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Complete description of the set of solutions to a strongly nonlinear O.D.E.

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Abstract

We give a complete description of the set of solutions to the boundary value problem

$$-\left(\varphi\left(u'\right)\right)' = f\left(u\right) \text{ in } (0,1)\,;\; u\left(0\right) = u\left(1\right) = 0$$

where φ is an odd increasing homeomorphism of $\mathbb{R}$ and $f \in C\left(\mathbb{R}, \mathbb{R}\right)$ is odd.

Key words. Strongly nonlinear O.D.E's, Sturm-liouville b.v.p's, Rabinowitz global bifurcation theory.

$\mathcal{A}.\mathcal{M}.\mathcal{S}$. Subject Classifications: 34B15, 34C15

1 Introduction

The purpose of this paper is to give a complete description of the set of solutions to the boundary value problem

$$\begin{cases} -\left(\varphi\left(u'\right)\right)' = f\left(u\right) \text{ in } (0,1) \\ u\left(0\right) = u\left(1\right) = 0 \end{cases} \tag{1}$$

where φ is an odd increasing homeomorphism of $\mathbb{R}$ and f is an odd function of $C\left(\mathbb{R}, \mathbb{R}\right)$.

By a solution of (1), we mean a function $u \in C^1\left([0,1]\right)$ satisfying $\left(\varphi\left(u'\right)\right)' = -f\left(u\right)$ in $(0,1)$ and the Dirichlet conditions $u\left(0\right) = u\left(1\right) = 0$.

Note that the differential operator $u \to \left(\varphi\left(u'\right)\right)'$ is linear if and only the function $x \to \varphi\left(x\right)$ is linear, hence the ODE in (1) is said strongly nonlinear.

This work is motivated by the previous ones done in [13], [14], [15], [8] and essentially by [16].

In [16] García-Huidobro & Ubilla study problem (1) under the following hypothesis on the functions f and φ

$$\lim_{x\to 0} \frac{\varphi\left(\sigma x\right)}{\varphi\left(x\right)} = \sigma^{q-1} \text{ for some } q > 1 \text{ and for all } \sigma \in (0,1)\,,$$

$$\lim_{x\to+\infty}\frac{\varphi(\sigma x)}{\varphi(x)}=\sigma^{p-1} \text{ for some } p>1 \text{ and for all } \sigma\in(0,1),$$

$$\lim_{x\to 0}\frac{f(x)}{\varphi(x)}=a \text{ and } \lim_{x\to+\infty}\frac{f(x)}{\varphi(x)}=A.$$

Using time-maps approach they give a multiplicity result when a and A lie in some resonance intervals.

In this work we will replace the growing conditions on φ and f at 0 and $+\infty$ by global conditions on the convexity of φ and f. These new conditions which will play significant role in the proof of existence of solutions as well as in the proof of uniqueness of these solutions in some areas of $C^1([0,1])$, can appear very restrictive. However we think that this condition is usual, indeed this kind of assumption is often met in the literature when an exactitude result is aimed (see [3], [6] and [18]).

Our strategy is as follows:

In a first stage, we locate the possible solutions of problem (1) in some subsets A_k^ν, (where for $k\in\mathbb{N}^*$ and $\nu=+,-$ A_k^ν is defined in section 2) of $C^1([0,1])$ and we give some properties of these solutions. An immediate consequence of these results is: $u\in A_k^+$ is solution to problem (1) if and only if u is a positive solution to the problem

$$\begin{cases} -(\varphi(u'))'=f(u) \text{ in } \left(0,\frac{1}{2k}\right)\\ u(0)=u'\left(\frac{1}{2k}\right)=0.\end{cases} \tag{2}$$

Then we associate to problem (2) the auxiliary Sturm-Liouville problem

$$\begin{cases} -v''(x)=\widetilde{f}\left(\int_0^x\psi(v'(t))\,dt\right) \text{ in } \left(0,\frac{1}{2k}\right)\\ v(0)=v'\left(\frac{1}{2k}\right)=0.\end{cases} \tag{3}$$

such that u is positive solution to problem (2) if and only if $v(x)=\int_0^x\varphi(u'(t))\,dt$ is a positive solution to the auxiliary Sturm-Liouville problem (3). Thus we are brought to investigate a nonlinear Sturm-Liouville problem for which after addition of a linear part containing a real parameter existence of a positive solution will be proved by the use of Rabinowitz global bifurcation theory (see [19], [20] and [21]).

At the end, we will use assumptions (5) and (7) to prove uniqueness of the solution in each subset A_k^ν.

The paper is organized as follows: Section 2 is devoted to the statement of the main results and some necessary notations. In section 3 we expose some preliminary results we need in the proof of the principal results. In the last section we give the proofs of main results.

2 Notations and main results

In the following we denote by $\mathbb{E}=C^1([a,b])$ with its norm $\|u\|_1=\|u\|_0+\|u'\|_0$

Let, for any integer $k \geq 1$ and $a < b$

$$S_k^+ = \left\{ \begin{array}{c} u \in \mathbb{E} : u \text{ admits exactly } (k-1) \text{ zeros in }]a,b[\\ \text{all are simple, } u(a) = u(b) = 0 \text{ and } u'(a) > 0 \end{array} \right\}$$

$S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$.

Let u be a function belonging to $C([a,b])$ which vanishes at x_1 and x_2 $(x_1 < x_2)$. If u does not vanish at any point of the open interval $I =]x_1, x_2[$ we call its restriction to this interval $I-$ hump of u. When there is no confusion we say a hump of u.

With this definition in mind, each function in S_k^+ has exactly k humps such that the first one is positive, the second is negative, and so on with alterations.

Let A_k^+ $(k \geq 1)$ the subset of S_k^+ composed by the functions u satisfying:

- Every hump of u is symmetrical about the center of the interval of its definition.
- Every positive (resp. negative) hump of u can be obtained by translating the first positive (resp. negative) hump.
- The derivative of each hump of u vanishes one and only one time.

Let $A_k^- = -A_k^+$ and $A_k = A_k^+ \cup A_k^-$.

We recall that the boundary value problem:

$$\left\{ \begin{array}{l} -u'' = \lambda u \text{ in } (a,b) \\ u(a) = u'(b) = 0 \end{array} \right.$$

has an increasing sequence of eigenvalues $(\mu_k([a,b]))_{k \geq 1}$ with $\mu_k([a,b]) = \dfrac{(2k-1)^2 \pi^2}{4(b-a)^2}$.

We will use in this work the so called Jensen inequality given by:

$$F\left(\frac{1}{b-a} \int_a^b u(t)\,dt \right) \geq \frac{1}{b-a} \int_a^b F(u(t))\,dt$$

where $F : \mathbb{R} \to \mathbb{R}$ is a concave function and u is a function in $C([a,b])$.

Moreover if $b - a < 1$ and $F(0) = 0$ then

$$F\left(\int_a^b u(t)\,dt \right) \geq \int_a^b F(u(t))\,dt \tag{4}$$

Let S be the set of solutions to problem (1), then our main results are :

Theorem 1 ***(Superlinear case) :***

Suppose the functions φ and f satisfy the following conditions:

$$\varphi \textit{ is concave on } \mathbb{R}^+, \tag{5}$$

$$\lim_{x\to 0}\frac{f(x)}{\varphi(x)}=0 \text{ and } \lim_{x\to+\infty}\frac{f(x)}{\varphi(x)}=+\infty, \tag{6}$$

$$\text{the function } s\to\frac{f(s)}{s} \text{ is increasing on } (0,+\infty) \tag{7}$$

Then

$$S\subset\{0\}\cup\left(\underset{k\geq 1}{\cup}A_k\right)$$

and for each integer $k\geq 1$ *there exists* $u_k\in A_k^+$ *such that*

$$S\cap A_k=\{\,u_k\,,-u_k\,\}.$$

Theorem 2 ***(Sublinear case) :***

Suppose the functions φ *and* f *satisfy the following conditions:*

$$\varphi \text{ is convex on } \mathbb{R}^+, \tag{8}$$

$$\lim_{x\to 0}\frac{f(x)}{\varphi(x)}=+\infty \text{ and } \lim_{x\to+\infty}\frac{f(x)}{\varphi(x)}=0, \tag{9}$$

$$f \text{ is increasing and concave in } \mathbb{R}^+. \tag{10}$$

Then

$$S\subset\{0\}\cup\left(\underset{k\geq 1}{\cup}A_k\right)$$

and for each integer $k\geq 1$ *there exists* $u_k\in A_k^+$ *such that*

$$S\cap A_k=\{\,u_k\,,-u_k\,\}.$$

Remark 1 *The above theorems give a complete description of the solution set of the problem* (1)*, indeed the theorems state that there is no solution except the trivial solution and those belonging to* $\underset{k\geq 1}{\cup}A_k$*, and in each* $A_k^{\pm}$ *there is exactly one solution.*

Remark 2 *Hypothesis* (7) *is similar to (3-3) assumed in [4]. To obtain the exact number of solutions to the boundary value problem*

$$\begin{cases} -u''=\lambda u+f(u) \text{ in } (0,1) \\ u(0)=u(1)=0 \end{cases} \tag{11}$$

according λ *in a resonance interval, the author assumed the function* $s\to\dfrac{f(s)}{s}$ *and* $s\to\dfrac{-f(-s)}{s}$ *are increasing on* $(0,+\infty)$.

Note that, hypothesis (7) *implies that* f *is increasing, and if* f *is convex then hypothesis* (7) *is satisfied.*

In the sublinear case, hypothesis (10) *implies that the function* $s\to\dfrac{f(s)}{s}$ *is decreasing on* $(0,+\infty)$.

3 Some preliminary results:

In this section we give some lemmas which will be crucial for the proof of our main results.

Consider the boundary value problem

$$\begin{cases} -(\varphi(u'))' = g(u) \text{ in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{12}$$

where φ is an odd increasing homeomorphism of $\mathbb{R}$ and g is a function in $C(\mathbb{R}, \mathbb{R})$ satisfying

$$xg(x) > 0 \text{ for all } x \in \mathbb{R}^*. \tag{13}$$

We define a solution of problem (12) to be a function $u \in \mathbb{E}$ satisfying $(\varphi(u'))' = -g(u)$ in (a,b) and $u(a) = u(b) = 0$.

If u is a solution to problem (12), then there exists a real constant $C \geq 0$ such that

$$\Psi(\varphi(u'(x))) + G(u(x)) = C \text{ for all } x \in [a,b] \tag{14}$$

where $G(x) = \displaystyle\int_0^x g(t)\,dt$, $\Psi(x) = \displaystyle\int_0^x \psi(t)\,dt$ with $\psi = \varphi^{-1}$.

Note that Ψ the Legendre transform of the convex function Φ where $\Phi(s) = \int_0^s \varphi(t)\,dt$, is even , $\Psi(0) = 0$ and $\Psi(s) > 0$ for all $s \neq 0$.

Then the first result in this section is:

Lemma 3 *Suppose that hypothesis* (13) *holds true. If u is a nontrivial solution to problem* (12)*, then there exists an integer $k \geq 1$ such that $u \in A_k$.*

Proof. Let u be a nontrivial solution to problem (12). We begin the proof by showing $u'(a) \neq 0$.

Let us suppose the contrary. Then, if we put $x = a$ in equation (14), we get $C = 0$. Thus, for any $x \in [0,1]$, $G(u(x)) = -\Psi(\varphi(u'(x))) \leq 0$. Since G is strictly positive on $\mathbb{R}^*$ and $G(0) = 0$, $u(x) = 0$ for all $x \in [a,b]$. This is impossible since u is a nontrivial solution.

Now, let us show that u has a finite number of zeros. Suppose the contrary and let (z_n) the infinite sequence of zeros of u and z_* an accumulate point of (z_n). Then we have

$$u(z_*) = u'(z_*) = \lim_{n\to\infty} \frac{u(z_n) - u(z_*)}{z_n - z_*} = 0.$$

Again, putting $x = z_*$ in equation (14) we get the same contradiction as above.

Let z_1 and z_2 two consecutive zeros of u, and suppose that $u > 0$ in (z_1, z_2) and y_* is a critical point of u in (z_1, z_2). It follows from equation (12) that $(\varphi(u'))' = -g(u)$ in (z_1, z_2). Since φ is an increasing odd homeomorphism of $\mathbb{R}$, $u' > 0$ in (z_1, y_*), $u' < 0$ in (y_*, z_2) and $u'(y_*) = 0$. Thus y_* is the unique critical point of u at which u reach its maximum value.

Let

$$\rho = u\left(y_*\right) = \max_{x\in(z_1,\, z_2)} u\left(x\right)$$

It follows from equation (14) that

$$u'\left(t\right) = \psi\left(\Psi_+^{-1}\left(G\left(\rho\right) - G\left(u\left(t\right)\right)\right)\right) \text{ for all } t \in [z_1,\ y_*] \tag{15}$$

and

$$u'\left(t\right) = -\psi\left(\Psi_+^{-1}\left(G\left(\rho\right) - G\left(u\left(t\right)\right)\right)\right) \text{ for all } t \in [y_*, z_2] \tag{16}$$

where Ψ_+^{-1} is the inverse of Ψ on $\mathbb{R}^+$. Then

$$x - z_1 = \int_0^{u(x)} \frac{du\left(t\right)}{u'\left(t\right)} = \int_0^{u(x)} \frac{du\left(t\right)}{\psi\left(\Psi_+^{-1}\left(G\left(\rho\right) - G\left(u\left(t\right)\right)\right)\right)} \text{ for all } x \in [z_1, y_*] \tag{17}$$

and

$$z_2 - x = -\int_0^{u(x)} \frac{du\left(t\right)}{u'\left(t\right)} = \int_0^{u(x)} \frac{du\left(t\right)}{\psi\left(\Psi_+^{-1}\left(G\left(\rho\right) - G\left(u\left(t\right)\right)\right)\right)} \text{ for all } x \in [y_*, z_2] \tag{18}$$

Putting $x = y_*$ in equations (17) and (18) we get

$$y_* - z_1 = \int_0^{\rho} \frac{du\left(t\right)}{\psi\left(\Psi_+^{-1}\left(G\left(\rho\right) - G\left(u\left(t\right)\right)\right)\right)} = z_2 - y_*$$

which yields

$$y_* = \frac{z_1 + z_2}{2}.$$

For the symmetry of the $(z_1, z_2)-$hump of u about $\dfrac{z_1 + z_2}{2}$, it suffices to show that for all $x \in [z_1, z_2]$ $u\left(z_1 + z_2 - x\right) = u\left(x\right)$.This becomes very easy if we observe that $x = (z_1 + z_2) - (z_1 + z_2 - x)$ and make use of equations (17) and (18), then we get: in each of the cases $x \in \left[z_1, \dfrac{z_1 + z_2}{2}\right]$ or $x \in \left[\dfrac{z_1 + z_2}{2}, z_2\right]$

$$\begin{aligned} x - z_1 &= z_2 - (z_1 + z_2 - x) = \int_0^{u(x)} \frac{du\left(t\right)}{\psi\left(\Psi_+^{-1}\left(G\left(\rho\right) - G\left(u\left(t\right)\right)\right)\right)} \\ &= \int_0^{u(z_1+z_2-x)} \frac{du\left(t\right)}{\psi\left(\Psi_+^{-1}\left(G\left(\rho\right) - G\left(u\left(t\right)\right)\right)\right)} \end{aligned}$$

which leads to $u\,(z_1+z_2-x)=u\,(x)$ for all $x\in[\,z_1,\ z_2]$[1].

It remains to show that if $z_3<z_4$ are two consecutive zeros of u and $u>0$ in $[\,z_3,\ z_4]$, then $u_{[\,z_3,\ z_4]}$ is the translation of $u_{[\,z_1,\ z_2]}$.

To do this it suffices to prove that $u\,(z_3+(x-z_1))=u\,(x)$ for all $x\in[z_1,z_2]$.

Putting respectively $x=\dfrac{z_1+z_2}{2}$ and $x=\dfrac{z_3+z_4}{2}$ in equation (14) we deduce

$$C=G\left(u\left(\frac{z_1+z_2}{2}\right)\right)=G\left(u\left(\frac{z_3+z_4}{2}\right)\right)$$

Since G is strictly increasing on $(0,+\infty)$, $u\left(\dfrac{z_1+z_2}{2}\right)=u\left(\dfrac{z_3+z_4}{2}\right)$.

Making use of equations (17) and (18), we get :

$$\begin{aligned}
z_4-\frac{z_3+z_4}{2} &= \frac{z_4-z_3}{2}\\
&= \int_0^{u\left(\frac{z_3+z_4}{2}\right)} \frac{du\,(t)}{\psi\left(\Psi_+^{-1}\left(G\left(u\left(\frac{z_3+z_4}{2}\right)\right)-G\left(u\,(t)\right)\right)\right)}\\
&= \int_0^{u\left(\frac{z_1+z_2}{2}\right)} \frac{du\,(t)}{\psi\left(\Psi_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right)-G\left(u\,(t)\right)\right)\right)}\\
&= z_2-\frac{z_1+z_2}{2}=\frac{z_2-z_1}{2}
\end{aligned}$$

which yields $z_3+(z_2-z_1)=z_4$.

If we set $v\,(x)=u\,(z_3+(x-z_1))$ for all $x\in[z_1,z_2]$, then we have

$$\begin{aligned}
v\,(z_1) &= u\,(z_3)=0\\
v\,(z_2) &= u\,(z_4)=0
\end{aligned}$$

Observe that u and v are solutions of the problem

$$\begin{cases} -\left(\varphi\,(w')\right)'=g\,(w) \ \text{ in } \ (z_1,z_2)\\ w\,(z_1)=w\,(z_2)=0\end{cases}$$

So, for any $x\in\left[z_1,\dfrac{z_1+z_2}{2}\right]$, we have:

$$\begin{aligned}
x-z_1 &= \int_0^{u(x)} \frac{du\,(t)}{\psi\left(\Psi_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right)-G\left(u\,(t)\right)\right)\right)}\\
&= \int_0^{v(x)} \frac{dv\,(t)}{\psi\left(\Psi_+^{-1}\left(G\left(v\left(\frac{z_1+z_2}{2}\right)\right)-G\left(v\,(t)\right)\right)\right)}
\end{aligned}$$

[1]We have $\int_0^a f\,(t)\,dt=\int_0^b f\,(t)\,dt$ with $f>0$.

which leads to $v(x) = u(x)$ for all $x \in \left[z_1, \frac{z_1+z_2}{2}\right]$.

Using the symmetry of the function u we deduce that $v(x) = u(x)$ for all $x \in [z_1, z_2]$. This completes the proof of the lemma. ■

Lemma 4 *Suppose that hypothesis* (13) *holds true and g is odd. If $u \in A_k^+$ (resp. A_k^-) is solution to problem* (12) *with $k \geq 2$ then the first negative (resp. positive) hump of u is a translation of the first negative (resp. positive) of $(-u)$.*

Proof. Let $u \in A_k^+$ be a solution to problem (12) and $(z_i)_{i=0}^{i=k}$ the finite sequence of zeros of u such that $0 = z_0 < z_1 < z_2 < \cdots < z_k = 1$.

Since the positive (resp. negative) humps of u are translations of the first positive (resp. negative) hump one, it suffices to prove that $u_{[z_1,z_2]}$ is a translation of $-u_{[0,z_1]}$.

Let us prove that the two humps have the the same length.. Putting $x = \frac{z_1}{2}$ and $x = \frac{z_1+z_2}{2}$ in (14) we get

$$C = G\left(u\left(\tfrac{z_1}{2}\right)\right) = G\left(u\left(\tfrac{z_1+z_2}{2}\right)\right).$$

Since G is even and increasing in $\mathbb{R}^+$

$$u\left(\tfrac{z_1}{2}\right) = -u\left(\tfrac{z_1+z_2}{2}\right).$$

Set $\rho = u\left(\frac{z_1}{2}\right) = -u\left(\frac{z_1+z_2}{2}\right)$, as in the proof of Lemma 3

$$\frac{z_1}{2} = \int_0^{\rho} \frac{ds}{\psi\left(\Psi_+^{-1}\left(G(\rho) - G(s)\right)\right)}$$

and

$$\begin{aligned}\frac{z_2 - z_1}{2} = \frac{z_1+z_2}{2} - z_1 &= \int_{u\left(\frac{z_1+z_2}{2}\right)}^{0} \frac{ds}{\psi\left(\Psi_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right) - G(s)\right)\right)} \\ &= \int_0^{-u\left(\frac{z_1+z_2}{2}\right)} \frac{ds}{\psi\left(\Psi_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right) - G(s)\right)\right)} \\ &= \int_0^{\rho} \frac{ds}{\psi\left(\Psi_+^{-1}\left(G(\rho) - G(s)\right)\right)} = \frac{z_1}{2}\end{aligned}$$

which leeds to

$$z_2 - z_1 = z_1.$$

Setting $v(x) = -u(z_1 + x)$ for all $x \in [0, z_1]$ and arguing as in the proof of Lemma 3, we get $u(x) = v(x) = -u(z_1 + x)$ for all $x \in [0, z_1]$. So the lemma is proved ■

Lemma 5 *Suppose that hypothesis* (13) *holds true. If $u_1 \neq u_2$ are two positives solutions of problem* (12)*, then $u_1$and u_2 are ordered, namely $u_1{<}u_2$ in (a, b) or $u_1{<}u_2$ in (a, b).*

Proof. Let u_1 and u_2 be two solutions of the lemma.

We have

• either $u_1'(a) = u_2'(a)$

• or $u_1'(a) \neq u_2'(a)$.

Assume first that the first situation holds. We deduce from equation (14) :

$$G\left(u_1\left(\tfrac{a+b}{2}\right)\right) = \Psi\left(\varphi\left(u_1'(a)\right)\right) = G\left(u_2\left(\tfrac{a+b}{2}\right)\right) = \Psi\left(\varphi\left(u_2'(a)\right)\right).$$

Since G is strictly increasing, we get $u_1\left(\dfrac{a+b}{2}\right) = u_2\left(\dfrac{a+b}{2}\right)$.

Let $\rho = u_1\left(\dfrac{a+b}{2}\right) = u_2\left(\dfrac{a+b}{2}\right)$, then (17) written for u_1 and u_2 gives:

$$\begin{aligned} x-a &= \int_0^{u_1(x)} \frac{du(t)}{\psi\left(\Psi_+^{-1}\left(G(\rho)-G(u_1(t))\right)\right)} \\ &= \int_0^{u_2(x)} \frac{du(t)}{\psi\left(\Psi_+^{-1}\left(G(\rho)-G(u_2(t))\right)\right)} \quad \text{for any } x \in \left[a, \frac{a+b}{2}\right]. \end{aligned}$$

Hence, $u_1(x) = u_2(x)$ for all $x \in \left[a, \dfrac{a+b}{2}\right]$. Since u_1 and u_2 are in A_1^+; namely u_1 and u_2 are symmetrical about $\frac{a+b}{2}$, $u_1(x) = u_2(x)$ for all $x \in [a,b]$, which contradicts the statement of the lemma.

Now, suppose that $u_1'(a) < u_2'(a)$. Since u_1 and u_2 are symmetrical about $\dfrac{a+b}{2}$, we will prove that $u_1(x) < u_2(x)$ for all $x \in \left(a, \dfrac{a+b}{2}\right]$.

Let $A = \left\{x \in \left(a, \frac{a+b}{2}\right], u_1(x) = u_2(x)\right\}$. Assume $A \neq \emptyset$ and let $x_0 = \inf A$ and $u = u_1 - u_2$.

Then $x_0 > a$, indeed if $x_0 = a$ and (x_n) is a sequence such that $\lim\limits_{n\to+\infty} x_n = x_0$, we get :

$$0 < u_2'(a) - u_1'(a) = \lim_{n\to+\infty} \frac{u_2(x_n) - u_1(x_n)}{x_n - a} = 0$$

which is impossible.

Thus, let (y_n) be a sequence in (a, x_0) such that $\lim\limits_{n\to+\infty} y_n = x_0$. We get:

$$u'(x_0) = \lim_{n\to+\infty} \frac{u(y_n) - u(x_0)}{y_n - x_0} = \lim_{n\to+\infty} \frac{u(y_n)}{y_n - x_0} \leq 0$$

then,

$$0 \leq u_2(x_0) \leq u_1(x_0).$$

Using again (14), we obtain:

$$\begin{gathered} \Psi\left(\varphi\left(u_1'(a)\right)\right) - \Psi\left(\varphi\left(u_1'(x_0)\right)\right) = G\left(u_1(x_0)\right) \\ = G\left(u_1(x_0)\right) = \Psi\left(\varphi\left(u_2'(a)\right)\right) - \Psi\left(\varphi\left(u_2'(x_0)\right)\right) \end{gathered}$$

so

$$0 > \Psi\left(\varphi\left(u_1'\left(a\right)\right)\right) - \Psi\left(\varphi\left(u_2'\left(a\right)\right)\right) = \Psi\left(\varphi\left(u_1'\left(x_0\right)\right)\right) - \Psi\left(\varphi\left(u_2'\left(x_0\right)\right)\right) \geq 0$$

which is impossible, therefore $A = \emptyset$ ■.

4 Proof of the main results

Since the function f is odd and satisfies hypothesis (13), it leads from lemma 3 any non trivial solution to problem (1) belongs to $\underset{k\geq 1}{\cup} A_k$.

4.1 Existence of solutions :

It arises from lemmas 3 and 4: to get a solution belonging to A_k^+ (resp. A_k^-) to problem (1) it suffices to prove that the problem

$$\begin{cases} -\left(\varphi\left(u'\left(x\right)\right)\right)' = f\left(u\left(x\right)\right) \text{ in } \left(0, \frac{1}{2k}\right) \\ u\left(0\right) = u'\left(\frac{1}{2k}\right) = 0. \end{cases} \tag{19}$$

admits a positive (resp. negative) solution.[2]

Set $f^+ = \max\left(f, 0\right)$ and consider the boundary value problem

$$\begin{cases} -v''\left(x\right) = f^+\left(\int_0^x \psi\left(v'\left(t\right)\right)dt\right) \text{ in } \left(0, \frac{1}{2k}\right) \\ v\left(0\right) = v'\left(\frac{1}{2k}\right) = 0. \end{cases} \tag{20}$$

Observe that if v is a positive solution to problem (20) if and only if $u\left(x\right) = \int_0^x \psi\left(u'\left(t\right)\right)dt$ is a positive solution to the problem $(19)^3$.

Hence, we are brought to look for positive solutions to the problem

$$\begin{cases} -v''\left(x\right) = f^+\left(\int_0^x \psi\left(v'\left(t\right)\right)dt\right) \text{ in } \left(0, a\right) \\ v\left(0\right) = v'\left(a\right) = 0 \end{cases} \tag{21}$$

where $a \in \left(0, 1\right)$.

Consider the boundary value problem

$$\begin{cases} -v''\left(x\right) = \lambda v\left(x\right) + f^+\left(u\left(x\right)\right) \text{ in } \left(0, a\right) \\ v\left(0\right) = v'\left(a\right) = 0. \end{cases} \tag{22}$$

where λ is a real parameter and $u\left(x\right) = \int_0^x \psi\left(v'\left(t\right)\right)dt$.

We mean by a solution of problem (22) a pair $\left(\lambda, v\right) \in \mathbb{R}\times C^1\left(\left[0, a\right]\right)$ satisfying $-v''\left(x\right) = \lambda v\left(x\right) + f^+\left(\int_0^x \psi\left(v'\left(t\right)\right)dt\right)$ $x \in \left(0, a\right)$ and the boundary conditions $v\left(0\right) = v'\left(a\right) = 0$.

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[2] Any positive solution of (19) is concave. to see that one can use (14).

[3] Any solution of (20) is concave.

4.1.1 Existence in the superlinear case:

Let $\varepsilon > 0$, we deduce from assumption (6) existence of $\delta > 0$ such that

$$\text{for all } x \in \mathbb{R},\ |x| < \delta \text{ implies } |f(x)| < \varepsilon |\varphi(x)| = \varepsilon\varphi(|x|).$$

Since ψ is an odd increasing function on $\mathbb{R}^+$, we have for $v \in C^1([0,a])$ and for all $x \in [0,a]$

$$\begin{aligned}\left|\int_0^x \psi(v'(t))\,dt\right| \leq \int_0^x |\psi(v'(t))|\,dt &= \int_0^x \psi(|v'(t)|)\,dt \\ &\leq \psi(\|v\|_1)\end{aligned}$$

Thus, if $\eta := \varphi(\delta)$ then for all $v \in C^1([0,a])$

$$\|v\|_1 < \eta \text{ implies } \left|\int_0^x \psi(v'(t))\,dt\right| \leq \delta \text{ for all } x \in [0,a]$$

then

$$\begin{aligned}\left|f\left(\int_0^x \psi(v'(t))\,dt\right)\right| &= f\left(\left|\int_0^x \psi(v'(t))\right|dt\right) \\ &\leq \varepsilon\varphi\left(\left|\int_0^x \psi(v'(t))\,dt\right|\right) \\ &\leq \varepsilon\varphi(\psi(\|v\|_1)) \\ &\leq \varepsilon\|v\|_1\end{aligned}$$

which means $f(u) = \circ(\|v\|_1)$ and $f^+(u) = \circ(\|v\|_1)$

Therefore, Rabinowitz global bifurcation theory (see [19] and [20]) states: the pair $(\lambda_1, 0)$ is a bifurcation point for a component $\mathbb{S}_1^+ \subset \mathbb{R}\times \tilde{S}{}_1^+$ of positive solutions to (22) which is unbounded in $\mathbb{R}\times C^1([0,a])$ where $\lambda_1 = \mu_1([0,a])$ and

$$\tilde{S}{}_1^+ = \left\{v \in C^1([0,a]) : v(0) = v'(a) = 0 \text{ and } v > 0 \text{ in } (0,a)\right\}.$$

Thus, to prove existence of a positive solution to problem (21) it suffices to show the following

Theorem 6 *$\mathbb{S}_1^+$ crosses $\{0\} \times C^1([0,a])$.*

Before proving theorem 6, we need the following lemma:

Lemma 7 *If $(\lambda, v) \in \mathbb{S}_1^+$ then $\lambda < \lambda_1$.*

Proof. Let Φ be the first positive eigenfunction of

$$\begin{cases} -\Phi'' = \lambda_1\Phi \text{ in } (0,a) \\ \Phi(0) = \Phi'(a) = 0. \end{cases}$$

Multiplying (22) by Φ and integrating on $(0,a)$ we get:

$$-\int_0^a v''\Phi = \lambda\int_0^a v\Phi + \int_0^a f^+(u)\,\Phi$$

Then, two integrations by parts give

$$(\lambda_1 - \lambda) \int_0^a v\Phi = \int_0^a f^+ (u)\, \Phi > 0$$

which leads to

$$\lambda < \lambda_1.$$

■

Proof of theorem 6

Suppose the contrary, and let $(\lambda_n, v_n) \subset \mathbb{S}_1^+$an unbounded sequence in $\mathbb{R}\times C^1([0,a])$ and set $u_n(x) = \int_0^x \psi(v_n'(t))\, dt$. An immediate consequence of Lemma 7 is: $0 < \lambda_n < \lambda_1$ and (v_n) is unbounded in $C^1([0,a])$.

First Let us prove that v_n is unbounded with the respect of the C^0 norm. Suppose the contrary; Since $-v_n'' = \lambda_n v_n + f(u_n)$ and v_n'' is unbounded[4] with the respect of the C^0 norm, u_n is unbounded with the respect of the C^0 norm on $[0,a]$.

Let for any $R > 0$ $J_n = \{x \in [0,a] : \varphi(u_n(x)) \geq R\}$.

We claim that there exist $R_0 > 0$ such that $l(J_n) \leq \dfrac{1}{2a}$ .This is due to:

Denote by θ_n the real number belonging to $[0,a]$ such that $\varphi(u_n(\theta_n)) = R$ and let Φ_n and $\lambda_{1,n}$ be respectively the first positive eigenfunction and the first eigenvalue of the problem

$$\begin{cases} -v'' = \lambda v \text{ in } (\theta_n, a) \\ v(\theta_n) = v'(a) = 0. \end{cases}$$

Multiplying (22) by Φ_n and integrating between θ_n and a we get

$$\int_{\theta_n}^a -v_n''\Phi_n = \lambda_n \int_{\theta_n}^a v_n\Phi_n + \int_{\theta_n}^a f^+(u_n)\,\Phi_n.$$

After two integrations by parts we obtain:

$$\lambda_{1,n} \int_{\theta_n}^a v_n\Phi_n \geq \lambda_n \int_{\theta_n}^a v_n\Phi_n + \int_{\theta_n}^a f^+(u_n)\,\Phi_n. \tag{23}$$

We deduce from hypothesis (6) that $\lim\limits_{x\to+\infty} \dfrac{f^+(\psi(x))}{x} = +\infty$, so for $M = \dfrac{\pi^2}{a^2}$ there exists $R_0 > 0$ such that

$$x \geq R_0 \text{ implies } f^+(\psi(x)) \geq Mx.$$

[4]Otherwise v_n' will be bounded on $[0,a]$ with the respect of the C^0 norm, and then v_n with the respect of the C^1 norm.

Thus, we deduce from (23):

$$\begin{aligned}(\lambda_{1,n}-\lambda_n)\int_{\theta_n}^{a} v_n\Phi_n &\geq \int_{\theta_n}^{a}\left(f^{+}\circ\psi\right)\left(\varphi\left(u_n\right)\right)\Phi_n \\ &\geq M\int_{\theta_n}^{a}\varphi\left(u_n\right)\Phi_n\end{aligned}\quad. \tag{24}$$

Since φ is concave, Jensen inequality (4) leads to

$$\varphi\left(u_n\left(x\right)\right)\geq v_n\left(x\right)\ \text{ for all } x\in\left[\theta_n,a\right].$$

Thus, we deduce from (24):

$$\left(\lambda_{1,n}-\left(\lambda_n+M\right)\right)\int_{\theta_n}^{a} v_n\Phi_n\geq 0.$$

then

$$\frac{\pi^2}{\left(a-\theta_n\right)^2}\geq\left(\lambda_n+M\right)$$

finally

$$l\left(J_n\right)=\left(\frac{1}{2}-\theta_n\right)\leq\frac{1}{2a} \tag{25}$$

Now let us return to the equation satisfied by u_n. We have

$$-\left(\varphi\left(u_n'\right)\right)'=\lambda_n v_n+f^{+}\left(u_n\right)\ \text{ in }\ (0,a)$$

Multiplying by u' and integrating $[x,a]$, we get

$$\Psi\left(\varphi\left(u_n'\left(x\right)\right)\right)=F^{+}\left(\rho_n\right)-F^{+}\left(u_n\left(x\right)\right)+\lambda_n\int_{x}^{a} v_n u_n'\ \text{for all } x\in[0,a]$$

where $\rho_n=u_n\left(a\right)$ and $F^{+}\left(x\right)=\int_0^x f^{+}\left(t\right)dt$.
Then as in the proof of Lemma 3 we obtain

$$\begin{aligned}\theta_n=\int_0^{R_0}\frac{du_n\left(t\right)}{u_n'\left(t\right)}&=\int_0^{R_0}\frac{ds}{\psi\left(\Psi_{+}^{-1}\left(F^{+}\left(\rho_n\right)-F^{+}\left(s\right)+\lambda_n\int_x^{\frac{1}{2}}v_n u_n'\right)\right)}\\ &\leq\int_0^{R_0}\frac{ds}{\psi\left(\Psi_{+}^{-1}\left(F^{+}\left(\rho_n\right)-F^{+}\left(s\right)\right)\right)}\end{aligned} \tag{26}$$

Thus, on one hand, since $\dfrac{1}{\psi\left(\Psi_{+}^{-1}\left(F^{+}\left(\rho_n\right)-F^{+}\left(s\right)\right)\right)}$ is bounded in $[0,R_0]$ and $\lim\limits_{n\to\infty}\rho_n=+\infty$.

$$\lim_{n\to+\infty}\theta_n=\lim_{n\to+\infty}\int_0^{R_0}\frac{ds}{\psi\left(\Psi_{+}^{-1}\left(F^{+}\left(\rho_n\right)-F^{+}\left(s\right)\right)\right)}=0$$

and on the other hand it arises from (25) $\theta_n \geq \dfrac{1}{2a}$ which is impossible and v_n is unbounded in $C^0([0,a])$.

Now arguing as above, let for any $R > 0$ $J_n = \{x \in [0,a] : \ v_n(x) \geq R\}$, $R_0 > 0$ such that $l(J_n) \leq \dfrac{1}{2a}$ and θ_n the real number belonging to $[0,a]$ such that $v_n(\theta_n) = R_0$.

Thus, in one hand

$$R_0 = \int_0^{\theta_n} v_n'(t)\,dt \geq \frac{1}{2a} v_n'(\theta_n) \tag{27}$$

and on the other hand,

$$\begin{aligned} v_n\left(\frac{1}{2}\right) &= \int_0^a v_n'(t)\,dt = \int_0^{\theta_n} v_n'(t)\,dt + \int_{\theta_n}^a v_n'(t)\,dt \\ &\leq R_0 + \frac{1}{2a} v_n'(\theta_n) \end{aligned} \tag{28}$$

which is impossible because from (27) we deduce that $v_n'(\theta_n)$ is bounded and (28) leads to $v_n'(\theta_n)$ is unbounded. This completes the proof of theorem 6. ■

4.1.2 Existence in the sublinear case:

Let $\varepsilon > 0$, we deduce from hypothesis (9) existence of $\chi > 0$ such that

$$x > \chi \text{ implies } f^+(x) < \varepsilon\varphi(x).$$

Note that since ψ is concave and increasing, and f is increasing

$$\begin{array}{c} f^+\left(\int_0^x \psi(v'(t))\,dt\right) \leq f^+(\psi(v(x))) \\ \leq f^+\left(\psi\left(\|v\|_1\right)\right) \text{ for all } x \in [0,a] \end{array}.$$

Thus if $\eta = \varphi(\chi)$, then for all $v \in C^1([0,a])$ and for all $x \in [0,a]$

$$\|v\|_1 > \eta \text{ implies } f^+\left(\int_0^x \psi(v'(t))\,dt\right) < \varepsilon \|v\|_1$$

and $f^+\left(\int_0^x \psi(v'(t))\,dt\right) = \circ\left(\|v\|_1\right)$

Therefore, Rabinowitz global bifurcation theory states (see [21]): the pair $(\lambda_1, +\infty)$ is a bifurcation point for a component $\mathbb{S}_1^+ \subset \mathbb{R}\times \tilde{S}{}_1^+$ of positive solutions to (22) such that:

If Ω is a neighborhood of $(\lambda_1, +\infty)$ whose projection on $\mathbb{R}$ is bounded and whose projection on $C^1([0,a])$ is bounded away from 0 then either

1. $\mathbb{S}_1^+ \backslash \Omega$ is bounded in $\mathbb{R}\times C^1([0,a])$, in which a case $\mathbb{S}_1^+ \backslash \Omega$ meets $\mathbb{R}\times\{0\}$ or
2. $\mathbb{S}_1^+ \backslash \Omega$ is unbounded in $\mathbb{R}\times C^1([0,a])$. Moreover if $\mathbb{S}_1^+ \backslash \Omega$ has a bounded projection on $\mathbb{R}$ then $\mathbb{S}_1^+ \backslash \Omega$ meets $(\mu_k([0,a]), +\infty)$ with $k \geq 2$.

Thus, to prove existence of a positive solution to problem (21) it suffices to show the following

Theorem 8 $\mathbb{S}_1^+$ *crosses* $\{0\} \times C^1([0,a])$.

Proof of theorem 8:

To obtain theorem 8 it suffices to prove that if Ω is as above, then $\mathbb{S}_1^+ \setminus \Omega$ don't meet $(\mu_k([0,a]), +\infty)$ with $k \geq 2$ and don't meet $\mathbb{R}^+ \times \{0\}$.

Let Φ be the first positive eigenfunction of

$$\begin{cases} -\Phi'' = \lambda_1 \Phi \text{ in } (0,a) \\ \Phi(0) = \Phi'(a) = 0. \end{cases}$$

and $(\lambda, v) \in \mathbb{S}_1^+$. Arguing as in the proof of lemma 7 we get

$$\lambda < \lambda_1$$

which means that $\mathbb{S}_1^+ \setminus \Omega$ don't meet $(\mu_k([0,a]), +\infty)$ with $k \geq 2$.

Now suppose that (λ_n, v_n) is a sequence in $\mathbb{S}_1^+$ converging[5] to $(\lambda^*, 0)$ with $\lambda_n > 0$. Multiplying (22) by Φ and integrating on $(0,a)$ we get

$$(\lambda_1 - \lambda_n) \int_0^a v_n \Phi = \int_0^a f^+(u_n) \Phi$$

where $u_n(x) = \int_0^x \psi(v_n'(t))\, dt$.

Using the concavity of f we get

$$(\lambda_1 - \lambda_n) \int_0^a v_n(t) \Phi(t)\, dt \geq \int_0^a \left(\int_0^t f^+(\psi(v_n'(s))) \Phi(s)\, ds \right) dt.$$

We deduce from hypothesis (9) that $\lim\limits_{x \to 0} \dfrac{f^+(\psi(x))}{x} = +\infty$ and for $M = \dfrac{\pi^2}{a^2}$, there exist $\delta > 0$ such that

$$0 \leq x < \delta \text{ implies } f^+(\psi(x)) > Mx.$$

Hence, For n large

$$f^+(\psi(v_n'(s))) \geq M v_n'(s)$$

and

$$(\lambda_1 - \lambda_n - M) \int_0^a v_n(t) \Phi(t)\, dt \geq 0.$$

This is impossible since

$$\lambda_1 - \lambda_n - M < \lambda_1 - M < 0.$$

which completes the proof of theorem 8.

[5] $v_n \to 0$ with the respect of the C^1 norm.

4.2 Uniqueness in $\mathbf{A}_k^{\pm}$

We will expose in this paragraph the proof of uniqueness in $A_k^{\pm}$ in the superlinear case. The other case will be treated similarly.

We deduce from Lemma 3 and Lemma 4 that: to show uniqueness of the solution to problem (1) in each $\mathrm{A}_k^{\pm}$, it suffices to show that the boundary value problem

$$\begin{cases} -(\varphi(u'))' = f(u) \text{ in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{29}$$

has a unique solution in A_1^{+}.

Now, if u and v are two solutions in A_1^{+} to problem (29), then we have

$$\int_a^b -(\varphi(u'))' v + (\varphi(v'))' u = \int_a^b f(u) v - f(v) u$$

or

$$2\int_a^{\frac{a+b}{2}} \left(\frac{\varphi(u')}{u'} - \frac{\varphi(v')}{v'}\right) u'v' = \int_a^b \left(\frac{f(u)}{u} - \frac{f(v)}{v}\right) uv. \tag{30}$$

First we deduce from Lemma 5 that u and v are ordered and from assumption (7) that f is increasing in $\mathbb{R}^{+}$. Then, if we suppose $u < v$ in $(0,1)$ we get $(\varphi(u') - \varphi(v'))' = -(f(u) - f(v)) < 0$ in $\left[a, \frac{a+b}{2}\right)$, namely $u' < v'$ in $\left[a, \frac{a+b}{2}\right)$.

In one hand, it follows from assumption (7) that

$$\int_a^b \left(\frac{f(u)}{u} - \frac{f(v)}{v}\right) uv < 0. \tag{31}$$

In the other hand, the concavity of φ involve that the function $s \to \dfrac{\varphi(s)}{s}$ is decreasing on $(0,+\infty)$, then

$$\int_a^{\frac{a+b}{2}} \left(\frac{\varphi(u')}{u'} - \frac{\varphi(v')}{v'}\right) u'v' > 0. \tag{32}$$

Inequalities (31) and (32) contradict equation (32), so uniqueness of the solution to problem (29) is proved. ■

References

[1] Addou I. & A.Benmezai, *Exact number of positive solution for a class of quasi-linear boundary value problems,* Dynamic Syst. Appl. 8 (1999), 147 – 180.

[2] Addou I. & A.Benmezai, *Boundary value problem for onedimensional p-Laplacian with even superlinearity,* Electron. J. Diff. Eqns. (1999) N°$9, 1-29$.

[3] Ambrosetti A., *On the exact number of positive solutions of convex nonlinear problem*, Boll. U.M.I. (5) 15-A, 610-615.

[4] Berestycki H., *Le nombre de solutions de certains problèmes semi-lineaires elleptiques,* J. Funct. Anal. 40, (1981), 1-29.

[5] Boccardo L., P. Drábek, D. Giachi & M. Kučera, *Generalization of the Fredholm alternative for nonlinear differential operators, Nonlinear Anal. T.M.A. vol.10 N*$°10\ (1986), 1083-1103$.

[6] Castro A., S. Gadam & R.Shivaji, *Evolution of positive solution curve in semi positone problems with concave nonlinearities*, J. Math. Annal. Appl., Vol 245 N° 1 (2000), 282-293.

[7] Crandall M.G. & P.H. Rabinowitz, *Bifurcation from simple eigenvalues,* J. Funct. Anal. 8 (1971), 321-340.

[8] Dambrosio W., *Multiple solutions of weakly- coupled systems with p-laplacian operator*, Result Math. 36 (1999), 34-54.

[9] Dang H., K. Schmitt & R. Shivaji, *On the number of solutions of B.V.P. involving p-Laplacian,* Electron. J. Diff. Eqns. (1996) N°$1, 1-9$.

[10] De-Coster C., *Pairs of positive solutions for the one dimensional p-Laplacian,* Nonlinear Anal. T.M.A., 23 (1994) , $669-681$.

[11] Delpino M.A & R.F. Manásevich, *A homotopic deformation along p of the Leray-Schauder degree result and existence for* $\left(|u'|^{p-2}u'\right)' + f(t,u),\ u(0) = u(T) = 0$ $p>1$, J. Diff. Eqns 80 (1989) , 1-13.

[12] Delpino M.A & R.F. Manásevich, *Global bifurcation from the eigenvalues of the p-Laplacian,* J. Diff. Eqns 92 (1991) , 226-251.

[13] García-Huidobro M., R.F. Manásevich & F. Zanolin, *A Fredholm-like result for strongly nonlinear second order O.D.E.'s,* J. Diff. Eqns 114 (1994) , 132-167.

[14] García-Huidobro M., R.F. Manásevich & F. Zanolin, *On a pseudo Fučík spectrum for strongly nonlinear second order O.D.E.'s and an existence result,* J. Comput. Appl. Math. 52 (1994) , 219-239.

[15] García-Huidobro M., R.F. Manásevich & F. Zanolin, *Strongly nonlinear second order O.D.E.'s with rapidly growing terms,* J.Math. Anal. Appl. 202 (1996) , 1-26.

[16] García-Huidobro M. & P.Ubilla, *Multiplicity of solutions for a class of nonlinear second-order equations,* Nonlinear Anal. T.M.A., Vol. 28, N° 9 (1997), 1509-1520.

[17] Guedda M. & L. Veron, *Bifurcation phenomena associated to the p-Laplacian operator,* Trans. Amer. Soc. 114 (1994), 132-167

[18] Laetsch T, *The number of solutions of a nonlinear two point boundary value problem,* Indiana Univ. Math. J., Vol. 20 (1979), 1-13.

[19] Rabinowitz P.H., *Nonlinear Sturm-Liouville problems for second order ordinary differential equations,* Comm. Pure Appl. Math. 23 (1970), 939 − 962.

[20] Rabinowitz P.H., *Some global results for nonlinear eigenvalue problems,* J. Funct. Anal. 7 (1971), 487 − 513.

[21] Rabinowitz P.H., *On bifurcation from infinity,* J. Diff. Eqns. 14 (1973), 462 − 475.

[22] Ubilla P., *Multiplicity results for the 1-dimensional generalized p-Laplacian,* J. Math. App. 190 (1995), 611-623.