Electronic Journal of Qualitative Theory of Differential Equations
2003, No. **14**, 1-18; **http://www.math.u-szeged.hu/ejqtde/**

A FUNCTIONAL INTEGRAL INCLUSION INVOLVING CARATHEODORIES

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Abstract

In this paper the existence of extremal solutions of a functional integral inclusion involving Carathéodory is proved under certain monotonicity conditions. Applications are given to some initial and boundary value problems of ordinary differential inclusions for proving the existence of extremal solutions. Our results generalize the results of Dhage [8] under weaker conditions and complement the results of O'Regan [16].

Key words and phrases: Fixed point theorem, integral inclusion and existence theorem.
AMS (MOS) Subject Classifications: 47H10

1 Introduction

The topic of differential and integral inclusions is of much interest in the subject of set-valued analysis. The existence theorems for the problems involving the inclusions are generally obtained under the assumption that the set-function in question is either lower or upper semi-continuous on the domain of its definition. See Aubin and Cellina [2] and the references therein. Recently the Carathéodory condition of multi-function has become most common in the literature while proving the existence theorems. But the study of such inclusions for the existence of extremal solutions is rare in the literature. Most recently the present author has obtained some results for differential inclusion in this direction with some stronger order relation of the multi-function. Therefore it is of interest to discuss the existence theorems for the extremal solutions under the weaker order relation of multi-function. In this paper we study the functional integral inclusions involving Carathédory .

Let $\mathbb{R}$ denote the real line. Let E be a Banach space with the norm $\|\cdot\|_E$ and let 2^E denote the class of all non-empty closed subsets of E. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the integral inclusion

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t, s)F(s, x(\eta(s)))\, ds \tag{1}$$

for $t \in J$, where $\sigma, \eta : J \to J$, $q : J \to E$, $k : J \times J \to \mathbb{R}$, are continuous and $F : J \times E \to 2^E$.

By the solution of the integral inclusion (1) we mean a continuous function $x : J \to E$ such that

$$x(t) = q(t) + \int_0^{\sigma(t)} k(t, s)v(s)\, ds$$

for some $v \in \mathcal{B}(J, \mathbb{R})$ satisfying $v(t) \in F(t, x(\eta(t)))$, $\forall\, t \in J$, where $\mathcal{B}(J, E)$ is the space all E-valued Bochner integrable functions on J.

The integral inclusion (1) has been studied recently by O'Regan [16] for the existence result under Carathéodory condition of F. In the present work we discuss the existence of extremal solutions of the integral inclusion (1) under certain monotonicity condition of the set-function F. In the following section we prove some fixed point theorems for monotone increasing set-maps on ordered Banach spaces.

2 Preliminaries

Let X be a Banach space with a norm $\|\cdot\|$ and let 2^X denote the class of all non-empty closed subsets of X. A correspondence $T : X \to 2^X$ is called a multi-valued mapping (in short multi-map) and a point $u \in X$ is called a fixed point of T if $u \in Tu$. Let $T(A) = \cup_{x \in A} Tx$. T is said to have closed (resp. convex and compact) values if Tx is closed (resp. convex and compact) subset of X for each $x \in X$. T is called bounded on bounded subset B of X if $T(B)$ is bounded subset of X. T is called upper semi-continuous if for any open subset G of X the set $\{x \in X \mid Tx \subset G\}$ is open in X. T is said to be totally bounded if for any bounded subset B of X, $\overline{T(B)}$ is totally bounded subset of X. Again T is called completely continuous if it is upper semi-continuous and totally bounded on X. Finally T is called compact if $T(X)$ is a compact subset of X. It is known that if the multi-valued map T is totally bounded with non-empty compact values then T is upper semi-continuous if and only if T has a closed graph (that is $x_n \to x_*, y_n \to y_*, y_n \in Tx_n \Rightarrow y_* \in Tx_*$).

A Kuratowski measure of noncompactness α of a bounded set A in X is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf \left\{ r > 0 : A = \bigcup_{i=1}^{n} A_i,\ \ \mathrm{diam}(A_i) \le r,\ \ \forall i \right\}. \tag{2}$$

The function α enjoys the following properties:

(α_1) $\alpha(A) = 0 \Leftrightarrow A$ is precompact.

(α_2) $\alpha(A) = \alpha(\overline{A}) = \alpha(\overline{co}A)$, where $\overline{A}$ and $\overline{co}$ denote respectively the closure and the closed convex hull of A.

(α_3) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(α_4) $a(A \cup B) = max\{\alpha(A), \alpha(B)\}$.

(α_5) $\alpha(\lambda A) = |\lambda|\alpha(A), \forall \lambda \in \mathbb{R}$.

(α_6) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

The details of measures of noncompactness and their properties appear in Banas and Goebel [4].

Definition 2.1 *A mapping $T : X \to 2^X$ is called condensing (countably condensing) if for any bounded (resp. bounded and countable) subset A of X, $T(A)$ is bounded and $\alpha(T(A)) < \alpha(A),\ \ \alpha(A) > 0$.*

A subset A of X is called countable if there exists a one-to-one correspondence $f : \mathbb{N} \to A$, where $\mathbb{N}$ is the set of natural numbers. The element $a = f(1) \in A$ is called the first element of A. A multi-valued mapping $T : X \to X$ is said to satisfy **Condition** D if for any countable subset A of X,

$$A \subseteq \overline{co}(\{a\} \cup T(A)) \Rightarrow \overline{A} \text{ is compact} \tag{3}$$

where a is the first element of A. The fixed point theory for the single-valued mappings T on a Banach space X satisfying **Condition** D have been discussed in Dhage [9].

Obviously every completely continuous multi-valued mapping T on a bounded subset of X into X satisfies **Condition** D.

A non-empty closed subset K of X is called a cone if it satisfies (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K, \forall \lambda \in \mathbb{R}^+$ and (iii) $-K \cap K = 0$, where 0 is a zero element of X. We define an order relation $\leq$ in X by

$$x \leq y \text{ iff } y - x \in K. \tag{4}$$

Let $\overline{x}, \overline{y} \in X$ be such that $\overline{x} \leq \overline{y}$. Then by the order interval $[\overline{x}, \overline{y}]$ we mean a set in X defined by

$$[\overline{x}, \overline{y}] = \{x \in X \mid \overline{x} \leq x \leq \overline{y}\}. \tag{5}$$

A cone K in X is called normal if the norm $\|\cdot\|$ is semi- monotone on K, i.e. for given $x, y \in K$, with $x \leq y$ there exists a constant $N > 0$ such that $\|x\| \leq N\|y\|$. It is known that if the cone K in X is normal, then every order- bounded set in X is bounded in norm. The details of cones and their properties may be found in Amann [3], Deimling [6] and Heikkila and Lakshmikantham [13].
Let $A, B \in 2^X$. Then by $A \leq B$ we mean the following :

$$\left.\begin{array}{r} \text{For every } \ a \in A \text{ there is a point } \ b \in B \text{ such that } \ a \leq b \\ \text{and for every } \ b' \in B \text{ there is a point } \ a' \in A \text{ such that } \ a' \leq b'. \end{array}\right\} \tag{6}$$

For any $A \in 2^X$ we denote

$$\vee A = \inf\{x \in X \,|\, a \leq x, \forall\ a \in A\}$$

and

$$\wedge A = \sup\{x \in X \,|\, a \geq x, \forall\ a \in A\}.$$

Definition 2.2 *A multi-valued mapping $T : X \to 2^X$ is called weakly isotone increasing (resp. weakly isotone decreasing) if $x \leq Tx$ (resp. $x \geq Tx$), $\forall x \in X$. A weakly isotone multi-valued map is one which is either weakly isotone increasing or weakly isotone decreasing on X.*

Definition 2.3 *A mapping $T : X \to 2^X$ is called isotone increasing if for any $x, y \in X$, $x \leq y$, we have that $Tx \leq Ty$.*

3 Fixed Point Theory

Before going to the main results of this paper, we give a useful definition.

Definition 3.1 *A multi-valued mapping $T : X \to 2^X$ is called* **Chandrabhan** *if A is a countable subset of X, then*

$$A \subseteq \overline{co}(C \cup T(A)) \Rightarrow \overline{A} \ \text{ is compact} \tag{7}$$

where C is a precompact subset of A.

Notice that a mapping $T : X \to 2^X$ satisfying Condition D is Chandrabhan, but the converse may not be true.

Theorem 3.1 *Let $T : X \to 2^X$ be an upper semi-continuous and Chandrabhan map with closed values. Further if T is weakly isotone, then T has a fixed point.*

Proof. Suppose first that T is isotone increasing on X. Let $x \in X$ be arbitrary and define a sequence $\{x_n\} \subset X$ as follows. Let $x_0 = x$. Since T is isotone increasing, we have $x_o \leq Tx_0$. Then there is a point, say $x_1 \in Tx_0$ such that $x_0 \leq x_1$. Continuing in this way, by induction we have a sequence $\{x_n\}$ in X such that

$$x_0 = x, \;\; x_{n+1} \in Tx_n, \;\; n \geq 0 \tag{8}$$

and

$$x_0 \leq x_1 \leq x_2, \leq \ldots \leq x_n \leq \ldots .$$

Take $A = \{x_0, x_1, \ldots\}$. Then A is countable and $a = x_0$ is its first element. Thus we have

$$\begin{aligned} A &= \{x_0, x_1, \ldots\} \\ &= \{x_0\} \cup \{x_1, x_2, \ldots\} \\ &\subseteq \{a\} \cup T(\{x_0, x_1, \ldots\}) \\ &= \{a\} \cup T(A) \\ &\subset \overline{co}(\{a\} \cup T(A)). \end{aligned}$$

Since T is Chandrabhan, we have that $\overline{A}$ is compact. Therefore the sequence $\{x_n\}$ has a convergent subsequence converging to a point $x^* \in X$. From (2.3) it follows that $\{x_n\}$ is monotone increasing and so the whole sequence converges to x^*. By the upper semi-continuity of T,

$$x^* = \lim_n x_{n+1} \in \lim_n T(x_n) = T(\lim_n x_n) = Tx^*.$$

Similarly if T is weakly isotone decreasing, then it can be proved that T has a fixed point. This completes the proof. $\square$

Corollary 3.1 *Let S be a closed and bounded subset of a Banach space X and let $T : S \to 2^S$ be an upper semi-continuous and countably condensing map with closed values. Further if T is weakly isotone, then T has a fixed point.*

Proof. To conclude, we simply show that T is a Chandrabhan map on A. Let A be a countable subset of S. Suppose that

$$A \subseteq \overline{co}(C \cup T(A))$$

where C is a precompact subset of A. We show that A is precompact. If not, then $\alpha(A) > 0$. Now by the properties $(\alpha_2) - (\alpha_4)$ of α, one has

$$\begin{aligned} \alpha(A) &\leq \alpha(C \cup T(A)) \\ &= \max\{\alpha(C), \alpha(T(A)\} \\ &< \max\{0, \alpha(A)\} \\ &= \alpha(A) \end{aligned}$$

which is a contradiction. Hence A is precompact and $\overline{A}$ is compact in view of completeness of S. Thus T is Chandrabhan and now the conclusion follows by an application of Theorem 3.1. $\square$

Corollary 3.2 *Let S be a closed and bounded subset of a Banach space X and let $T : S \to 2^S$ be a completely continuous multi-mapping with closed values. Further if T is weakly isotone, then T has a fixed point.*

Theorem 3.2 *Let $T : [\overline{x}, \overline{y}] \to 2^{[\overline{x},\overline{y}]}$ be a Chandrabhan multi-map such that*

(a) T is upper semi-continuous ,and

(b) T is isotone increasing.

Then T has a least fixed point x_* and a greatest fixed point x^* in $[\overline{x}, \overline{y}]$. Moreover the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_{n+1} = \wedge T x_n, \quad n \geq 0 \;\; \textit{with} \;\; x_0 = \overline{x} \tag{9}$$

and

$$y_{n+1} = \vee T y_n, \quad n \geq 0 \;\; \textit{with} \;\; y_0 = \overline{y} \tag{10}$$

converge to x_* and x^* respectively.

Proof. Define the sequences $\{x_n\}$ and $\{y_n\}$ in $[\overline{x}, \overline{y}]$ by (9) and (10) respectively. As Tx is closed subset of $[\overline{x}, \overline{y}]$, $\vee Tx, \wedge Tx \in Tx$ for each $x \in [\overline{x}, \overline{y}]$, and so the sequences $\{x_n\}$ and $\{y_n\}$ are well defined. Since T is isotone increasing, the sequence $\{x_n\}$ is monotone increasing in $[\overline{x}, \overline{y}]$. Let $A = \{x_0, x_1, \ldots\}$. Then A is countable and $a = x_0$ is its first element. Thus we have

$$\begin{aligned} A &= \{x_0, x_1, \ldots\} \\ &= \{x_0\} \cup \{x_1, x_2, \ldots\} \\ &= \{a\} \cup T(A). \end{aligned}$$

Since T satisfies Condition D, A is compact. Hence the sequence $\{x_n\}$ converges to a point $x_* \in [\overline{x}, \overline{y}]$. By the upper semi-continuity of T, $x_* \in T x_{x_*}$. Again consider a sequence $\{y_n\} \subseteq [\overline{x}, \overline{y}]$ defined by (10). Since T is isotone increasing, we have that $\{y_n\}$ is a monotone decreasing sequence in $[\overline{x}, \overline{y}]$. Then following the above arguments, it can be proved that the sequence $\{y_n\}$ converges to a point x^* in $[\overline{x}, \overline{y}]$. By the upper semi-continuity of T, we obtain $x^* \in Tx^*$. Finally we show that x_* and x^* are respectively the least and the greatest fixed points of T in $[\overline{x},\overline{y}]$. Let $z\in[\overline{x},\overline{y}]$ be any fixed point of T in $[\overline{x},\overline{y}]$. Then by the isotonicity of T, one has

$$\overline{x}=x_0\leq x_1\leq\ldots\leq x_n\leq z\leq y_n\leq\ldots\leq y_1\leq y_0=\overline{y}$$

$\forall n\in\mathbb{N}$. Passing to the limit as $n\to\infty$, we obtain $x_*\leq z\leq x^*$. This completes the proof. $\square$

Corollary 3.3 *Let $T:[\overline{x},\overline{y}]\to 2^{[\overline{x},\overline{y}]}$ be a continuous and countably condensing. Further if T is isotone increasing and the cone K in X is normal, then T has a least fixed point x_* and a greatest fixed point x^* in $[\overline{x},\overline{y}]$. Moreover the sequences $\{x_n\}$ and $\{y_n\}$ defined by (9) and (10) converge respectively to x_* and x^*.*

Proof. We simply show that T satisfies condition D on $[\overline{x},\overline{y}]$. Let A be a countable set in $[\overline{x},\overline{y}]$, which is bounded in view of the normality of the cone K in X. Suppose that $A\subseteq\{a\}\cup T(A)$, where a is a first element of A. We show that A is precompact. Suppose not. Then by the properties (α_2)-(α_4),

$$\begin{aligned}\alpha(A) &\leq \alpha(\{a\}\cup T(A))\\ &= max\{\alpha(\{a\}),\alpha(T(A)\}\\ &< max\{0,\alpha(A)\}\\ &= \alpha(A)\end{aligned}$$

which is a contradiction. Hence A is precompact and $\overline{A}$ is compact in view of the completeness of X. Now the desired conclusion follows by an application of Theorem 3.2. $\square$

Corollary 3.4 *Let $T:[\overline{x},\overline{y}]\to 2^{[\overline{x},\overline{y}]}$ be a completely continuous multi-mapping. Further if T is isotone increasing and the cone K in X is normal, then T has a least fixed point x_* and a greatest fixed point x^* in $[\overline{x},\overline{y}]$. Moreover the sequences $\{x_n\}$ and $\{y_n\}$ defined by (9) and (10) converge to x_* and x^* respectively.*

Remark 3.1 Note that when $T=\{f\}$, a singe-valued mapping, the results of this section reduce to the fixed point results of Dhage proved in [9].

4 Existence of extremal solutions

In this section we shall prove the existence of the extremal solutions of integral equation (1) between the given lower and upper solutions on J and using the Carathéodory's conditions of F.

Define a norm $\|\cdot\|$ in $AC(J,E)$ by

$$\|x\| = \sup_{t\in J} \|x(t)\|_E. \tag{11}$$

Clearly $C(J,E)$ is a Banach space with this supremum norm. We introduce an order relation $\leq$ in $AC(J,E)$ with the help of the cone K in $C(J,E)$ defined by the cone

$$K_{AC} = \{x \in C(J,E) | x(t) \in K_E, \ \ \forall t \in J\}, \tag{12}$$

where K_E is a cone in E.

The following result is crucial in the sequel.

Lemma 4.1 *(*Martin [15]*) Let $T > 0$ and let E be a Banach space. Then the following statements hold.*

i) If $A \subset C([0,T],E)$ is bounded, then

$$\sup_{t\in[0,T]} \alpha(A(t)) \leq \alpha(A([0,T])) \leq \alpha(A).$$

ii) If $A \subseteq C([0,T],E)$ is bounded and equi-continuous, then

$$\alpha(A) = \sup_{t\in[0,T]} \alpha(A(t)) = \alpha(A([0,T])).$$

We also need the following definitions in the sequel.

Definition 4.1 *A multi-valued map $F : J \to 2^E$ is said to be measurable if for any $y \in X$, the function $t \to d(y, F(t)) = inf\{\|y-x\|_E : x \in F(t)\}$ is measurable.*

Denote

$$\|F(t,x)\| = \{\|u\|_E : u \in F(t,x)\}$$

and

$$\||F(t,x)\|| = sup\{\|u\|_E : u \in F(t,x)\}.$$

Definition 4.2 *A multi-valued function $\beta : J \times E \to 2^E$ is called Carathéodory if*

i) $t \to \beta(t,x)$ is measurable for each $x \in E$, and *ii)* $x \to \beta(t, x)$ *is an upper semi-continuous almost everywhere for* $t \in J$.

Definition 4.3 *A Carathéodory multi-function* $F(t, x)$ *is called* L^1*-Carathéodory if for every real number* $r > 0$ *there exists a function* $h_r \in L^1(J, \mathbb{R})$ *such that*

$$\|\|F(t, x)\|\| \leq h_r(t) \ \ a.e. \quad t \in J$$

for all $x \in E$ *with* $\|x\|_E \leq r$.

Denote

$$S^1_F(x) = \{v \in \mathcal{B}(J, E) | v(t) \in F(t, x(\eta(t))) \ \text{ a.e. } \ t \in J\}.$$

Then we have the following lemmas due to Lasota and Opial [14].

Lemma 4.2 *If* $diam(E) < \infty$ *and* $F : J \times E \to 2^E$ *is* L^1*-Carathéodory, then* $S^1_F(x) \neq \emptyset$ *for each* $x \in E$.

Lemma 4.3 *Let* E *be a Banach space,* F *an Carathéodory multi-map with* $S^1_F \neq \emptyset$ *and let* $\mathcal{L} : L^1(J, E) \to C(J, E)$ *be a continuous linear mapping. Then the operator*

$$\mathcal{L} \circ S^1_F : C(J, E) \to 2^{C(J,E)}$$

is a closed graph operator on $C(J, E) \times C(J, E)$.

Following Dhage and Kang [10] and Dhage et.al. [11], we have

Definition 4.4 *A function* $a \in C(J, E)$ *is called a lower solution of integral inclusion (1) if it satisfies* $a(t) \leq q(t) + \int_0^{\sigma(t)} k(t, s)v(s)\, ds$ *for all* $v \in \mathcal{B}(J, E)$ *such that* $v(t) \in F(t, a(\eta(t)))$ *a.e.* $t \in J$. *Similarly a function* $b \in C(J, E)$ *is called an upper solution of integral inclusion (1) if it satisfies* $b(t) \geq q(t) + \int_0^{\sigma(t)} k(t, s)v(s)\, ds$ *for all* $v \in \mathcal{B}(J, E)$ *such that* $v(t) \in F(t, b(\eta(t)))$ *a.e.* $t \in J$

Definition 4.5 *A multi-function* $F(t, x)$ *is said to be nondecreasing in* x *almost everywhere for* $t \in J$ *if for any* $x, y \in E$ *with* $x \leq y$ *we have that* $F(t, x) \leq F(t, y)$ *for almost everywhere* $t \in J$.

We consider the following set of hypotheses in the sequel.

(H$_0$) The function $k(t,s)$ is continuous and nonnegative on $J\times J$ with

$$K=\sup_{t,s\in J} k(t,s).$$

(H$_1$) The multi-valued function $F(t,x)$ is Carathéodory.

(H$_2$) For any countable and bounded set A of E, $\alpha(F(J\times A))\le\lambda\alpha(A)$, for some real number $\lambda>0$.

(H$_3$) The function $F(t,x)$ is nondecreasing in x almost everywhere for $t\in J$.

(H$_4$) $S^1_F(x)\neq\emptyset$ for each $x\in C(J,E)$.

(H$_5$) The integral inclusion (1) has a lower solution a and upper solution b with $a\le b$.

(H$_6$) The function $t\to \||F(t,a(\eta(t)))\||_E+\||F(t,b(\eta(t))\||_E$ is Lebesgue integrable on J.

Remark 4.1 *Note that if hypothesis (H$_3$) and (H$_4$) hold, then the multi-map S^1_F : $C(J,E)\to 2^{C(J,E)}$ is isotone increasing.*

Remark 4.2 *Suppose that hypotheses (H$_1$), (H$_3$), (H$_5$) hold and define a function $h:J\to\mathbb{R}$ by*

$$h(t)=\||F(t,a(\eta(t)))\||_E+\||F(t,b(\eta(t))\||_E \tag{13}$$

$\forall\, t\in J$. Then the function h is Lebesgue integrable on J and

$$\|F(t,x(\eta(t)))\|_E\le h(t) \tag{14}$$

$\forall\, t\in J$ and $\forall x\in[a,b]$.

Theorem 4.1 *Assume that hypotheses (H$_0$)-(H$_5$) hold. If $\lambda K<1$ and the cone K_E in E is normal, then integral inclusion (1) has a minimal solution x_* and a maximal solution x^* in $[a,b]$. Moreover the sequences $\{x_n\}$ and $\{y_n\}$ defined by*

$$x_{n+1}(t)=q(t)+\int_0^t k(t,s)(\wedge F(s,x_n(\eta(s))))\,ds,\quad n\ge 0\ \text{with}\ x_0=a \tag{15}$$

and

$$y_{n+1}(t)=y_0+\int_0^t k(t,s)(\vee F(s,y_n(\eta(s))))\,ds,\ \ n\ge 0\ \text{with}\ y_0=b \tag{16}$$

converge respectively to x_* and x^*.

Proof. Let $X = C(J, E)$ and consider the order interval $[a, b]$ in X which is well defined in view of (H_5). Define a mapping $T : [a, b] \to 2^X$ by

$$Tx = \left\{ u : u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(s)\,ds,\ \ v \in S_F^1(x) \right\}. \tag{17}$$

We shall show that T satisfies all the conditions of Corollary 3.3 on $[a, b]$. First we show that T is isotone increasing on $[a, b]$. Let $x, y \in [a, b]$ be such that $x \leq y$. Then by (H_3) and Remark 4.1 we get

$$\begin{aligned} Tx &= \left\{ u : u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v_1(s)\,ds,\ \ v_1 \in S_F^1(x) \right\} \\ &\leq \left\{ u : u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v_2(s)\,ds,\ \ v_2 \in S_F^1(y) \right\} \\ &= Ty \end{aligned}$$

$\forall\, t \in J$. As a result $Tx \leq Ty$. Therefore T is isotone increasing on $[a, b]$. Next let $x \in [a, b]$ be arbitrary. Then we have $a \leq x \leq b$. By (H_3)-(H_5) we obtain

$$a \leq Ta \leq Tx \leq Tb \leq b.$$

Hence T defines a mapping $T : [a, b] \to 2^{[a,b]}$. Again the cone K_C in $C(J, E)$ is normal. To see this, let $x, y \in K_C$ be such that $0 \leq x \leq y$. Then we have $0 \leq x(t) \leq y(t), \forall\, t \in J$. Since the cone K_E in E is normal, there is a constant $N > 0$ such that $\|x(t)\|_E \leq N\|y(t)\|_E$, $\forall\, t \in J$. This further in view of relation (4) implies that $\|x\| \leq N\|y\|$ and so the cone K_C is normal in $C(J, E)$. As a result the order interval $[a, b]$ is a norm-bounded subset of $C(J, E)$. Finally we show that T is a countably condensing mapping on $[a, b]$. Let $A \subseteq [a, b]$ be countable. Then for $t \in J$, we have

$$\begin{aligned} \alpha(T(A(t))) &\leq \alpha\left(\cup \left\{ q(t) + \int_0^{\sigma(t)} k(t,s)F(s, x(\eta(s)))\,ds : x \in A \right\} \right) \\ &\leq \alpha(\{q(t)\}) + \alpha\left(\cup \left\{ \int_0^{\sigma(t)} k(t,s)F(s, x(\eta(s)))\,ds : x \in A \right\} \right) \\ &\leq \alpha\left(\int_0^{\sigma(t)} k(t,s)F(s, A(\eta(s)))\,ds \right) \\ &= \int_0^{\sigma(t)} k(t,s)\alpha(F(s, A(\eta(s))))\,ds \\ &\leq K \int_0^{\sigma(t)} \alpha(F(J \times A(J))\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq K\int_0^{\sigma(t)} \lambda\alpha(A(J))\,ds\\
&\leq \lambda K\alpha(A(J))
\end{aligned}
$$

where $A(J)=\cup_{s\in J}\{\phi(s):\phi\in A\}$. Now an application of Lemma 4.1 yields that for each $t\in J$, we have

$$\alpha(T(A(t)))\leq \lambda K\alpha(A). \tag{18}$$

Next we show that $T(A)$ is a uniformly bounded and equi-continuous set in $[a,b]$. To see this, let $u\in Tx$ be arbitrary. Then there is a $v\in S_F^1(x)$ such that

$$u(t)=q(t)+\int_0^{\sigma(t)} k(t,s)v(s)\,ds.$$

Hence

$$
\begin{aligned}
\|u(t)\|_E &\leq \|q(t)\|_E+\int_0^{\sigma(t)}\|k(t,s)v(s)\|_E\,ds\\
&\leq \|q(t)\|_E+K\int_0^{\sigma(t)}\|F(s,x(\eta(s)))\|_E\,ds\\
&\leq \|q(t)\|_E+K\int_0^{\sigma(t)} h(s)\,ds\\
&= \|q(t)\|_E+K\|h\|_{L^1}
\end{aligned}
$$

for all $x\in A$ and so $T(A)$ is a uniformly bounded set in $[a,b]$. Again let $t,\tau\in J$, then for any $y\in T(A)$ one has

$$
\begin{aligned}
\|y(t)-y(\tau)\|_E &\leq \|q(t)-q(\tau)\|_E+\|\int_0^{\sigma(t)} k(t,s)v(s)\,ds-\int_0^{\sigma(\tau)} k(\tau,s)v(s)\,ds\|_E\\
&\leq \|q(t)-q(\tau)\|_E+\|\int_0^{\sigma(t)} k(t,s)v(s)\,ds-\int_0^{\sigma(t)} k(\tau,s)v(s)\,ds\|_E\\
&\quad +\|\int_0^{\sigma(t)} k(\tau,s)v(s)\,ds-\int_0^{\sigma(\tau)} k(\tau,s)v(s)\|_E\\
&\leq \|q(t)-q(\tau)\|_E+\|\int_0^{\sigma(t)} [k(t,s)-k(\tau,s)]v(s)\,ds\|_E\\
&\quad +\|\int_{\sigma(\tau)}^{\sigma(t)} |k(\tau,s)|v(s)\,ds\|_E\\
&\leq \|q(t)-q(\tau)\|_E+\left|\int_0^{\sigma(t)} |k(t,s)-k(\tau,s)|\ \|v(s)\|_E\,ds\|_E\right|\\
&\quad +\left|\int_{\sigma(\tau)}^{\sigma(t)} |k(\tau,s)|\ \|v(s)\|_E\,ds\right|
\end{aligned}
$$

for some $v \in S^1_F(x)$. This further implies that

$$\begin{aligned}\|y(t) - y(\tau)\|_E &\leq \|q(t) - q(\tau)\|_E + \int_0^{\sigma(t)} |k(t,s) - k(\tau,s)| h(s)\, ds \\ &\quad + |p(t) - p(\tau)|\end{aligned}$$

where $p(t) = K \int_0^{\sigma(t)} h(s)\, ds$.

Notice that p and σ are continuous functions on J, so p is uniformly continuous on J. As a result we have that

$$\|y(t) - y(\tau)\|_E \to 0 \ \text{ as } \ t \to \tau.$$

This shows that $T(A)$ is an equi-continuous set in X. Now an application of Lemma 4.1 (ii) to (18) yields that

$$\alpha(T(A)) \leq \lambda K \alpha(A)$$

where $\lambda K < 1$. This shows that T is a countably condensing multi-valued mapping on $[a,b]$. Next we show that T is a upper semi-continuous multi-valued mapping on $[a,b]$. Let $\{x_n\}$ be a sequence in $[a,b]$ such that $x_n \to x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in Tx_n$ and $y_n \to y_*$. We shall show that $y_* \in Tx_*$.

Since $y_n \in Tx_n$, there exists a $v_n \in S^1_F(x_n)$ such that

$$y_n(t) = q(t) + \int_0^{\sigma(t)} k(t,s) v_n(s)\, ds, \ \ t \in J.$$

We must prove that there is a $v_* \in S^1_F(x_*)$ such that

$$y_*(t) = q(t) + \int_0^{\sigma(t)} k(t,s) v_*(s) ds, \ \ t \in J.$$

Consider the continuous linear operator $\mathcal{L} : L^1(J, \mathrm{I\!R}) \to C(J,E)$ defined by

$$\mathcal{L}v(t) = \int_0^{\sigma(t)} k(t,s) v(s) ds, \ \ t \in J.$$

Now $\|y_n - q(t) - (y_* - q(t))\|_E \to 0$ as $n \to \infty$.

Next we show that $\mathcal{L} \circ S^1_F$ is a closed graph operator on $C(J,E) \times C(J,E)$. From the definition of $\mathcal{L}$ we have

$$y_n(t) - q(t) \in \mathcal{L} \circ S^1_F(x_n).$$

Since $y_n \to y_*$, there is a point $v_* \in S_F^1(x_*)$ such that

$$y_*(t) = q(t) + \int_0^t k(t,s)v_*(s)\,ds, \;\; t \in J.$$

This shows that T is an upper semi-continuous operator on $[a, b]$.

Next we show that Tx is closed subset of X for each $x \in X$. To finish, it is enough to show that the values of the operator S_F^1 are closed in $\mathcal{B}(J, \mathbb{R})$. Let $\{\omega_n\}$ be a sequence in $\mathcal{B}(J, \mathbb{R})$ such that $\omega_n \to \omega$. Then $\omega_n \to \omega$ in measure. So there exists a subsequence S of the positive integers such that $\omega_n \to \omega$ a.e. $n \to \infty$ through S. Since the hypothesis (H_4) holds, the values of S_F^1 are closed in $\mathcal{B}(J, \mathbb{R})$. Thus for each $x \in X$, Tx is a non-empty, closed and bounded subset of X.

Thus T satisfies all the conditions of Corollary 3.3 and therefore an application of it yields that T has a least fixed point x_* and a greatest fixed point x^* in $[a, b]$. This further implies that x_* and x^* are respectively the minimal and maximal solutions of integral inclusion (1) on J, and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (15) and (16) converge respectively to x_* and x^*. This completes the proof. □

5 Applications

In this section we obtain the existence theorems for extremal solutions to initial and boundary value problems of ordinary differential inclusions by the applications of the main existence result of the previous section.

5.1. Initial Value Problem: Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the initial value problem (in short IVP) of ordinary functional differential inclusion,

$$\left.\begin{array}{l} x' \in F(t, x(\eta(t))) \;\text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{array}\right\} \tag{19}$$

where $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ and $\eta : J \to J$ is continuous.

By a solution of the IVP (19) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies the relations in (19) on J, that is, there exists a $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, x(\eta(t)))$ for all $t \in J$ such that $x'(t) = v(t)$ a.e. $t \in J$ and $x(0) = x_0$, where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

The IVP (19) has been studied in the literature for various aspects of the solution. See Aubin and Cellina [2], Deimling [7] and the references therein.

Now $AC(J, \mathbb{R})$ is a Banach space with respect to the norm $\|\cdot\|_C$ given by

$$\|x\|_C = \sup\{|x(t)|\, t \in J\}.$$

Define an order relation $\leq$ in $AC(J, \mathbb{R})$ by the cone K_{AC} in $AC(J, R)$ given by

$$K_{AC} = \{x \in AC(J, \mathbb{R}) |\, x(t) \geq 0,\ \forall\ \in J\} \tag{20}$$

It is known that the cone K_{AC} is normal in $AC(J, \mathbb{R})$.

Definition 5.1 *A function $a \in AC(J, \mathbb{R})$ is called a lower solution of IVP (19) if for all $v \in S^1_F(x), x \in AC(J, \mathbb{R})$ we have $a'(t) \leq v(t)$ for all $t \in J$ with $x(0) = x_0$. Similarly an upper solution b of IVP (19) is defined.*

We need the following hypothesis in the sequel.

(H_6) The IVP (19) has a lower solution a and an upper solution b on J with $a \leq b$.

Theorem 5.1 *Assume that hypotheses (H_1)-(H_4) and (H_6) hold. Then the functional differential inclusion (19) has a maximal and a minimal solution on J.*

Proof: A function $x : J \to \mathbb{R}$ is a solution of the IVP (19) if and only if it is a solution of the integral inclusion

$$x(t) - x_0 \in \int_0^t F(s, x(\eta(s)))\, ds, \;\; t \in J. \tag{21}$$

Now the desired conclusion follows by an application of Theorem 4.1 with $q(t) = x_0$, $\sigma(t) = t$ for all $t \in J$ and $k(t, s) = 1\ \forall\ t, s \in J$, since $AC(J, \mathbb{R}) \subset BM(J, \mathbb{R})$. $\square$

5.2. Boundary Value Problems: Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the first and second boundary value problems (in short BVPs) of functional differential inclusions

$$\left.\begin{aligned} & x''(t) \in F(t, x(\eta(t))), a.e. t \in J \\ & x(0) = 0 = x(1) \end{aligned}\right\} \tag{22}$$

and

$$\left.\begin{aligned} & x''(t) \in F(t, x(\eta(t))), a.e. t \in J \\ & x(0) = 0 = x'(1) \end{aligned}\right\} \tag{23}$$

where $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ and $\eta : J \to J$ is continuous.

Definition 5.2 *By a solution of the BVP (22) or (23) we mean a function* $x \in AC^1(J, \mathrm{I\!R})$ *such that* $x''(t) = v(t)$ *for all* $t \in J$ *for some* $v \in S^1_F(x)$ *, where* $AC^1(J, \mathrm{I\!R})$ *is the space of all continuous real-valued functions whose first derivative exists and is absolutely continuous on J.*

The study of BVPs (19) and (21) for existence result has been made in Halidias and Papageorgiou [12] via the method of Lower and upper solutions. In the present paper, we shall discuss the existence of extremal solutions between the given lower and upper solutions.

Define an order relation in $AC^1(J, \mathrm{I\!R})$ by the cone K_{AC} given by (20). Clearly K_{AC} is a normal cone in $AC^1(J, \mathrm{I\!R})$.

A solution x_M of BVP (22) or (23) is called maximal if for any solution x of such BVP, $x(t) \le x_M(t)$ for all $t \in J$. Similarly a minimal solution of BVP (22) or (23) is defined.

Definition 5.3 *A function* $a \in AC^1(J, \mathrm{I\!R})$ *is called a lower solution of BVP (22) if for all* $v \in S^1_F(x), x \in AC(J, \mathrm{I\!R})$ *we have* $a''(t) \le v(t)$ *for all* $t \in J$ *with* $a(0) = 0 = a(1)$. *Again a function* $b \in AC^1(J, \mathrm{I\!R})$ *is called an upper solution of BVP (22) if for all* $v \in S^1_F(x), x \in AC(J, \mathrm{I\!R})$ *we have* $b''(t) \ge v(t)$ *for all* $t \in J$ *with* $b(0) = 0 = b(1)$. *Similarly an upper and lower solution of BVP (23) are defined.*

We consider the following hypothesis:

(H_7) The BVP (22) has a lower solution a and an upper solution b with $a \le b$.

(H_8) The BVP (23) has a lower solution a and an upper solution b with $a \le b$.

Theorem 5.2 *Assume that hypotheses* (H_1)-(H_4) *and* (H_7) *hold. Then BVP (22) has a minimal and a maximal solution on J.*

Proof: A function $x : J \to R$ is a solution of BVP (22) if and only if it is a solution of the integral inclusion

$$x(t) \in \int_0^1 G(t,s)F(s, x(\eta(s))))\, ds, \quad t \in J \tag{24}$$

where $G(t, s)$ is a Green's function associated with the homogeneous linear BVP

$$x''(t) = 0, \quad \text{a.e. } t \in J$$

$$x(0) = 0 = x(1).$$

It is known that $G(t, s)$ is a continuous and nonnegative real-valued function on $J \times J$. Now an application of Theorem 4.1 with $q(t) = 0$, $\sigma(t) = 1$ for all $t \in J$ and $k(t, s) = G(t, s), \forall\, t, s \in J$ yields that BVP (22) has a minimal and a maximal solution on J, $AC1(J, \mathrm{I\!R}) \subset BM(J, \mathrm{I\!R})$. □

Theorem 5.3 *Assume that hypotheses (H_1)-(H_4) and (H_8) hold. Then BVP (23) has a minimal and a maximal solution on J.*

Proof: A function $x : J \to \mathrm{I\!R}$ is a solution of BVP (23) if and only if it is a solution of the integral inclusion

$$x(t) \in \int_0^1 H(t, s) F(s, x(\eta(s)))\, ds, \ \in J,$$

where $H(t, s)$ is a Green's function for the BVP

$$\left.\begin{array}{l} x''(t) = 0 \text{ a.e. } \ t \in J, \\ x(0) = 0 = x'(1). \end{array}\right\} \tag{25}$$

It is known that $H(t, s)$ is a continuous and nonnegative real-valued function on $J \times J$. Now an application of Theorem 4.1 with $q(t) = 0$, $\sigma(t) = 1$ for all $t \in J$ and $k(t, s) = H(t, s), \forall\, t, s \in J$ yields that BVP (23) has a minimal and a maximal solution on J, $AC^1(J, \mathrm{I\!R}) \subset BM(J, \mathrm{I\!R})$. □

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