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NONNEGATIVE SOLUTIONS OF PARABOLIC OPERATORS WITH LOWER-ORDER TERMS

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Abstract

We develop the harmonic analysis approach for parabolic operator with one order term in the parabolic Kato class on $C^{1,1}$-cylindrical domain Ω. We study the boundary behaviour of nonnegative solutions. Using these results, we prove the integral representation theorem and the existence of nontangential limits on the boundary of Ω for nonnegative solutions. These results extend some first ones proved for less general parabolic operators.

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1. INTRODUCTION

In this paper we are interested in some aspects of the theory of the differential parabolic operator

$$L = \frac{\partial}{\partial t} - \operatorname{div}(A(x,t)\nabla_x) + B(x,t).\nabla_x$$

defined on $\Omega = D\times]0,T[$, where D is a bounded $C^{1,1}$-domain of $\mathbf{R^n}$ and $0 < T < \infty$. The matrix $A(x,t)$ is assumed to be real, symmetric, uniformly elliptic, i.e. $\frac{1}{\mu}I \leq A(x,t) \leq \mu I$ for some $\mu \geq 1$, with Lipschitz coefficients. The vector $B(x,t)$ is assumed to be in the parabolic Kato class as introduced by Zhang in [15], i.e. $B \in L^1_{loc}$ and satisfies $\lim_{h\to 0} N_h^\alpha(B) = 0$, where

$$\begin{aligned} N_h^\alpha(B) &= \sup_{x,t} \int_{t-h}^{t} \int_D |B(y,s)| \frac{1}{(t-s)^{\frac{n+1}{2}}} \exp\left(-\alpha \frac{|x-y|^2}{t-s}\right) dyds \\ &+ \sup_{y,s} \int_{s}^{s+h} \int_D |B(x,t)| \frac{1}{(t-s)^{\frac{n+1}{2}}} \exp\left(-\alpha \frac{|x-y|^2}{t-s}\right) dxdt \end{aligned}$$

for some constant $\alpha > 0$.

In fact, the real starting points of this work are the famous papers [10] of Kemper, [5] of Fabes, Garofalo and Salsa, [9] of Heurteaux and [12] of Nyström. We recall here that, as was initially studied for the Laplace operator by Hunt and Wheeden in [7] and [8], the notion of kernel function, the integral representation theorem and the existence of nontangential limit at the boundary for nonnegative solutions (Fatou's theorem) for the heat equation have been developed by Kemper in [10] on Lipschitz domains. In his work an important role was played by the invariance of the heat equation under translations. The results of Hunt and Wheeden have been later extended to more general elliptic equations by Ancona in [1] and Gaffarelli, Fabes, Mortola and Salsa in [6]. In [5], Fabes, Garofalo and Salsa are interested in the same problem for parabolic operators in divergence form with measurable coefficients on Lipschitz cylinders. When they attempted to adapt the techniques of [6] for their case, an interesting difficulty occurs, namely to prove the "doubling" property, which was essential for the proof of Fatou's theorem and which is equivalent to the existence of a "backward" Harnack inequality for nonnegative solutions (we refer the reader to [5] for more details). By proving some boundary Harnack principles for nonnegative solutions, they succeeded in resolving the problem for parabolic operators with time independent coefficients and they established all of Kemper's results in this case. In [9], Heurteaux took up the same problem for parabolic operators in divergence form with Lipschitz coefficients on more general Lipschitz domains, and by a straightforward adaptation of the idea of Ancona [1], he was able to extend the results of Fabes, Garofalo and Salsa to his situation. Recently, Nyström studied in [12] parabolic operators in divergence form with measurable coefficients on Lipschitz domains and he proved among other things, the existence and uniqueness of a kernel function and established the integral representation theorem.

In this paper, our aim is to investigate the above mentioned results for our operator. The main difficulty is created by the lower order term where we cannot benefit from results proved for L having adjoint companions as is the case of operators in divergence form in [5], [9], [10] and [12]. To overcome this difficulty our idea is based on the Green function estimates proved by the author in [13] and the Harnack inequality recently proved by Zhang in [15], under the above assumptions. Our method seems to be new and applies to similar parabolic operators and our results include their counterparts for the elliptic operator $\operatorname{div}(A(x)\nabla_x)+B(x).\nabla_x$ with B in the elliptic Kato class, i.e. $B \in L^1_{loc}(D)$ and satisfies $\lim_{\alpha\to 0}\sup_x \int_{|x-y|\leq\alpha} \frac{|B(y)|}{|x-y|^{n-1}}dy = 0$, which was studied by several authors. Our paper is organized as follows.

In Section 1, we give some notations and we state some known results that will be used throughout this paper. In Section 2, basing on the Green function estimates (Theorem 2.2, below), we prove a boundary Harnack principle and a comparison theorem for nonnegative L-solutions vanishing on a part of the parabolic boundary $\partial_p\Omega$ of Ω. In Section 3, using the previous results and the Harnack inequality (Theorem 2.1, below), we characterize the Martin boundary of the cylinder Ω with respect to the class of parabolic operators L that we deal with. More precisely, we prove that for every point $Q \in \partial_p\Omega$ there exists a unique (up to a multiplicative constant) minimal nonnegative L-solution, and then the Martin boundary of Ω with respect to L is homomorphic (or identical) to the parabolic boundary $\partial_p\Omega$ of Ω. In Section 4, we are able to define the kernel function and prove, basing on the previous results, the integral representation theorem for nonnegative L-solutions on Ω. In particular, we deduce a Fatou type theorem for our operator by proving that any nonnegative L-solution on Ω has a nontangential limit at the boundary except for a set of zero L-parabolic measure.

2. NOTATIONS AND KNOWN RESULTS

Let G be the L-Green function on $\Omega = D\times]0,T[$. We simply denote by G_A the function $G(\cdot,\cdot;y,s)$ if $A=(y,s)\in\Omega$.

A point $x \in \mathbf{R^n}$ will be also denoted by (x', x_n) with $x' \in \mathbf{R^{n-1}}$ and $x_n \in \mathbf{R}$, when we need.

For $x \in D$, let $d(x)$ denotes the distance from x to the boundary ∂D of D. For an open set Ω of $\mathbf{R^{n+1}}$, let $\partial_p\Omega$ be the parabolic boundary of Ω, i.e. $\partial_p\Omega$ is the set of points on the boundary of Ω which can be connected to some interior point of Ω by a closed curve having a strictly increasing t-coordinate. For an arbitrary set Σ in Ω and a function u on Ω, we denote by $R^\Sigma u$ the nonnegative L-superparabolic envelope of u with respect to Σ which also called the "reduct" of u with respect to Σ, and defined by

$$R^\Sigma u = \inf \{ v : v \text{ nonnegative } L - \text{supersolution on } \Omega \text{ with } v \geq u \text{ on } \Sigma \} .$$

We next recall some known results that will be used in this work.

Theorem 2.1.(Harnack inequality [15]). *Let* $0 < \alpha < \beta < \alpha_1 < \beta_1 < 1$ *and* $\delta \in (0,1)$ *be given. Then there are constants* $C > 0$ *and* $r_0 > 0$ *such that for all* $(x,s) \in \mathbf{R^n} \times \mathbf{R}$, *all positive* $r < r_0$ *and all nonnegative weak* L*-solutions* u *in* $B(x,r) \times [s-r^2, s]$, *one has*

$$\sup_{\Omega^-} u \leq C \inf_{\Omega^+} u,$$

where $\Omega^- = B(x,\delta r) \times [s-\beta_1 r^2, s-\alpha_1 r^2]$ *and* $\Omega^+ = B(x,\delta r) \times [s-\beta r^2, s-\alpha r^2]$. *All constants depend on* B *only in terms of the rate of convergence of* $N_h^\alpha(B)$ *to zero when* $h \to 0$.

Theorem 2.2.(Green function estimates [13]). *There exist positive constants* k, c_1 *and* c_2 *depending only on* n, μ, T, D *and on* B *only in terms of the rate of convergence of* $N_h^\alpha(B)$ *to zero when* $h \to 0$ *such that*

$$\frac{1}{k}\varphi(x,y,t-s)\frac{\exp\left(-c_2\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}} \leq G(x,t;y,s) \leq k\varphi(x,y,t-s)\frac{\exp\left(-c_1\frac{|x-y|^2}{t-s}\right)}{(t-s)^{n/2}}$$

for all x, $y \in D$ *and* $0 < s < t \leq T$, *where* $\varphi(x,y,u) = \min\left(1, \frac{d(x)}{\sqrt{u}}, \frac{d(y)}{\sqrt{u}}, \frac{d(x)d(y)}{u}\right)$.

Theorem 2.3.(Minimum principle). *Let* Ω *be a bounded open set of* $\mathbf{R^{n+1}}$ *and* u *an* L*-supersolution in* Ω *satisfying* $\liminf_{z \to z_0} u \geq 0$ *for all* $z_0 \in \partial_p\Omega$. *Then* $u \geq 0$ *in* Ω.

3. BOUNDARY BEHAVIOUR

We prove in this section a boundary Harnack principle and a comparison theorem for nonnegative L-solutions vanishing on a part of the parabolic boundary, which will be used in the next section to characterize the Martin boundary of $\Omega = D\times]0,T[$.

D is a $C^{1,1}$-bounded domain, then for each $z \in \partial D$ there exists a local coordinate system $(\xi', \xi_n) \in \mathbf{R^{n-1}} \times \mathbf{R}$, a function ψ on $\mathbf{R^{n-1}}$ and constants $c_0 > 0$ and $r_0 \in]0,1]$ such that
i) $\nabla_{\xi'}\psi$ is c_0-Lipschitz,
ii) $D \cap B(z,r_0) = B(z,r_0) \cap \{(\xi',\xi_n) : \xi_n > \psi(\xi')\}$, and
iii) $\partial D \cap B(z,r_0) = B(z,r_0) \cap \{(\xi',\xi_n) : \xi_n = \psi(\xi')\}$.
By compactness of ∂D, the constants c_0 and r_0 can be chosen independent of $z \in \partial D$.

For $Q \in \mathbf{R^{n+1}}$, $r > 0$ and $h > 0$, we denote by $T_Q(r,h)$ the cylinder

$$T_Q(r,h) = Q + \left\{(x', x_n, t) \in \mathbf{R^{n+1}} : |x'| < r,\ |t| < r^2,\ |x_n| < h\right\}.$$

We have the following result.

Theorem 3.1(Boundary Harnack principle). *Let $Q \in \partial D\times]0,T[$, $r \in]0,r_0]$ and $\lambda > 0$. Then there exists a constant $C > 0$ depending only on n, μ, λ, D, T and on B in terms of the rate of convergence of $N_h^\alpha(B)$ to zero when $h \to 0$ such that for all nonnegative L-solutions u on $\Omega \setminus T_Q(\frac{r}{2}, \lambda\frac{r}{2})$ continuously vanishing on $\partial_p\Omega \setminus T_Q(\frac{r}{2}, \lambda\frac{r}{2})$, we have*

$$u(M) \le C\, u(M_r)$$

for all $M \in \Omega \setminus T_Q(r, \lambda r)$, where $M_r = Q + (0, \lambda r, r^2)$.

Proof. Without loss of generality we assume $Q = (0,0,S) \equiv (0,\psi(0),S)$, where ψ as defined above is the function which, after a suitable rotation, describes ∂D as a graph around $(0,0)$. In view of the minimum principle, it suffices to prove the theorem for $M \in \Omega \cap \partial T_Q(r, \lambda r)$.
We first consider the particular case $u = G_A$ with $A \in \Omega \cap T_Q(\frac{r}{2}, \lambda\frac{r}{2})$.
We write

$$A = Q + (y,s) \equiv Q + (y', y_n, s) \text{ with } |y'| \le \frac{r}{2},\ 0 < y_n \le \lambda\frac{r}{2},\ |s| \le \frac{r^2}{4},$$

$$M = Q + (x,t) \equiv Q + (x', x_n, t), \quad \text{and} \quad M_r = Q + (0, \lambda r, r^2).$$

By Theorem 2.2, we have

$$\frac{G_A(M)}{G_A(M_r)} \le k^2 \left(\frac{r^2 - s}{t - s}\right)^{n/2+1} \frac{d(x)}{\lambda r} \exp\left(c_2 \frac{|\lambda r - y_n|^2 + |y'|^2}{r^2 - s} - c_1 \frac{|x - y|^2}{t - s}\right).$$

Using the fact that

$$\frac{|\lambda r - y_n|^2 + |y'|^2}{r^2 - s} \le \frac{4}{3}\lambda^2 + 1$$

and

$$\frac{d(x)}{\lambda r} \le \frac{x_n - \psi(x')}{\lambda r} \le \frac{x_n + |\nabla_{x'}\psi||x'|}{\lambda r} \le \frac{\lambda r + c_0 r}{\lambda r} = \frac{\lambda + c_0}{\lambda},$$

we have

$$\frac{G_A(M)}{G_A(M_r)} \le k_1 \left(\frac{r^2 - s}{t - s}\right)^{n/2+1} \exp\left(-c_1 \frac{|x - y|^2}{t - s}\right).$$

From the inequality $e^{-\alpha} \le (\frac{m}{\alpha e})^m$, for all $m > 0$, $\alpha > 0$, it follows that

$$\frac{G_A(M)}{G_A(M_r)} \le k_2 \min\left(\left(\frac{r^2 - s}{t - s}\right)^{n/2+1}, \left(\frac{r^2 - s}{|x - y|^2}\right)^{n/2+1}\right).$$

Since $M \in \Omega \cap \partial T_Q(r, \lambda r)$, we need to study the following three cases:
If $t = r^2$, $0 < x_n \le \lambda r$, and $|x'| \le r$, then

$$\frac{G_A(M)}{G_A(M_r)} \le k_2.$$

If $x_n = \lambda r$, $|t| \le r^2$, and $|x'| \le r$, then

$$\frac{G_A(M)}{G_A(M_r)} \le k_2 \left(\frac{r^2 - s}{|x_n - y_n|^2}\right)^{n/2+1} \le k_2 \left(\frac{5}{\lambda^2}\right)^{n/2+1} = C.$$

If $|x'| = r$, $0 < x_n < \lambda r$, and $|t| \le r^2$, then

$$\frac{G_A(M)}{G_A(M_r)} \le k_2 \left(\frac{r^2 - s}{|x' - y'|^2}\right)^{n/2+1} \le k_2 5^{n/2+1} = C.$$

Note that the same estimate holds when the pole A lies in $\Omega \cap T_Q(\varepsilon r, \lambda \varepsilon r)$ with $0 < \varepsilon < 1$. The constant C then depends also on ε.

For the general case, by considering the set $\Sigma = \Omega \setminus T_Q(\frac{2}{3}r, \frac{2}{3}\lambda r)$ we see that the function $v = R^\Sigma u$ is an L-potential on Ω with support in $\Omega \cap \partial\Sigma$ and then there exists a positive measure μ supported in $\Omega \cap \partial\Sigma$ such that $R^\Sigma u = \int_{\Omega\cap\partial\Sigma} G_A d\mu(A)$.
For all $M \in \Omega \setminus T_Q(r, \lambda r)$, we have

$$
\begin{aligned}
R^\Sigma u(M) &= \int_{\Omega\cap\partial\Sigma} G_A(M) d\mu(A) \\
&\leq C \int_{\Omega\cap\partial\Sigma} G_A(M_r) d\mu(A) \\
&= C R^\Sigma u(M_r) \\
&= C u(M_r),
\end{aligned}
$$

which completes the proof. $\square$

In the sequel, for $\lambda > 0$, we denote by $\mathcal{C}_\lambda$ the set

$$
\mathcal{C}_\lambda = \left\{ (x,t) \in \mathbf{R^{n+1}} : t > \sup\left(|x'|^2, \frac{|x_n|^2}{\lambda^2} \right) \right\}.
$$

We next have the following result.

Theorem 3.2(Comparison theorem). *Let $Q \in \partial D \times]0, T[$, $\lambda > 0$, and for $\rho > 0$ denote $M_\rho = Q + (0, \lambda\rho, \rho^2)$. Then there exists a constant $C > 0$ depending only on n, μ, λ, D, T and on B in terms of the rate of convergence of $N_h^\alpha(B)$ to zero when $h \to 0$ such that for all $r \in]0, \frac{r_0}{4}]$ and for any two nonnegative L-solutions u, v on $\Omega \setminus T_Q(r, \lambda r)$ continuously vanishing on $\partial_p\Omega \setminus T_Q(r, \lambda r)$, we have*

$$
\frac{u(M)}{u(M_{2r})} \leq C \frac{v(M)}{v(M_{2r})},
$$

for all $M \in [\Omega \cap (Q + \mathcal{C}_\lambda) \cap T_Q(r_0, \lambda r_0)] \setminus T_Q(2r, 2\lambda r)$.

Proof. Without loss of generality we assume $Q = (0, 0, S)$. We first prove the estimate for $u = G_A$ and $v = G_B$ with $A, B \in \Omega \cap T_Q(r, \lambda r)$.
Let

$$
M = Q + (x', x_n, t) \quad \text{with} \quad |x'| \leq r_0,\ 0 < x_n \leq \lambda r_0,\ 4r^2 < t \leq r_0^2,
$$

$$
\text{and} \quad t \geq \sup\left(|x'|^2, \frac{|x_n|^2}{\lambda^2} \right).
$$

Put

$$A = Q + (y', y_n, s) \quad \text{with} \quad |y'| \leq r,\ 0 < y_n \leq \lambda r,\ |s| \leq r^2,$$

and

$$B = Q + (z', z_n, \rho) \quad \text{with} \quad |z'| \leq r,\ 0 < z_n \leq \lambda r,\ |\rho| \leq r^2.$$

By Theorem 2.2, we have

$$\begin{aligned}
\frac{G_A(M)G_B(M_{2r})}{G_A(M_{2r})G_B(M)} \quad &\leq \quad k\left[\frac{(t-\rho)(4r^2-s)}{(t-s)(4r^2-\rho)}\right]^{n/2+1} \\
&\quad \times \exp\left(c_2(\frac{|x-z|^2}{t-\rho} + \frac{|y'|^2 + |2\lambda r - y_n|^2}{4r^2-s})\right).
\end{aligned}$$

Using the fact that

$$\frac{t-\rho}{t-s} = 1 + \frac{s-\rho}{t-s} \leq 1 + \frac{2r^2}{3r^2} = \frac{5}{3},$$

$$\frac{4r^2-s}{4r^2-\rho} \leq \frac{5r^2}{3r^2} = \frac{5}{3},$$

$$\begin{aligned}
\frac{|x-z|^2}{t-\rho} \quad &\leq \quad 2\frac{|x|^2}{t-\rho} + 2\frac{|z|^2}{t-\rho} \\
&= \quad 2\frac{t}{t-\rho}\frac{|x|^2}{t} + 2\frac{|z|^2}{t-\rho} \\
&\leq \quad 2(1+\frac{r^2}{3r^2})(1+\lambda^2) + 2\frac{(1+\lambda^2)r^2}{3r^2} \\
&= \quad \frac{10}{3}(1+\lambda^2),
\end{aligned}$$

and

$$\frac{|y'|^2 + |2\lambda r - y_n|^2}{4r^2-s} \leq \frac{r^2 + 4\lambda^2 r^2}{3r^2} = \frac{1+4\lambda^2}{3}$$

hold, we obtain

$$\frac{G_A(M)G_B(M_{2r})}{G_A(M_{2r})G_B(M)} \leq C.$$

For the general case, by considering the set $\Sigma = \Omega \setminus T_Q(r, \lambda r)$ we see that the functions $R^\Sigma u$ and $R^\Sigma v$ are two L-potentials on Ω with support in $\partial T_Q(r, \lambda r)$

and then there exist two positive measures σ and ν supported in $\partial T_Q(r, \lambda r)$ such that $R^\Sigma u = \int_{\partial T_Q(r,\lambda r)} G_A d\sigma(A)$ and $R^\Sigma v = \int_{\partial T_Q(r,\lambda r)} G_B d\nu(B)$.
From the last inequality we then deduce

$$\int\int G_A(M)G_B(M_{2r})d\sigma(A)d\nu(B) \leq C \int\int G_B(M)G_A(M_{2r})d\sigma(A)d\nu(B),$$

which means

$$R^\Sigma u(M)R^\Sigma v(M_{2r}) \leq CR^\Sigma v(M)R^\Sigma u(M_{2r}),$$

and so the required estimate follows from the equalities $R^\Sigma u = u$ on Σ and $R^\Sigma v = v$ on Σ.

4. MINIMAL NONNEGATIVE L-SOLUTIONS

In this section we exploit the results of Section 2 to characterize the Martin boundary of Ω. More precisely we show that for every point $Q \in \partial_p\Omega$ there exists a unique (up to a multiplicative constant) minimal nonnegative L-solution, and then the Martin boundary is identical to $\partial_p\Omega$.
We first introduce the notion of minimal nonnegative L-solution.

Definition 4.1. A nonnegative L-solution u on a given domain Ω of $\mathbf{R^{n+1}}$ is called *minimal* if every L-solution v on Ω satisfying the inequalities $0 \leq v \leq u$ is a constant multiple of u.

In view of a limiting argument given by Lemma 2.1 in [15], we may assume that $|B| \in L^\infty$. We denote by $\mathcal{H}$ the set of L-solutions on Ω. We recall that $(\Omega, \mathcal{H})$ is a $\mathcal{P}$-Bauer space in the sense of [4] and any minimal nonnegative L-solution is the limit of a sequence of extreme potentials (see [11], Lemma 1.1). Note that an extreme potential is a potential with point support, and by Theorem III in [2] any two potentials in the whole space $\mathbf{R^n} \times \mathbf{R}$ with the same point support are proportional. Since the hypothesis of proportionality is satisfied if and only if it is satisfied locally (see [11] Lemma 1.3), this property holds in Ω. It follows that every minimal nonnegative L-solution is the limit of a sequence $c_kG(x, t; y_k, s_k)$ for some sequence of poles $(y_k, s_k) \subset \Omega$ and constants $c_k \in \mathbf{R}_+$. By compactness of $\overline{\Omega}$, it is clear that if $h(x, t) = \lim_{k\to+\infty} c_kG(x, t; y_k, s_k)$ is a minimal nonnegative L-solution, then there exists a subsequence of $((y_k, s_k))_k$ which converges to a point $(y, s) \in \partial_p\Omega$. The reverse of this result constitutes the object of the following theorem.

Theorem 4.2. *For each point $Q = (y, s) \in \partial_p\Omega$, there exist sequences $((y_k, s_k))_k$ convergent to Q and $(c_k)_k$ in $\mathbf{R}_+$ such that the function $h(x,t) = \lim_{k\to+\infty} c_k G(x, t; y_k, s_k)$ is a minimal nonnegative L-solution.*

Proof. Case 1: $y \in \partial D$ and $s > 0$.
Consider a sequence $(A_n)_n \subset \Omega$ convergent to Q and put $\varphi_{A_n} = \dfrac{G_{A_n}}{G_{A_n}(M_{r_0})}$, where $M_{r_0} = Q + (0, \lambda r_0, r_0^2)$.
By Theorem 3.1, there exists a constant $C = C(n, \mu, \lambda, T, B) > 0$ such that for all $r \in]0, r_0]$, $n \geq n(r) \in \mathbf{N}$, we have

$$\varphi_{A_n}(M) \leq C\, \varphi_{A_n}(M_r),$$

for all $M \in \Omega \setminus T_Q(r, \lambda r)$.
On the other hand by the Harnack inequality (Theorem 2.1), there exists a constant $C' = C'(n, \mu, \lambda, T, B) > 0$ such that

$$\varphi_{A_n}(M_r) \leq C' \varphi_{A_n}(M_{r_0}) = C'.$$

Therefore, for all $r \in]0, r_0]$, $n \geq n(r) \in \mathbf{N}$, we have

$$\varphi_{A_n}(M) \leq CC',$$

for all $M \in \Omega \setminus T_Q(r, \lambda r)$.
This means that $(\varphi_{A_n})_n$ is locally uniformly bounded and then it has a subsequence converging to a nonnegative L-solution φ on Ω vanishing on $\partial\Omega \setminus \{Q\}$. To prove that φ is minimal, denote by $C_Q(\Omega)$ the set of all nonnegative L-solutions on Ω vanishing on $\partial\Omega \setminus \{Q\}$. We will show that $C_Q(\Omega)$ is a half-line generated by a minimal nonnegative L-solution. Using the Harnack inequality and Theorem 3.1 again we see that $C_Q(\Omega)$ is a convex cone with compact base $\mathcal{B} = \{u \in C_Q(\Omega) : u(M_{r_0}) = 1\}$, and by the Krein-Milman theorem it is generated by the extremal elements of $\mathcal{B}$ which are the minimal nonnegative L-solutions. To complete the proof, it suffices to prove that two minimal nonnegative L-solutions in Ω are proportional.

Recall that if h is a minimal nonnegative L-solution in Ω and $E \subset \Omega$, then $\hat{R}_h^E = h$ or $\hat{R}_h^E$ is an L-potential of Ω, where $\hat{R}_h^E$ is the lower semi-continuous regularization of R_h^E. We say E is thin at h if $\hat{R}_h^E$ is an L-potential of Ω. Using Theorem 3.1 and Theorem 3.2 we prove as in [9] (Proposition 4.2) that $\Omega \cap (Q + \mathcal{C}_\lambda)$ is not thin at h.

Let h_1, h_2 two minimal nonnegative L-solutions of $C_Q(\Omega)$. By Theorem 3.2, there exists $C = C(n, \lambda, \mu, T, B) > 0$ such that

$$\frac{h_1(M)}{h_1(M_{2r})} \leq C \frac{h_2(M)}{h_2(M_{2r})},$$

for $0 < r \leq \dfrac{r_0}{4}$, $M \in [\Omega \cap (Q + \mathcal{C}_\lambda) \cap T_Q(r_0, \lambda r_0)] \setminus T_Q(2r, 2\lambda r)$, and this also gives

$$\frac{h_1(M)}{h_1(M_{r_0})} \leq C^2 \frac{h_2(M)}{h_2(M_{r_0})},$$

for all $M \in \Omega \cap (Q + \mathcal{C}_\lambda) \cap T_Q(r_0, \lambda r_0)$.
Using the non-thinness of $\Omega \cap (Q + \mathcal{C}_\lambda)$ at h_1 and h_2 we see that the previous inequality holds on Ω which means $h_1 \leq \alpha h_2$, $\alpha \geq 0$, and consequently h_1, h_2 are proportional.

Case 2: $y \in \partial D$ and $s = 0$.
By the first case, there exists a minimal nonnegative L-solution $\widetilde{h}$ on $\widetilde{\Omega} = D \times]-1, T[$ vanishing on $\partial\widetilde{\Omega} \setminus \{Q\}$. In view of the minimum principle, $\widetilde{h}(x,t) = 0$ for $t < 0$. Clearly, the function $h \equiv \widetilde{h}_{/\Omega}$ is a minimal nonnegative L-solution on Ω.

Case 3: $y \in D$ and $s = 0$.
Let $h(x,t) = G(x,t;y,s)$. We prove that h is a minimal nonnegative L-solution on Ω. Let u be a nonnegative L-solution on Ω such that $u \leq h$. We define $\widetilde{u}$ by

$$\widetilde{u}(x,t) = \begin{cases} u(x,t) & \text{if} \quad 0 < t < T \\ 0 & \text{if} \quad -1 \leq t \leq 0. \end{cases}$$

Denote by $\widehat{u}$ the lower semi-continuous regularization of $\widetilde{u}$, then $\widehat{u}$ is a nonnegative L-superparabolic function on $\widetilde{\Omega} = D \times]-1, T[$ with harmonic support $\{(y,0)\}$ and $\widehat{u}(x,t) \leq \widetilde{G}(x,t;y,0)$, where $\widetilde{G}$ is the L-Green function of $\widetilde{\Omega}$. It follows that $\widehat{u}$ is an L-potential of $\widetilde{\Omega}$ with support $\{(y,0)\}$ and so there exists $C \geq 0$ such that $\widehat{u}(x,t) = C\,\widetilde{G}(x,t;y,0)$. This gives $u(x,t) = C\,G(x,t;y,0) = C\,h(x,t)$, and then h is minimal.

5. INTEGRAL REPRESENTATION AND NONTANGENTIAL LIMITS

Following the characterization of the Martin boundary in Section 3, we are now able to define the kernel function associated to our operator and the cylinder Ω. Let $Q_0 = (x_0, t_0)$ be a given point in Ω.

Definition 5.1. We say that a function $K : \Omega \to [0, +\infty]$ is an *L-kernel function at $Q = (y, s) \in \partial_p\Omega$ normalized at Q_0* if the following conditions are fulfilled:
i) $K(x, t) \geq 0$ for each $(x, t) \in \Omega$ and $K(Q_0) = 1$,
ii) $K(\cdot, \cdot)$ is an L-solution in Ω,
iii) $K(\cdot, \cdot) \in C(\overline{\Omega} \setminus \{Q\})$ and $\lim_{(x,t)\to(y_0,s_0)} K(x, t) = 0$ if $(y_0, s_0) \in \partial_p\Omega \setminus \{Q\}$,
iv) $K(\cdot, \cdot) \equiv 0$, if $s \geq t_0$.

It is clear that by means of Theorem 4.2, for each point $Q \in \partial_p\Omega$, there exists a unique L-kernel function at Q normalized at Q_0. We denote this unique kernel function by K_Q.

Note that from the proof of Theorem 4.2, $K_Q = \dfrac{G_Q}{G_Q(Q_0)}$, when $Q = (y, 0)$.

For $p \in \Omega \cap \{t < t_0\}$, we also denote by K_p the function $K_p = \dfrac{G_p}{G_p(Q_0)}$.

We have the following continuity property of the L-kernel function.

Proposition 5.2. *Under the previous notations we have*

$$\lim_{p\to p_0, p\in\overline{\Omega}} K_p(M) = K_{p_0}(M),$$

for all $p_0 \in \partial\Omega \cap \{t < t_0\}$ and all $M \in \Omega$.

Proof. Denote by $p_0 = (y, s)$. When $y \in D$ and $s = 0$, i.e. $p_0 = (y, 0)$, the proposition holds by the continuity of the Green function.
By considering $\widetilde{\Omega} = D\times] - 1, T[$ instead of $\Omega = D\times]0, T[$ it is enough to prove the proposition for $y \in \partial D$ and $s > 0$. In the sequel we treat this case.
Let $(q_n)_n \subset \overline{\Omega}$ be a sequence convergent to p_0. By Theorem 3.1 there exists $C > 0$ such that for n sufficiently large and r sufficiently small we have

$$K_{q_n}(M) \leq CK_{q_n}(M_r),$$

for all $M \in \Omega \setminus T_{p_0}(r, \lambda r)$, where $M_r = p_0 + (0, \lambda r, r^2)$.
We deduce from the minimum principle that

$$K_{q_n}(M) \leq CK_{q_n}(M_r)h_r(M),$$

holds for all $M \in \Omega \setminus T_{p_0}(r, \lambda r)$, where h_r is the L-parabolic measure of $\partial T_{p_0}(r, \lambda r)$ in $\Omega \setminus \overline{T}_{p_0}(r, \lambda r)$.

On the other hand, there exists, by the Harnack inequality (Theorem 2.1), a constant $C' > 0$ such that

$$K_{q_n}(M_r) \le C' K_{q_n}(Q_0) = C'.$$

Combining the two previous inequalities, we obtain

$$K_{q_n}(M) \le CC' h_r(M),$$

for all $M \in \Omega \setminus T_{p_0}(r, \lambda r)$.
This inequality proves that any adherence value of (K_{q_n}) is an element of $C_{p_0}(\Omega)$, the cone of nonnegative L-solutions vanishing on $\partial_p \Omega \setminus \{p_0\}$ which is a half-line generated by K_{p_0}. The sequence $(K_{q_n})_n$ is locally uniformly bounded, hence it has adherence values and by evaluating at Q_0, we conclude that K_{p_0} is the only adherence value of $(K_{q_n})_n$. Thus $(K_{q_n}(M))_n$ converges to $K_{p_0}(M)$ for all $M \in \Omega$.

Theorem 5.3 (Integral representation). *Let u be a nonnegative L-solution in $\Omega = D\times]0, T[$. Then there exists two unique positive Borel measures μ_1, μ_2 on $\partial D\times]0, t_0[$ and D, respectively, such that*

$$u(x,t) = \int_{\partial D \times]0,t[} K_{(y,s)}(x,t) \mu_1(dy, ds) + \int_D G(x,t;y,0)\mu_2(dy),$$

for all $(x,t) \in \Omega \cap \{t < t_0\}$.

Proof. Let E be the real vector space generated by the differences of any two nonnegative L-solutions on Ω. E endowed with the topology of uniform convergence on compact subdomains is a locally convex vector space which is metrizable. The set $\mathcal{C} = \{u_{/\{t \le t_0\}} : u \in E, u \ge 0\}$ is a convex cone which is reticulate for the natural order and $\mathcal{B} = \{u \in \mathcal{C} : u(Q_0) = 1\}$ is a base of $\mathcal{C}$ which is compact and metrizable. Note that the extremal elements of $\mathcal{B}$ are exactly the minimal nonnegative L-solutions on $\Omega \cap \{t \le t_0\}$ normalized at Q_0. To clarify this point, let u be an extremal element of $\mathcal{B}$ and v an L-solution satisfying $0 \le v \le u$ with $v \ne 0$ and $v \ne u$. Then $v(Q_0) \ne 0$ and $v(Q_0) \ne 1$ and the equality

$$u = v(Q_0)\frac{v}{v(Q_0)} + (1 - v(Q_0))\,\frac{u - v}{1 - v(Q_0)}$$

implies

$$u = \frac{v}{v(Q_0)} = \frac{u-v}{1-v(Q_0)},$$

which means $v = v(Q_0)u$.
Conversely, let $u \in \mathcal{B}$ be a minimal nonnegative L-solution and suppose that there exist $\alpha \in]0,1[$, u_1, $u_2 \in \mathcal{B}$ such that $u = \alpha u_1 + (1-\alpha)u_2$; then $\alpha u_1 \leq u$ and $(1-\alpha)u_2 \leq u$ which implies $u_1 = \beta_1 u$ and $u_2 = \beta_2 u$ for some β_1, $\beta_2 \in \mathbf{R}_+$. By evaluating at Q_0, we have $\beta_1 = \beta_2 = 1$ and so $u = u_1 = u_2$.
Denote by $\mathcal{E}$ the set of extremal elements of $\mathcal{B}$. By the Choquet theorem, for any $u \in \mathcal{B}$ there exists a unique positive Radon measure μ supported in $\mathcal{E}$ such that $u = \int_{\mathcal{E}} h d\mu(h)$. This also implies

$$u(M) = \int_{\mathcal{E}} h(M) d\mu(h),$$

for all $M \in \Omega$, since the map $h \to h(M)$ is a continuous linear form.
On the other hand, the Martin boundary of $\Omega \cap \{t \leq t_0\}$ is $\Delta = \partial_p \Omega \cap \{t \leq t_0\}$ and by Proposition 5.2, the kernel functions K_Q are continuous as a functions of Q. Therefore the map $\Delta \to \mathcal{E}$, $Q \to K_Q$, is a homeomorphism which transforms μ into a positive Radon measure ν on Δ. Hence, the previous equality gives

$$u(M) = \int_{\Delta} K_Q(M) d\nu(Q),$$

for all $M \in \Omega \cap \{t \leq t_0\}$.

By proportionality this representation holds for any nonnegative L-solution on Ω. Using the fact that $\Delta = (\partial D \times [0, t_0[) \cup (D \times \{0\})$ and denoting by $\mu_1 = \nu_{/(\partial D \times [0,t_0[)}$ and $\mu_2 = \dfrac{1}{G(x_0, t_0; \cdot, \cdot)} \nu_{/(D \times \{0\})}$, we obtain the equality asserted in the theorem. □

At this point we have all the tools we need to study nontangential limits for nonnegative L-solutions on the boundary of Ω. Basing on the integral representation theorem and the abstract Fatou's theorem [14] we may prove in the same way as in [9] (Theorem 6.2) the existence of nontangential limits for nonnegative L-solutions in Ω (Theorem 5.6, below). Since the theory is by now standard we will not give the details of the proof but we only state the result. We first introduce the notion of nontangential limit.

Definition 5.4. Let Ω be an open set of $\mathbf{R^{n+1}}$ and $(Q_n)_n$ a sequence of points in Ω. We say that $(Q_n)_n$ *converges nontangentially* to a point $Q \in \partial\Omega$, if $\lim\limits_{n\to+\infty} Q_n = Q$ and $\inf\limits_n \dfrac{d(Q_n, \partial\Omega)}{d(Q_n, Q)} > 0$, where d is the parabolic distance which is defined by $d((x,t),(y,s)) = |x-y| + |t-s|^{1/2}$.

Definition 5.5. Let Ω be an open set of $\mathbf{R^{n+1}}$ and u a function defined on Ω. We say that u has a *nontangential limit* $l \in \mathbf{R}$ at $Q \in \partial\Omega$, if for any sequence $(Q_n)_n \subset \Omega$ converging nontangentially to Q, one has $\lim\limits_{n\to+\infty} u(Q_n) = l$.

We have the following interesting result.

Theorem 5.6. *Let u be a nonnegative L-solution in $\Omega = D\times]0,T[$. Then u has a finite nontangential limit for $d\mu^{(x_0,t_0)}$-almost every point $Q \in \partial\Omega$, where $d\mu^{(x_0,t_0)}$ denotes the L-parabolic measure associated with a given point (x_0,t_0) in Ω.*

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