_c^+=0$. Then $\lim_{t\to+\infty}\Psi(t)x_0=0$ if and only if $x_0\in E_s^+$.*

(ii) Assume $n_c^-=0$. Then $\lim_{t\to-\infty}\Psi(t)x_0=0$ if and only if $x_0\in E_u^-$.

Proof We prove only (i), the second statement can be verified similarly. For $x_0\in E_s^+$ we have $P^+x_0=x_0$, hence setting $s=0$ in (12)

$$|\Psi(t)x_0|\le K\mathrm{e}^{-\alpha t}|x_0| \qquad t>0.$$

Thus for $x_0\in E_s^+$ we have proved $\lim_{t\to+\infty}\Psi(t)x_0=0$.

Let us assume now $\lim_{t\to+\infty}\Psi(t)x_0=0$. Since $x_0=P^+x_0+(I-P^+)x_0$ and $P^+x_0\in E_s^+$ we have

$$0=\lim_{t\to+\infty}\Psi(t)x_0=\lim_{t\to+\infty}\Psi(t)P^+x_0+\lim_{t\to+\infty}\Psi(t)(I-P^+)x_0=\lim_{t\to+\infty}\Psi(t)(I-P^+)x_0 \tag{14}$$

Applying (13) to $\Psi(s)(I-P^+)x_0$ and setting $t=0$ we get

$$|(I-P^+)x_0|L^{-1}\mathrm{e}^{\beta s}\le|\Psi(s)(I-P^+)x_0| \qquad s>0.$$

Therefore (14) can be satisfied only in the case $(I-P^+)x_0=0$, which means that $x_0\in E_s^+$, what we had to prove. □

Corollary 1 *Let us assume $n_c^+=n_c^-=0$. The homogeneous equation (11) has a unique solution in C_0 (it is $x\equiv0$) if and only if $\dim(E_s^+\cap E_u^-)=0$.*

Proof Let us assume $\dim(E_s^+\cap E_u^-)=0$. If for a solution $x=\Psi(\cdot)x_0$ of (11) we have $\lim_{\pm\infty}x=0$, then according to Proposition 2 $x_0\in E_s^+\cap E_u^-$, hence $x_0=0$, that is $x\equiv0$.

Let us now assume $\dim(E_s^+\cap E_u^-)>0$. Let $x_0\in E_s^+\cap E_u^-$, $x_0\neq0$. Then $x=\Psi(\cdot)x_0$ is a nonzero solution of (11) and $\lim_{\pm\infty}x=0$. □

We will need the following three lemmas. The first is a generalization of Proposition 2 to the inhomogeneous equation.

Lemma 4 *Let $y \in C_0$, $x_0 \in \mathbb{C}^n$ and let x be given by (10).*

(i) Assume $n_c^+ = 0$. Then $\lim_{+\infty} x = 0$ if and only if

$$(I - P^+)x_0 = -\int_0^{+\infty} (I - P^+)\Psi(s)^{-1}y(s)ds \tag{15}$$

(ii) Assume $n_c^- = 0$. Then $\lim_{-\infty} x = 0$ if and only if

$$P^- x_0 = \int_{-\infty}^{0} P^- \Psi(s)^{-1}y(s)ds \tag{16}$$

Proof First we show that

$$\lim_{t\to+\infty} \Psi(t)P^+x_0 + \int_0^t \Psi(t)P^+\Psi(s)^{-1}y(s)\mathrm{d}s = 0 \qquad \text{for all } x_0 \in \mathbb{C}^n. \tag{17}$$

Applying (12) for $P = P^+$ and $s = 0$ we obtain

$$|\Psi(t)P^+x_0| \le K\mathrm{e}^{-\alpha t}|x_0| \to 0 \qquad \text{as } t \to +\infty.$$

Now we prove the convergence of the second term. Let $\varepsilon > 0$ be an arbitrary positive number. Since $\lim_{+\infty} y = 0$, there exists $t_1 > 0$ such that, for $t > t_1$ we have $|y(t)| < \varepsilon\alpha/2K$. Let $t_2 > t_1$ be a number for which

$$\frac{K}{\alpha}\|y\|\mathrm{e}^{\alpha(t_1 - t)} < \frac{\varepsilon}{2} \qquad \text{for all } t > t_2.$$

Then for $t > t_2$ we have

$$\left|\int_0^t \Psi(t)P^+\Psi(s)^{-1}y(s)\mathrm{d}s\right| \le \int_0^{t_1} |\Psi(t)P^+\Psi(s)^{-1}y(s)|\mathrm{d}s + \int_{t_1}^{+\infty} |\Psi(t)P^+\Psi(s)^{-1}y(s)|\mathrm{d}s \le$$

$$\|y\|K\mathrm{e}^{-\alpha t}\int_0^{t_1} \mathrm{e}^{\alpha s}\mathrm{d}s + \frac{\varepsilon\alpha}{2K}K\mathrm{e}^{-\alpha t}\int_{t_1}^{+\infty} \mathrm{e}^{\alpha s}\mathrm{d}s < \varepsilon.$$

Thus (17) is verified. Similarly we can prove

$$\lim_{t\to-\infty}\left(\Psi(t)(I - P^-)x_0 + \int_0^t \Psi(t)(I - P^-)\Psi(s)^{-1}y(s)\mathrm{d}s\right) = 0 \qquad \text{for all } x_0 \in \mathbb{C}^n. \tag{18}$$

Now we show that (15) is equivalent to

$$\lim_{t\to+\infty}\left(\Psi(t)(I - P^+)x_0 + \int_0^t \Psi(t)(I - P^+)\Psi(s)^{-1}y(s)\mathrm{d}s\right) = 0. \tag{19}$$

Applying (13) for $P = P^+$ and $t = 0$ we can see that the integral in (15) is convergent, since y is a bounded function. Let us assume first that (15) holds. Then

$$\Psi(t)(I - P^+)x_0 + \int_0^t \Psi(t)(I - P^+)\Psi(s)^{-1}y(s)\mathrm{d}s = -\int_t^{+\infty} \Psi(t)(I - P^+)\Psi(s)^{-1}y(s)\mathrm{d}s.$$

According to (13)

$$\left|\int_t^{+\infty}\Psi(t)(I-P^+)\Psi(s)^{-1}y(s)\mathrm{d}s\right| \le \max_{(t,+\infty)}|y(t)|\int_t^{+\infty}L\mathrm{e}^{-\beta(s-t)}\mathrm{d}s = \max_{(t,+\infty)}|y(t)|\frac{L}{\beta}\to 0 \quad \text{as } t\to+\infty.$$

Now let us assume that (19) holds. Let us introduce x^* by

$$(I-P^+)x^* := \int_0^{+\infty}(I-P^+)\Psi(s)^{-1}y(s)\mathrm{d}s$$

Then

$$\Psi(t)(I-P^+)x_0+\int_0^t\Psi(t)(I-P^+)\Psi(s)^{-1}y(s)\mathrm{d}s = \Psi(t)(I-P^+)(x_0+x^*)-\int_t^{+\infty}\Psi(t)(I-P^+)\Psi(s)^{-1}y(s)\mathrm{d}s.$$

We have seen that the second term tends to zero as $t\to+\infty$, hence

$$\lim_{t\to+\infty}\Psi(t)(I-P^+)(x_0+x^*)=0$$

implying $x_0+x^*=0$ according to Proposition 2, hence $x_0=-x^*$, yielding (15).

Similarly we can prove that (16) is equivalent to

$$\lim_{t\to-\infty}\left(\Psi(t)P^-x_0+\int_0^t\Psi(t)P^-\Psi(s)^{-1}y(s)\mathrm{d}s\right)=0. \tag{20}$$

Now the proof of statement (i) of the Lemma follows from the equation

$$x(t)=\Psi(t)P^+x_0+\int_0^t\Psi(t)P^+\Psi(s)^{-1}y(s)\mathrm{d}s+\Psi(t)(I-P^+)x_0+\int_0^t\Psi(t)(I-P^+)\Psi(s)^{-1}y(s)\mathrm{d}s.$$

According to (17) the first and second terms tend to zero as $t\to+\infty$. Hence $\lim_{+\infty}x=0$ if and only if the sum of the third and fourth term tends to zero as $t\to+\infty$. According to (19) this sum tends to zero if and only if that (15) holds, what we had to prove.

The proof of statement (ii) of the Lemma follows similarly from (18) and (20). □

Lemma 5 *Let us assume $n_c^+=n_c^-=0$. The following two statements are equivalent.*

(i) For any $a\in E_u^+$ and any $b\in E_s^-$ there exists a unique solution x_0 of

$$(I-P^+)x_0=a, \qquad P^-x_0=b.$$

(ii) $E_s^+\oplus E_u^-=\mathbb{C}^n$

Proof

Let us assume that (i) holds. First we show that $E_s^+\cap E_u^-=\{0\}$. Let $z\in E_s^+\cap E_u^-$, then $(I-P^+)z=0$ and $P^-z=0$, hence applying (i) with $a=b=0$ we get $z=0$. Now we show that for any $z\in\mathbb{C}^n$ there exist $z_1\in E_s^+$, $z_2\in E_u^-$, such that $z=z_1+z_2$. Let z_1 be the solution of (i) with $a=0$, $b=P^-z$, that is

$$(I-P^+)z_1=0, \qquad P^-z_1=P^-z.$$

Let z_2 be the solution of (i) with $a=(I-P^+)z$, $b=0$, that is

$$(I-P^+)z_2=(I-P^+)z, \qquad P^-z_2=0.$$

Then $(I-P^+)z=(I-P^+)(z_1+z_2)$ and $P^-z=P^-(z_1+z_2)$, hence the uniqueness implies $z=z_1+z_2$.

Now let us assume that (ii) holds. First we show that the solution in (i) is unique. Let us assume that there are two solutions x_0' and x_0''. Introducing $x_0 = x_0' - x_0''$ we have $P^- x_0 = 0$ and $(I - P^+)x_0 = 0$. Hence $x_0 \in E_u^- \cap E_s^+$, yielding $x_0 = 0$, that is $x_0' = x_0''$. Now we show that the solution x_0 in (i) exists. Let $z_1 \in E_s^+$, $z_2 \in E_u^-$, such that $a - b = z_1 + z_2$ and let $x_0 = b + z_2 = a - z_1$. Then $(I - P^+)x_0 = (I - P^+)(a - z_1) = a$, since $P^+ a = 0$ and $(I - P^+)z_1 = 0$. Similarly $P^- x_0 = P^-(b + z_2) = b$. □

The following lemma is proved in [16] Prop. 2.1 and in [5] p. 19, but in order to make the paper self-contained we present a short proof.

Lemma 6 *Let us assume $n_c^+ = n_c^- = 0$ and $E_s^+ \oplus E_u^- = \mathbb{C}^n$. Then system (11) possesses an exponential dichotomy in $\mathbb{R}$ with a projection P^*, for which $\mathrm{Im}\, P^* = E_s^+$, $\ker P^* = E_u^-$.*

Proof Assumption $E_s^+ \oplus E_u^- = \mathbb{C}^n$ implies that there exists a projection P^*, for which $\mathrm{Im}\, P^* = E_s^+$, $\ker P^* = E_u^-$. Namely, for an arbitrary vector $z \in \mathbb{C}^n$ we can define $P^* z$ as $P^* z = z_1$, where $z = z_1 + z_2$ with $z_1 \in E_s^+$, $z_2 \in E_u^-$. Since $\mathrm{Im}\, P^* = \mathrm{Im}\, P^+$, Lemma 2 and Lemma 3 imply that system (11) possesses an exponential dichotomy in $[0, +\infty)$ with the projection P^*. On the other hand $\mathrm{Im}\,(I - P^*) = \mathrm{Im}\,(I - P^-)$, hence Lemma 2 and Lemma 3 imply that system (11) possesses an exponential dichotomy in $(-\infty, 0]$ with the projection P^*. Since system (11) possesses an exponential dichotomy both in $[0, +\infty)$ and in $(-\infty, 0]$ with the projection P^*, according to Lemma 2 it possesses an exponential dichotomy in $\mathbb{R}$. □

Theorem 2 *Let us assume $n_c^+ = n_c^- = 0$.*

(i) If for any differentiable function $y \in C_0$, for which $\dot{y} \in C_0$ there exists a unique solution $x \in C_0$ of (9), then $E_s^+ \oplus E_u^- = \mathbb{C}^n$.

(ii) If $E_s^+ \oplus E_u^- = \mathbb{C}^n$, then system (9) has a unique solution $x \in C_0$ for any $y \in C_0$, and there exists $M > 0$, such that for any $y \in C_0$ and for the corresponding solution $x \in C_0$ the inequality $\|x\| \leq M\|y\|$ holds.

Proof (i) Let us assume that system (9) has a unique solution $x \in C_0$ for any differentiable function $y \in C_0$, for which $\dot{y} \in C_0$. Let $a \in E_u^+$ and $b \in E_s^-$ be arbitrary vectors. We will show that there exists a differentiable function $y \in C_0$, with $\dot{y} \in C_0$, such that

$$a = -\int_0^{+\infty} (I - P^+)\Psi(s)^{-1} y(s)\mathrm{d}s, \qquad b = \int_{-\infty}^0 P^- \Psi(s)^{-1} y(s)\mathrm{d}s. \tag{21}$$

Namely, let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying

$$h(0) = 0, \quad \dot{h}(0) = 0, \quad \int_0^{+\infty} h = -1, \quad \int_{-\infty}^0 h = 1, \quad |h(t)| \leq \mathrm{e}^{-(q+1)|t|}, \qquad |\dot{h}(t)| \leq r\mathrm{e}^{-(q+1)|t|} \text{ for } |t| > 1,$$

with some $r > 0$, and where $q \in \mathbb{R}$ is chosen to satisfy $\|A(t)\| \leq q$ for all $t \in \mathbb{R}$. It can be easily shown that there exists $k > 0$, such that $\|\Psi(t)\| \leq k\mathrm{e}^{qt}$ for all $t \in \mathbb{R}$. Let

$$y(t) = \begin{cases} \Psi(t)h(t)a, & \text{for } t \geq 0 \\ \Psi(t)h(t)b, & \text{for } t < 0 \end{cases}$$

Then y and $\dot{y}$ are continuous (also in zero, because their limits are zero from left and from right), and $y,\ \dot{y} \in C_0$, because for $t > 1$ we have

$$|y(t)| \leq \|\Psi(t)\||h(t)||a| \leq k\mathrm{e}^{qt}\mathrm{e}^{-(q+1)t}|a| = k|a|\mathrm{e}^{-t},$$

and

$$|\dot{y}(t)| \leq (\|\dot{\Psi}(t)\||h(t)| + \|\Psi(t)\||\dot{h}(t)|)|a| \leq (q + r)k|a|\mathrm{e}^{-t}.$$

Similar estimate can be derived for $t < -1$. Finally, we obtain

$$-\int_0^{+\infty} (I - P^+)\Psi(s)^{-1}y(s)\mathrm{d}s = -\int_0^{+\infty} (I - P^+)\Psi(s)^{-1}\Psi(s)h(s)a\mathrm{d}s = -a\int_0^{+\infty} h = a,$$

$$\int_{-\infty}^{0} P^-\Psi(s)^{-1}y(s)\mathrm{d}s = \int_{-\infty}^{0} P^-\Psi(s)^{-1}\Psi(s)h(s)b\mathrm{d}s = b\int_{-\infty}^{0} h = b.$$

Let $x \in C_0$ be the solution of (9) belonging to y. Then according to Lemma 4 for $x_0 = x(0)$ we have

$$a = (I - P^+)x_0, \qquad b = P^-x_0. \tag{22}$$

Hence Lemma 5 implies $E_s^+ \oplus E_u^- = \mathbb{C}^n$.

(ii) Now let us assume $E_s^+ \oplus E_u^- = \mathbb{C}^n$. Let $y \in C_0$ and a, b given by (21). According to Lemma 5 there exists a unique $x_0 \in \mathbb{C}^n$ satisfying (22), hence Lemma 4 implies $x \in C_0$. If $x^* \in C_0$ is another solution of (9), then $x^* - x \in C_0$ is a solution of (11). However, according to Corollary 1 $x^* - x \equiv 0$, that is $x^* \equiv x$.

Finally we prove the continuous dependence. According to Lemma 6 there exist a projection P^*, and positive numbers K, L, α, β, for which

$$\|\Psi(t)P^*\Psi(s)^{-1}\| \le K\mathrm{e}^{-\alpha(t-s)} \qquad \text{for } t \ge s,\ t, s \in \mathbb{R} \tag{23}$$

$$\|\Psi(t)(I - P^*)\Psi(s)^{-1}\| \le L\mathrm{e}^{-\beta(s-t)} \qquad \text{for } s \ge t,\ t, s \in \mathbb{R} \tag{24}$$

Repeating the proof of Lemma 4 replacing P^+ and P^- with P^* we get

$$(I - P^*)x_0 = -\int_0^{+\infty} (I - P^*)\Psi(s)^{-1}y(s)\mathrm{d}s, \qquad P^*x_0 = \int_{-\infty}^{0} P^*\Psi(s)^{-1}y(s)\mathrm{d}s.$$

Therefore from the variation of constant formula (10)

$$x(t) = \Psi(t)P^*x_0 + \int_0^t \Psi(t)P^*\Psi(s)^{-1}y(s)\mathrm{d}s + \Psi(t)(I - P^*)x_0 + \int_0^t \Psi(t)(I - P^*)\Psi(s)^{-1}y(s)\mathrm{d}s =$$

$$\int_{-\infty}^{t} \Psi(t)P^*\Psi(s)^{-1}y(s)\mathrm{d}s + \int_t^{+\infty} \Psi(t)(I - P^*)\Psi(s)^{-1}y(s)\mathrm{d}s.$$

From (23) and (24)

$$\left|\int_{-\infty}^{t} \Psi(t)P^*\Psi(s)^{-1}y(s)\mathrm{d}s\right| \le \|y\|K\mathrm{e}^{-\alpha t}\int_{-\infty}^{t} \mathrm{e}^{\alpha s}\mathrm{d}s = \|y\|\frac{K}{\alpha},$$

$$\left|\int_t^{+\infty} \Psi(t)(I - P^*)\Psi(s)^{-1}y(s)\mathrm{d}s\right| \le \|y\|L\mathrm{e}^{\beta t}\int_t^{+\infty} \mathrm{e}^{-\beta s}\mathrm{d}s = \|y\|\frac{L}{\beta},$$

hence $\|x\| \le (K/\alpha + L/\beta)\|y\|$. □

4 The spectrum of L

In Section 1 we have introduced the matrix functions A_λ, see (6). Since function Q tends to a limit at $\pm\infty$, the limits

$$A_\lambda^\pm = \lim_{t\to\pm\infty} A_\lambda(t)$$

exist. We have seen in Section 2 that the dimension of the stable, unstable and central subspaces of the matrices $A_\lambda^\pm$ play important role. Let us denote the number of eigenvalues (with multiplicity) of A_λ^+ with positive, negative, zero real part by $n_u^+(\lambda)$, $n_s^+(\lambda)$, $n_c^+(\lambda)$, respectively. We define $n_u^-(\lambda)$, $n_s^-(\lambda)$, $n_c^-(\lambda)$ similarly using A_λ^-.

Theorem 3 *Let us assume that at least one of the following two conditions holds:*

(a) $n_c^+(\lambda) > 0$ *and* $\int_0^{+\infty} |A_\lambda(t) - A_\lambda^+| < \infty$

(b) $n_c^-(\lambda) > 0$ *and* $\int_{-\infty}^0 |A_\lambda(t) - A_\lambda^-| < \infty$.

Then $\lambda \in \sigma(L)$.

Proof Let us assume the contrary, i.e. that λ is a regular value of L. Then according to (ii) of Lemma 1 for any differentiable function $y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$, for which $\dot{y} \in C_0(\mathbb{R}, \mathbb{C}^{2m})$, there exists a unique solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ of system (5); and there exists $M > 0$ such that for any y the inequality $\|x\| \leq M(\|y\| + \|\dot{y}\|)$ holds. However, Theorem 1 yields that this M cannot exist, which is a contradiction. $\square$

In the further considerations we deal with the case $n_c^+(\lambda) = 0 = n_c^-(\lambda)$. In this case we introduce the subspaces

$$E_s^+(\lambda), \quad E_u^+(\lambda), \quad E_s^-(\lambda), \quad E_u^-(\lambda)$$

in the same way as in Section 2. (Now they depend also on λ, because A depends on λ.)

Theorem 4 *Let us assume* $n_c^+(\lambda) = 0 = n_c^-(\lambda)$.

(i) λ *is an eigenvalue of* L *if and only if* $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) > 0$.

(ii) λ *is a regular value of* L *if and only if* $E_s^+(\lambda) \oplus E_u^-(\lambda) = \mathbb{C}^{2m}$.

Proof

(i) If λ is an eigenvalue of L, then there exists $V \in C_0(\mathbb{R}, \mathbb{C}^m)$, $V \neq 0$, for which $LV = \lambda V$. According to Proposition 1 $V' \in C_0(\mathbb{R}, \mathbb{C}^m)$, hence for $x = (V, V')^T$ we have $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ and it is a nonzero solution of $\dot{x}(t) = A_\lambda(t)x(t)$. Therefore the statement follows from Corollary 1.

If $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) > 0$, then according to Corollary 1 there is a nonzero solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ of $\dot{x}(t) = A_\lambda(t)x(t)$. Let $V = (x_1, \ldots, x_m)^T$, $U = (x_{m+1}, \ldots, x_{2m})^T$, then $U = V'$, hence V is twice differentiable, $V \neq 0$ (otherwise $U \equiv 0$ and $x \equiv 0$), and $LV = \lambda V$.

(ii) If λ is a regular value of L, then according to (ii) of Lemma 1 for any differentiable $y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ for which $\dot{y} \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ there exists a unique solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ of (5). Then Theorem 2 implies $E_s^+(\lambda) \oplus E_u^-(\lambda) = \mathbb{C}^{2m}$.

If $E_s^+(\lambda) \oplus E_u^-(\lambda) = \mathbb{C}^{2m}$, then according to Theorem 2 for any $y \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ there exists a unique solution $x \in C_0(\mathbb{R}, \mathbb{C}^{2m})$ of (5) and it depends continuously on y. Hence by (i) of Lemma 1 λ is a regular value of L. $\square$

Using that $E_s^+(\lambda) \oplus E_u^-(\lambda) = \mathbb{C}^{2m}$ is equivalent to $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) = 2m$ and $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) = 0$, the following statements are obvious consequences of the theorem above.

Corollary 2 *Let us assume* $n_c^+(\lambda) = 0 = n_c^-(\lambda)$.

1. *If* $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) > 2m$, *then* λ *is an eigenvalue of* L.
2. *If* $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) < 2m$, *then* $\lambda \in \sigma(L)$.
3. *If* $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) = 2m$ *and* $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) = 0$, *then* λ *is a regular value of* L.
4. *If* $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) = 2m$ *and* $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) > 0$, *then* λ *is an eigenvalue of* L.

Remark 1 If $n_c^+(\lambda) = 0 = n_c^-(\lambda)$, then the operator $L - \lambda I$ is Fredholm, and its Fredholm index is $\alpha(L - \lambda I) = \dim E_s^+(\lambda) + \dim E_u^-(\lambda) - 2m$, see [14, 16].

The dimension of $E_s^+(\lambda)$ and $E_u^-(\lambda)$ can be determined explicitly, because only the eigenvalues of the matrices $A_\lambda^\pm$ have to be determined to get these dimensions. However, to get $\dim(E_s^+(\lambda) \cap E_u^-(\lambda))$ the time dependent system must be solved numerically. This leads to the definition of the Evans function. Let

$$\Omega = \{\lambda \in \mathbb{C} \ : \ n_c^+(\lambda) = 0 = n_c^-(\lambda), \ n_s^+(\lambda) \neq 0 \neq n_u^-(\lambda), \text{ and } n_s^+(\lambda) + n_u^-(\lambda) = 2m\}.$$

For $\lambda \in \Omega$ let us denote the base of the subspace $E_s^+(\lambda)$ by $v_1^+, \ldots, v_{n_s^+}^+$, and the base of the subspace $E_u^-(\lambda)$ by $v_1^-, \ldots, v_{n_u^-}^-$. The assumption $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) > 0$ means that the two bases together give a linearly dependent system of vectors. The Evans function is defined as the determinant formed by these $2m$ vectors. That is the determinant is zero if λ is an eigenvalue.

Definition 2 The Evans function belonging to the operator L is $\mathcal{D} : \Omega \to \mathbb{C}$

$$\mathcal{D}(\lambda) = \det\left(v_1^+ \ldots v_{n_s^+}^+ v_1^- \ldots v_{n_u^-}^-\right)$$

We have proved that the eigenvalues are the zeros of the Evans function. It can be also shown that the multiplicity of an eigenvalue is equal to the multiplicity of the zero of the Evans function, and that the Evans function is an analytic function on the domain Ω [1]. Hence the zeros of $\mathcal{D}$ are isolated, that is in the domain where $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) = 2m$ there can be only isolated eigenvalues. This statement together with Corollary 2 enables us to determine the essential spectrum explicitly.

Corollary 3 *The essential spectrum of L is*

$$\sigma_e(L) = \{\lambda \in \mathbb{C} : \ n_c^+(\lambda) > 0 \ \textit{or} \ n_c^-(\lambda) > 0 \ \textit{or} \ n_s^+(\lambda) + n_u^-(\lambda) \neq 2m\}.$$

The bases of the stable and unstable subspaces can be determined numerically in the following way. We calculate the eigenvalues of A_λ^+ with negative real part, and its corresponding eigenvectors. Let us denote these eigenvalues by $\mu_1, \ldots, \mu_k$, and the eigenvectors by $u_1, \ldots, u_k$ (for short we used the notation $k = n_s^+(\lambda)$). Similarly, let us denote the eigenvalues of A_λ^- with positive real part by $\nu_1, \ldots, \nu_l$, and the corresponding eigenvectors by $v_1, \ldots, v_l$ (for short we used the notation $l = n_u^-(\lambda)$). Then choosing a sufficiently large number ℓ we solve the homogeneous equation $\dot{x}(t) = A_\lambda(t)x(t)$ in $[0, \ell]$ starting from the right end point with initial condition $x(\ell) = u_i e^{\mu_i \ell}$ for $i = 1, \ldots, k$. Hence we get $k = n_s^+(\lambda)$ linearly independent (approximating) solutions of the differential equations, therefore their values at 0 give a base of $E_s^+(\lambda)$. Similarly, solving the differential equation in $[-\ell, 0]$ we get a base of $E_u^-(\lambda)$, and the determinant defining the Evans function can be computed. We note that if ℓ is very large and there is a significant difference between the real parts of the eigenvalues $\mu_1, \ldots, \mu_k$, then the solution belonging to the eigenvalue with largest real part will dominate and the solutions starting from linearly independent initial conditions will be practically linearly dependent at zero. (Similar case can occur in $[-\ell, 0]$ as well.) To overcome this difficulty the problem can be extended to a wedge product space of higher dimension [3].

Now we show a method to determine the eigenvalues and eigenvectors of $A_\lambda^\pm$, which determine the dimensions of $E_s^+(\lambda)$ and $E_u^-(\lambda)$. We will deal with the two cases together, therefore for short we introduce

$$A_\lambda = \begin{pmatrix} 0 & I \\ D^{-1}(\lambda I - Q) & -cD^{-1} \end{pmatrix}, \tag{25}$$

where Q can be Q^+ or Q^-. Let us denote an eigenvalue of A_λ by μ and an eigenvector by $u = (u_1, u_2)^T$. That is

$$\begin{pmatrix} 0 & I \\ D^{-1}(\lambda I - Q) & -cD^{-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mu \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then $u_2 = \mu u_1$ and $D^{-1}(\lambda I - Q)u_1 - cD^{-1}u_2 = \mu u_2$, hence

$$D^{-1}(\lambda I - Q)u_1 - cD^{-1}\mu u_1 = \mu^2 u_1,$$

yielding

$$(\mu^2 D + \mu c I + Q - \lambda I)u_1 = 0.$$

Thus we have proved the following proposition.

Proposition 3 *The number μ is an eigenvalue and $u = (u_1, u_2)^T$ is an eigenvector of A_λ if and only if*

$$\det(\mu^2 D + \mu c I + Q - \lambda I) = 0 \tag{26}$$

and

$$u_1 \in \ker(\mu^2 D + \mu c I + Q - \lambda I), \qquad u_2 = \mu u_1.$$

Thus the eigenvalues of $A_\lambda^\pm$ are determined by equation (26) of degree $2m$. In the special case when Q is an upper or lower triangular matrix the l.h.s. of the equation is a product of m second degree polynomials, hence the solutions can be computed explicitly [20].

5 Applications

5.1 Case of a single equation, $m = 1$

Now let $U : \mathbb{R} \to \mathbb{R}$ be a solution of

$$U'' + cU' + f(U) = 0 \tag{27}$$

$$U(-\infty) = U_-, \quad U(+\infty) = U_+, \tag{28}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, $U_-, U_+ \in \mathbb{R}$ and it is assumed that $c \geq 0$. The stability of U is determined by the spectrum of the operator

$$L(V) = V'' + cV' + f'(U)V. \tag{29}$$

The function $q(t) = f'(U(t))$ is continuous and has limits at $\pm\infty$,

$$q^+ = f'(U_+), \qquad q^- = f'(U_-). \tag{30}$$

If U tends to the limits U_+ and U_- exponentially, then the integrals in the assumptions of Theorem 3 are convergent. Now $A_\lambda^\pm$ are 2-by-2 matrices and according to Proposition 3 their eigenvalues ($\mu_{1,2}$) are determined by the equation

$$\mu^2 + c\mu + q^\pm - \lambda = 0. \tag{31}$$

The essential spectrum can be determined by calculating the dimensions of $E_s^+(\lambda)$ and $E_u^-(\lambda)$. These dimensions can be easily determined from the sets where $n_c^+(\lambda) \geq 1$ and $n_c^-(\lambda) \geq 1$. These sets are formed by those values of λ to which $\mu = \mathrm{i}\omega$ is a solution of (31). Hence the set of λ values where $n_c^+(\lambda) \geq 1$ is the parabola

$$\mathcal{P}^+ = \{\lambda_1 + \mathrm{i}\lambda_2 \in \mathbb{C} \mid \lambda_1 = q^+ - \left(\frac{\lambda_2}{c}\right)^2\}. \tag{32}$$

It is easy to show that on the left hand side of the parabola $\dim E_s^+(\lambda) = 2$, and on the right hand side $\dim E_s^+(\lambda) = 1$, see Figure 1. Similarly, the set of λ values where $n_c^-(\lambda) \geq 1$ is the parabola

$$\mathcal{P}^- = \{\lambda_1 + \mathrm{i}\lambda_2 \in \mathbb{C} \mid \lambda_1 = q^- - \left(\frac{\lambda_2}{c}\right)^2\}. \tag{33}$$

It is easy to show that on the left hand side of the parabola $\dim E_u^-(\lambda) = 0$, and on the right hand side $\dim E_u^-(\lambda) = 1$, see Figure 1.

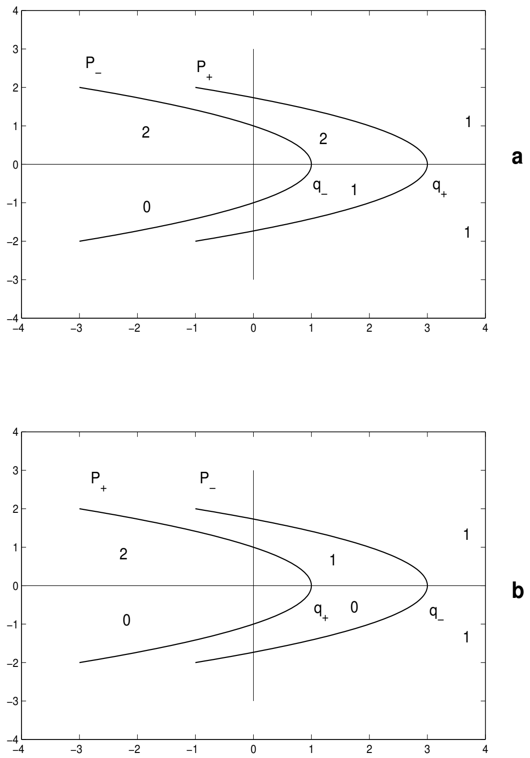

Figure 1. The parabolas given by (32) and (33) for $q_+ > q_-$ in (a) and for $q_- > q_+$ in (b). The dimensions of the subspaces $E_s^+(\lambda)$ and $E_u^-(\lambda)$ are shown, the upper numbers correspond to the former and the lower numbers correspond to the latter.

Now applying Theorem 3 and Corollary 2 we have the following results concerning the spectrum of L.

- Both parabolas belong to the essential spectrum of L.
- The domain lying on the left hand side of both parabolas consists of regular values of L.
- The domain lying on the right hand side of both parabolas contains all the isolated eigenvalues of L the remaining points of this domain (which are not isolated eigenvalues) are regular values of L.
- If $q^+ > q^-$, then the domain between the two parabolas is filled with eigenvalues.
- If $q^+ < q^-$, then the domain between the two parabolas is filled with points belonging to the essential spectrum, but they are not eigenvalues.

In this special case of $m = 1$ the location of the isolated eigenvalues with respect to the imaginary axis can also be determined. It can be shown that zero is a simple eigenvalue and all other isolated eigenvalues of L are negative (real) if and only if U is strictly monotone and $f'(U_-) < 0$, $f'(U_+) < 0$.

5.2 Flame propagation in a three variable model

Let us consider the travelling wave solutions of the problem

$$\begin{aligned}
\partial_\tau a &= L_A^{-1}\partial_x^2 a - af_1(b) \\
\partial_\tau w &= L_W^{-1}\partial_x^2 w - \beta w f_2(b) \\
\partial_\tau b &= \partial_x^2 b + af_1(b) - \alpha w f_2(b)
\end{aligned}$$

where $L_A,\ L_W,\ \alpha,\ \beta$ are positive constants ($L_A,\ L_W$ are the Lewis numbers) and

$$f_1(b) = \mathrm{e}^{(b-1)/\varepsilon b}, \quad f_2(b) = \mathrm{e}^{\mu(b-1)/\varepsilon b},$$

with some positive ε and μ. A travelling wave solution, propagating with velocity c, will also be denoted by (a, w, b), and satisfies the boundary conditions

$$a(y) \to 1, \quad w(y) \to 1, \quad b(y) \to 0 \qquad \text{as} \qquad y \to -\infty, \tag{34}$$

$$a'(y) \to 0, \quad w'(y) \to 0, \quad b'(y) \to 0 \qquad \text{as} \qquad y \to +\infty. \tag{35}$$

Here a is the concentration of the fuel, w is the concentration of an inhibitor species and b is scaled temperature. The travelling wave solution describes a flame propagating with velocity c. The number and stability of travelling waves of this system was investigated in [19]. We proved that the solutions a, w and b have limits at $+\infty$, that are denoted by a_+, w_+ and b_+. If $b_+ = 0$, then we refer to the travelling wave solution as a pulse solution. If $b_+ > 0$ then we call it a front solution. In the latter case $a_+ = 0 = w_+$. It was also shown that a saddle-node bifurcation may occur and there can be 1, 2 or 3 travelling wave solutions. The stability of these solutions can also change through Hopf bifurcation. The saddle-node and Hopf bifurcation curves were determined numerically. Here we only show how the method described in the previous Section works for this system to determine the essential spectrum of the corresponding linearizdimen ed operator. The results obtained by the Evans function method will be only cited from [19].

The operator corresponding to a travelling wave solution (a, w, b) of the above system takes the form

$$LV = \left(\begin{array}{l} L_A^{-1}V_1'' - cV_1' - f_1(b)V_1 - af_1'(b)V_3 \\ L_W^{-1}V_2'' - cV_2' - \beta f_2(b)V_2 - \beta w f_2'(b)V_3 \\ V_3'' - cV_3' + f_1(b)V_1 - \alpha f_2(b)V_2 + (af_1'(b) - \alpha w f_2'(b))V_3 \end{array} \right) \tag{36}$$

We consider L as an operator defined for the C^2 functions in the space

$$C_0 = \{V : \mathbb{R} \to \mathbb{C}^3 \mid V \text{ is continuous, } \lim_{t\to\pm\infty} V(t) = 0\}$$

endowed with the supremum norm.

Now $A^\pm_\lambda$ are the following 6-by-6 matrices

$$A^\pm_\lambda = \left(\begin{array}{cc} 0 & I \\ D^{-1}(\lambda I - Q^\pm) & cD^{-1} \end{array} \right)$$

where Q^- is a 3×3 zero matrix,

$$Q^+ = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \text{ for pulses,} \qquad Q^+ = \left(\begin{array}{ccc} -q_1 & 0 & 0 \\ 0 & -\beta q_2 & 0 \\ q_1 & -\alpha q_2 & 0 \end{array} \right) \text{ for fronts,}$$

and $q_1 = f_1(1 - \alpha/\beta)$, $q_2 = f_2(1 - \alpha/\beta)$. According to Proposition 3 the eigenvalues $(\mu_1, \ldots, \mu_6)$ of $A^\pm_\lambda$ are determined by the equation

$$(L_A^{-1}\mu^2 - c\mu + Q^\pm_{11} - \lambda)(L_W^{-1}\mu^2 - c\mu + Q^\pm_{22} - \lambda)(\mu^2 - c\mu + Q^\pm_{33} - \lambda) = 0, \tag{37}$$

since $Q^\pm$ are lower triangular matrices. Therefore the set of those λ values for which $n^+_c(\lambda) \geq 1$, that is $\mu = \mathrm{i}\omega$ (for some $\omega \in \mathbb{R}$) is a solution of equation (37), consists of three parabolas, denoted by P_1^+, P_2^+, P_3^+, see Figure 2 (a) and (b). In Figure 2 (a) the parabolas are shown in the case of a pulse solution, in Figure 2 (b) the parabolas corresponding to a front solution are shown. We define the parabolas P_1^-, P_2^-, P_3^- similarly as the loci of those λ values for which $n^-_c(\lambda) \geq 1$, they are shown in Figure 2 (c). The parabolas are given explicitly by

$$P_1^+ = \{\lambda \in \mathbb{C} \ : \mathrm{Re}\lambda = -q_1 - \frac{(\mathrm{Im}\lambda)^2}{L_A c^2}\}, \qquad P_1^- = \{\lambda \in \mathbb{C} \ : \mathrm{Re}\lambda = -\frac{(\mathrm{Im}\lambda)^2}{L_A c^2}\},$$

$$P_2^+ = \{\lambda \in \mathbb{C} \ : \mathrm{Re}\lambda = -\beta q_2 - \frac{(\mathrm{Im}\lambda)^2}{L_W c^2}\}, \qquad P_2^- = \{\lambda \in \mathbb{C} \ : \mathrm{Re}\lambda = -\frac{(\mathrm{Im}\lambda)^2}{L_W c^2}\},$$

$$P_3^+ = \{\lambda \in \mathbb{C} \ : \mathrm{Re}\lambda = -\frac{(\mathrm{Im}\lambda)^2}{c^2}\}, \qquad P_3^- = \{\lambda \in \mathbb{C} \ : \mathrm{Re}\lambda = -\frac{(\mathrm{Im}\lambda)^2}{c^2}\}.$$

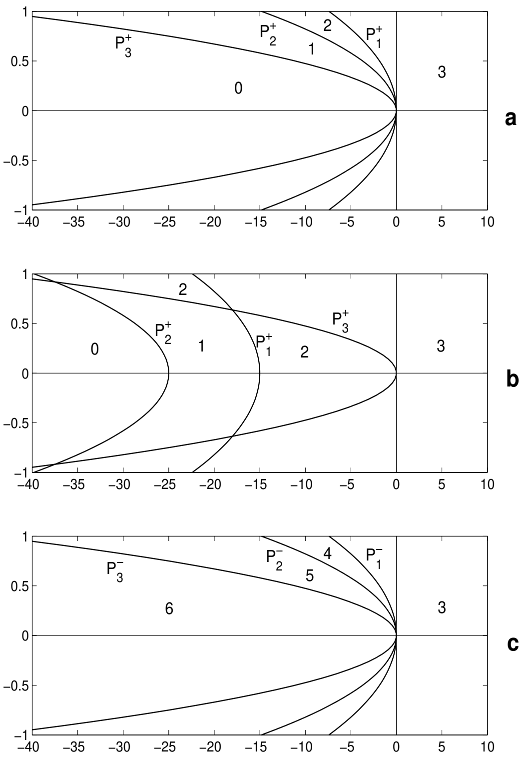

Figure 2. The parabolas determining the essential spectrum of the operator (36). The dimension of the subspace $E_s^+(\lambda)$ is shown in part (a) and (b). The dimension of the subspace $E_u^-(\lambda)$ is shown in part (c). (a) corresponds to pulses, (b) corresponds to fronts.

It can easily be seen that if λ is to the left of P_1^+, then both solutions for μ of equation $L_A^{-1}\mu^2 - c\mu + Q_{11}^{\pm} - \lambda = 0$ have positive real part, and if λ is to the right of P_1^+, then one of the solutions has positive real part, the other one has negative real part. The same is true for the other parabolas. Therefore we can determine, for any λ, the value of $n_s^+(\lambda)$, i.e. the dimension of $E_s^+(\lambda)$, and the value of $n_u^-(\lambda)$, i.e. the dimension of $E_u^-(\lambda)$. The values of these numbers are shown in Figure 2 in the different domains determined by the parabolas. Now applying Theorem 3 and Corollary 2 we have the following results concerning the spectrum of L.

Proposition 4 *1. All parabolas belong to the essential spectrum of L.*

2. The domain lying to the left of all parabolas consists of regular values.

3. The Evans function can be defined in the domain lying to the right of all parabolas.

4. In the case of fronts there are open domains filled with eigenvalues. In the case of pulses this kind of domain does not exist.

Proof We will use the dimensions of the spaces $E_s^+(\lambda)$, $E_u^-(\lambda)$ as they are given in Figure 2.

1. This statement is a direct consequence of Theorem 3.

2. If λ is in the domain lying to the left of all parabolas, then $\dim(E_s^+(\lambda)) = 0$, $\dim(E_u^-(\lambda) = 6$, hence conditions of statement 3 of Corollary 2 are fulfilled, thus λ is a regular value.

3. If λ is in the domain lying to the right of all parabolas, then $\dim(E_s^+(\lambda)) = 3$, $\dim(E_u^-(\lambda)) = 3$, hence $\lambda \in \Omega$, which is the domain of the Evans function.

4. In the case of fronts in the domain lying between P_1^+ and P_3^+ we have $\dim(E_s^+(\lambda)) = 2$ (see Figure 2b), $\dim(E_u^-(\lambda)) = 6$, hence conditions of Corollary 2 1. are fulfilled, thus any point λ of this domain is an eigenvalue, i.e. this domain is filled with eigenvalues. (We can find other domains where conditions of Corollary 2 1. are fulfilled.) In the case of pulses $P_i^+ = P_i^-$ for $i = 1, 2, 3$, hence

from Figure 2a and 2c we can see that $\dim(E_s^+(\lambda)) + \dim(E_u^-(\lambda)) = 6$ holds for any $\lambda \in \mathbb{C}$ except on the parabolas. Hence according to statements 3. and 4. of Corollary 2 any λ which is not on the parabolas is a regular value or an isolated eigenvalue.

We can decide whether there are eigenvalues with positive real part by computing the image of a half circle centred at the origin and lying in the right half plane under the Evans function $\mathcal{D}$. If the image winds around the origin, then by the argument principle there is (at least one) zero of $\mathcal{D}$ in the half circle. Choosing a sufficiently large half circle all the eigenvalues with positive real part are inside the half circle, because an estimate can be derived for the eigenvalues with positive real part. In [19] we computed the Evans function numerically and showed that Hopf bifurcation occurs for some value of L_A between 3 and 4. (The bifurcation value was determined more exactly.) For $L_A = 3$ the image of the half circle does not wind around the origin, hence there is no zero of $\mathcal{D}$ in the half circle. This value is below the Hopf bifurcation value. For $L_A = 4$ the image of the half circle winds twice around the origin, hence there are two zeros of $\mathcal{D}$ in the half circle. This value is above the Hopf bifurcation value. This shows that the Hopf bifurcation value of L_A is between 3 and 4.

5.3 The generalized KdV equation

Let us consider the travelling wave solutions of the problem

$$\partial_\tau u + \partial_x f(u) + \partial_x^3 u = 0 \tag{38}$$

where f is a twice differentiable convex function with $f(0) = 0 = f'(0)$ and $f(u)/u$ increasing. The motivating example is $f(u) = u^{p+1}/(p+1)$. This equation has a solitary (travelling) wave solution $u(\tau, x) = U(x - c\tau)$ for any $c > 0$, satisfying the boundary conditions $U(z) \to 0$ as $|z| \to \infty$, see e.g. [17]. (This solution can be given explicitly if $f(u) = u^{p+1}/(p+1)$).

The operator determining the stability of a travelling wave solution U takes the form

$$LV = V''' - cV' + (f'(U)V)' \tag{39}$$

The first order system corresponding to the third order equation $LV - \lambda V = W$ can be written in the form (5), where $x = (V, V', V'')^T$, $y = (0, 0, W)^T$ and

$$A_\lambda(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda - \dot{g}(t) & c - g(t) & 0 \end{pmatrix}$$

Here we used the notation $g(t) = f'(U(t))$.

Now $A_\lambda^\pm$ are the same matrices

$$A_\lambda^\pm = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & c & 0 \end{pmatrix}$$

The eigenvalues of $A_\lambda^\pm$ are determined by the characteristic polynomial

$$\mu^3 - \mu c - \lambda = 0.$$

Substituting $\mu = \mathrm{i}\omega$ into the characteristic equation we obtain $\mathrm{Re}\lambda = 0$, therefore the set of those λ values for which $n_c^\pm(\lambda) \geq 1$ is the imaginary axis.

Since $A_\lambda^+ = A_\lambda^-$, we have $E_s^+(\lambda) = E_s^-(\lambda)$ and $E_u^+(\lambda) = E_u^-(\lambda)$. Hence in the case $n_c^\pm(\lambda) = 0$ we have $\dim(E_s^+(\lambda)) + \dim(E_u^-(\lambda)) = 3$. In fact, as an elementary computation shows, for $\mathrm{Re}\lambda > 0$ we have $\dim(E_s^+(\lambda)) = 2$, $\dim(E_u^-(\lambda)) = 1$. For $\mathrm{Re}\lambda < 0$ we have $\dim(E_s^+(\lambda)) = 1$, $\dim(E_u^-(\lambda)) = 2$. Now applying Corollary 2 and Corollary 3 we have the following results concerning the spectrum of L.

Proposition 5 *1. The essential spectrum of L is the imaginary axis.*

2. If λ is not purely imaginary, then it is either a regular value or an isolated eigenvalue.

3. The Evans function can be defined either in the open right half complex plane or in the open left half plane.

In [17] a method is developped for the investigation of the behaviour of the Evans function on the positive half of the real line. The essence of the method is to compute the derivatives of the Evans function at zero and its limit at infinity. It is shown in [17] that $\mathcal{D}(\lambda) \to 1$ as $\lambda \to +\infty$ (and λ is real). It is also shown that $\mathcal{D}(0) = 0 = \mathcal{D}'(0)$ and that $\mathcal{D}''(0) < 0$ if $p > 4$ and $f(u) = u^{p+}/(p+1)$. Hence $\mathcal{D}(\lambda) < 0$ for small values of λ, therefore $\mathcal{D}$ has a positive real root, that is the solitary wave solution is unstable when $p > 4$.

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