A SYSTEM OF ABSTRACT MEASURE DELAY DIFFERENTIAL EQUATIONS

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Abstract

In this paper existence and uniqueness results for an abstract measure delay differential equation are proved, by using Leray-Schauder nonlinear alternative, under Carathéodory conditions.

Key words and phrases: Abstract measure delay differential equation, existence, fixed point.
AMS (MOS) Mathematics Subject Classifications: 34G99

1 Introduction

Functional differential equations with delay is a hereditary system in which the rate of change or the derivative of the unknown function or set-function depends upon the past history. The functional differential equations of neutral type is a hereditary system in which the derivative of the set-function is determined by the values of a state variable as well as the derivative of the state variable over some past interval in the phase space. Although the general theory and the basic results for differential equations have now been thoroughly investigated, the study of functional differential equations has not been complete yet. In recent years, there has been an increasing interest for such equations among the mathematicians of the world. The study of functional abstract measure differential equations is very rare.

The study of abstract measure delay differential equations was initiated by Joshi [6], Joshi and Deo [7] and Shendge and Joshi [11] and subsequently developed by Dhage [1]-[3]. Recently, the authors in [4] proved existence and uniqueness results for abstract measure differential equations, by using Leray-Schauder alternative [5], under Carathéodory conditions. In this paper, by using the same method, we extend the results of [4] to a system of abstract measure delay differential equations. In that our approach is different from that of Joshi [6]. The results of this paper complement and generalize the results of Joshi [6] on abstract measure delay differential equations under weaker conditions.

2 Preliminaries

Let $\mathbb{R}$ denote the real line, $\mathbb{R}^n$ an Euclidean space with repect to the norm $|\cdot|_n$ defined by

$$|x|_n = \max\{|x_1|, \dots, |x_n|\} \quad \text{for} \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \tag{1}$$

Let X be a real Banach space with any convenient norm $\|\cdot\|$. For any two points x, y in X, the segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X | z = x + r(y - x), 0 \le r \le 1\}.$$

Let x_0 and y_0 be two fixed points in X, such that $\overline{0y_0} \subset \overline{0x_0}$, where 0 is the zero vector of X. Let z be a point of X, such that $\overline{0x_0} \subset \overline{0z}$. For this z and $x \in \overline{y_0z}$, define the sets S_x and $\overline{S}_x$ as follows

$$S_x = \{rx : -\infty < r < 1\}$$

and

$$\overline{S}_x = \{rx : -\infty < r \le 1\}.$$

For $x_1, x_2 \in \overline{y_0z}$, we write $x_1 < x_2$ (or $x_2 > x_1$) if $\overline{y_0x_1} \subset \overline{y_0x_2}$. Let the positive number $\|x_0 - y_0\|$ be denoted by w. For each $x \in \overline{x_0z}$, $z > x_0$, let x_w denote that element of $\overline{y_0z}$ which

$$x_w < x, \quad \|x - x_w\| = w.$$

Note that, x_w and wx are not the same points, unless $w = 0$ and $x = 0$.

Let M denote the σ-algebra of all subsets of X so that (X, M) becomes a measurable space. Let $ca(X, M)$ be the space of all vector measures (signed measures) and define a norm $\|\cdot\|$ on $ca(X, M)$ by

$$\|p\| = |p|_n(X) \tag{2}$$

where $|p|_n$ is a total variation measure of p and is given by

$$|p|_n(X) = \sum_{i=1}^{\infty} |p(E_i)|_n, \ \ \forall E_i \subset X. \tag{3}$$

It is known that $ca(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ defined by (2). Let μ be a σ-finite measure on X and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we write $p << \mu$.

For a fixed $x_0 \in X$, let M_0 be the smallest σ-algebra on $\overline{S_{x_0}}$, containing $\{x_0\}$ and the sets S_x, $x \in \overline{y_0x_0}$. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form $\overline{S}_x$ for $x \in \overline{x_0z}$. Finally let $L^1_\mu(S_z, \mathbb{R})$ denote the space of all μ -integrable nonnegative real-valued functions h on S_z with the norm $\|\cdot\|_{L^1_\mu}$ defined by

$$\|h\|_{L^1_\mu} = \int_{S_z} |h(x)|\, d\mu.$$

3 Statement of the problem

Let μ be a σ-finite real measure on X. Given a $p \in ca(X, M)$ with $p << \mu$, consider the abstract measure delay differential equation (in short AMDDE), involving the delay w,

$$\begin{aligned} &\frac{dp}{d\mu} = f(x, p(\overline{S}_x), p(\overline{S}_{x_w})), \quad \text{a.e.} \quad [\mu] \quad \text{on} \quad \overline{x_0 z}, \\ &p(E) = q(E), \;\; E \in M_0, \end{aligned} \tag{4}$$

where q is a given known vector measure, $dp/d\mu$ is a Radon-Nikodym derivative of p with respect to μ and $f : S_z \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is such that $f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))$ is μ-integrable for each $p \in ca(S_z, M_z)$.

Definition 3.1 *Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)$ $(z > x)$ is said to be a solution of AMDDE (4) if*

(i) $p(E) = q(E), \;\; E \in M_0$,

(ii) $p << \mu$ *on* $\overline{x_0 z}$,

(iii) p *satisfies (4) a.e.* $[\mu]$ *on* $\overline{x_0 z}$.

Remark 3.1 *The AMDDE (4) is equivalent to the abstract measure integral equation*

$$p(E) = \begin{cases} \int_E f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))d\mu, & E \in M_z, \;\; E \subset \overline{x_0 z} \\ q(E), & E \in M_0. \end{cases}$$

A solution p of AMDDE (4) on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

In the following section we shall prove the main existence theorem for AMDDE (4) under suitable conditions on f. We shall use the following form of the Leray-Schauder's nonlinear alternative. See Dugundji and Granas [5].

Theorem 3.1 *Let $B(0, r)$ and $B[0, r]$ denote respectively the open and closed balls in a Banach space X centered at the origin 0 of radius r, for some $r > 0$. Let $T : B[0, r] \to X$ be a completely continuous operator. Then either*

(i) the operator equation $Tx = x$ has a solution in $B[0, r]$, or

(ii) there exists an $u \in X$ with $\|u\| = r$ such that $u = \lambda T u$ for some $0 < \lambda < 1$.

4 Existence and Uniqueness Theorems

We need the following definition in the sequel.

Definition 4.1 *A function $\beta : S_z \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is said to satisfy conditions of Carathéodory or simply it is Carathéodory if*

(i) $x \to \beta(x, y, z)$ *is μ-measurable for each* $(y, z) \in \mathbb{R}^n \times \mathbb{R}^n$.

(ii) $(y, z) \to \beta(x, y, z)$ *is continuous for almost everywhere μ on* $x \in \overline{x_0 z}$, *and*

(iii) for each given real number $\rho > 0$ there exists a function $h_\rho \in L^1_\mu(S_z, \mathbb{R})$ such that

$$|\beta(x, y, z)| \leq h_\rho(x) \ \text{a.e. } [\mu] \ x \in \overline{x_0 z}, \ \text{for each } y, z \in \mathbb{R} \text{ with } |y| \leq \rho, \ |z| \leq \rho.$$

We consider the following set of assumptions.

(A1) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the pseudo-metric d defined by

$$d(E_1, E_2) = |\mu|_n(E_1 \triangle E_2), \ \ E_1, E_2 \in M_z.$$

(A2) $\mu(\{x_0\}) = 0$.

(A3) q is continuous on M_z with respect to the Pseudo-metric d defined in (A1).

(A4) The function $f(x, y, z)$ is L^1_μ-Carathéodory.

(A5) There exists a function $\phi \in L^1_\mu(S_z, \mathbb{R}^+)$ such that $\phi(x) > 0$ a.e. $[\mu]$, $x \in S_z$ and a continuous and nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$|f(x, y, z)|_n \leq \phi(x)\psi(\max\{|y|_n, |z|_n\}) \ \text{a.e. } [\mu] \ \text{on } \overline{x_0 z}, \ \ \forall y \in \mathbb{R}^n, \forall z \in \mathbb{R}^n.$$

Theorem 4.1 *Suppose that assumptions (A1)–(A5) hold. Further if there exists a real number $r > 0$ such that*

$$r > \|q\| + \|\phi\|_{L^1_\mu}\psi(r) \tag{5}$$

then AMDDE (4) has a solution on M_z.

Proof. Let $X = ca(S_z, M_z)$ and consider an open ball $B(0, r)$ in $ca(S_z, M_z)$ centered at the origin and of radius r, where the real number $r > 0$ satisfies (5). Define an operator T from $B[0, r]$ into $ca(S_z, M_z)$ by

$$Tp(E) = \begin{cases} \int_E f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))d\mu, & E \in M_z, \ E \subset \overline{x_0 z} \\ q(E), & E \in M_0. \end{cases}$$

We shall show that the operator T satisfies all the conditions of Theorem 3.1 on $B[0,r]$.

Step I: First we show that T is continuous on $B[0,r]$. Let $\{p_n\}$ be a sequence of vector measures in $B[0,r]$ converging to a vector measure p. Then by Dominated Convergence Theorem,

$$\begin{aligned}\lim_n Tp_n(E) &= \lim_{n\to\infty}\int_E f(x,p_n(\overline{S}_x),p_n(\overline{S}_{x_w}))d\mu \\ &= \int_E f(x,p(\overline{S}_x),p(\overline{S}_{x_w}))d\mu \\ &= Tp(E)\end{aligned}$$

for all $E \in M_z, E \subset \overline{x_0 z}$. Similarly if $E \in M_0$, then

$$\lim_n Tp_n(E) = q(E) = Tp(E)$$

and, so T is a continuous operator on $B[0,r]$.

Step II: Next we show that $T(B[0,r])$ is a uniformly bounded and equi-continuous set in $ca(S_z, M_z)$. Let $p \in B[0,r]$ be arbitrary. Then we have $\|p\| \leq r$. Now by the definition of the map T one has

$$Tp(E) = \begin{cases} \int_E f(x,p(\overline{S}_x),p(\overline{S}_{x_w}))d\mu, & \text{if } E \in M_z, E \subset \overline{x_0 z} \\ q(E), & \text{if } E \in M_0. \end{cases}$$

Therefore for any $E = F \cup G, F \in M_0$ and $G \in M_z$, $G \subset \overline{x_0 z}$, we have

$$\begin{aligned}|Tp(E)|_n &\leq |q(E)|_n + \int_E |f(x,p(\overline{S}_x),p(\overline{S}_x))|_n d\mu \\ &\leq \|q\| + \int_E \phi(x)\psi(\max\{|p(\overline{S}_x)|_n, |p(\overline{S}_{x_w})|_n\})d\mu \\ &\leq \|q\| + \int_E \phi(x)\psi(\|p\|)d\mu \\ &\leq \|q\| + \|\phi\|_{L^1_\mu}\psi(r)\end{aligned}$$

for all $E \in M_z$. By definition of the norm $\|\cdot\|$ we have

$$\begin{aligned}\|Tp\| &= |Tp|_n(S_z) \\ &\leq \|q\| + \|\phi\|_{L^1_\mu}\psi(r).\end{aligned}$$

This shows that the set $T(B[0,r])$ is uniformly bounded in $ca(S_z, M_z)$.

Now we show that $T(B[0,r])$ is an equi-continuous set in $ca(S_z, M_z)$. Let $E_1, E_2 \in M_z$. Then there are sets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z$ with $G_1, G_2 \subset \overline{x_0 z}$, and

$$F_i \cap G_i = \emptyset, \quad i = 1, 2.$$

We know the set-identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1) \quad \text{and} \quad G_2 = (G_2 - G_1) \cup (G_2 \cap G_1). \tag{6}$$

Therefore we have

$$\begin{aligned} Tp(E_1) - Tp(E_2) &= q(F_1) - q(F_2) \\ &+ \int_{G_1 - G_2} f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))d\mu - \int_{G_2 - G_1} f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))d\mu. \end{aligned}$$

Since $f(x, y, z)$ is L^1_μ- Carathéodory, we have that

$$\begin{aligned} |Tp(E_1) - Tp(E_2)|_n &\leq |q(F_1) - q(F_2)|_n + \int_{G_1 \triangle G_2} |f(x, p(\overline{S}_x), p_n(\overline{S}_{x_w}))|_n d\mu \\ &\leq |q(F_1) - q(F_2)| + \int_{G_1 \triangle G_2} h_r(x)d\mu. \end{aligned}$$

Assume that $d(E_1, E_2) = |\mu|_n(E_1 \triangle E_2) \to 0$. Then we have $E_1 \to E_2$ and consequently $F_1 \to F_2$ and $|\mu|_n(G_1 \triangle G_2) \to 0$. From the continuity of q on M_0 it follows that

$$\begin{aligned} |Tp(E_1) - Tp(E_2)|_n &\leq |q(F_1) - q(F_2)|_n + \int_{G_1 \triangle G_2} h_r(x)d\mu \\ &\to 0 \text{ as } E_1 \to E_2. \end{aligned}$$

This shows that $T(B[0,r])$ is an equi-continuous set in $ca(S_z, M_z)$. Thus $T(B[0,r])$ is uniformly bounded and equi-continuous set in $ca(S_z, M_z)$, so it is compact in the norm topology on $ca(S_z, M_z)$. Now an application of Arzelá-Ascoli Theorem yields that $T(B[0,r])$ is a compact subset of $ca(S_z, M_z)$. As a result T is a continuous and totally bounded operator on $B[0,r]$. Hence an application of Theorem 3.1 yields that either $x = Tx$ has a solution or the operator equation $x = \lambda Tx$ has a solution u with $\|u\| = r$ for some $0 < \lambda < 1$. We shall show that this later assertion is not possible. We assume the contrary. Then there is an $u \in X$ with $\|u\| = r$ satisfying $u = \lambda Tu$ for some $0 < \lambda < 1$. Now for any $E \in M_z$, we have $E = F \cup G$, where $F \in M_0$ and $G \subset \overline{x_0 z}$ satisfying $F \cap G = \emptyset$.

Now

$$u(E) = \lambda Tu(E) = \begin{cases} \lambda q(F), & F \in M_0 \\ \lambda \int_G f(x, u(\overline{S}_x), u(\overline{S}_{x_w}))d\mu, & G \in M_z, G \subset \overline{x_0 z}. \end{cases}$$

Therefore

$$\begin{aligned} |u(E)|_n &= |\lambda q(F)|_n + \left|\lambda \int_G f(x, u(\overline{S}_x), u(\overline{S}_{x_w}))d\mu\right| \\ &\leq \|q\| + \left|\int_G |f(x, u(\overline{S}_x), u(\overline{S}_{x_w}))|_n d\mu\right| \\ &\leq \|q\| + \int_G \phi(x)\psi(\max\{|p(\overline{S}_x)|_n, |u(\overline{S}_{x_w})|_n\})d\mu \\ &\leq \|q\| + \int_G \phi(x)\psi(\|u\|)d\mu \\ &= \|q\| + \|\phi\|_{L^1_\mu}\psi(\|u\|). \end{aligned}$$

This further implies that

$$\|u\| = |u|_n(S_z) \leq \|q\| + \|\phi\|_{L^1_\mu}\psi(\|u\|).$$

Substituting $\|u\| = r$ in the above inequality, this yields

$$r \leq \|q\| + \|\phi\|_{L^1_\mu}\psi(r),$$

which is a contradiction to the inequality (5).

Hence the operator equation $p = Tp$ has a solution v with $\|v\| \leq r$. Consequently the AMDDE (4) has a solution $p = p(S_{x_0}, q)$ in $B[0, r]$. This completes the proof.

To prove the uniqueness theorem, we consider the following AMDDE

$$\left.\begin{aligned} &\frac{dr}{d\mu} = g(x, r(\overline{S}_x), r(\overline{S}_{x_w})) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z} \\ &r(E) = 0, \quad E \in M_0, \end{aligned}\right\} \tag{7}$$

where $g : S_z \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $g(x, r(\overline{S}_x), r(\overline{S}_{x_w}))$ is μ-integrable for each $r \in ca(S_z, M_z)$ with $r \geq 0$, and $g(x, y, z)$ is nondecreasing in y, z almost everywhere $[\mu]$ on $\overline{x_0 z}$.

Theorem 4.2 *Assume that the function g satisfies all the conditions of theorem 4.1 with the function f replaced by g. Suppose further that*

$$|f(x, y, z) - f(x, y_1, z_1)|_n \leq g(x, |y - y_1|_n, |z - z_1|_n) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}$$

and the identically zero measure is the only solution of AMDDE (7) on M_z. Then AMDDE (4) has at most one solution on M_z.

Proof. Suppose that AMDDE (4) has two solutions, namely p_1 and p_2 on M_z. Then we have

$$p_1(E) = q(F) + \int_G f(x, p_1(\overline{S}_x), p_1(\overline{S}_{x_w}))d\mu$$

and

$$p_2(E) = q(F) + \int_G f(x, p_2(\overline{S}_x), p_2(\overline{S}_{x_w}))d\mu,$$

for all $E \in M_z$ with $E = F \cup G$, $F \in M_0$, $G \subset \overline{x_0 z}$ and $F \cap G = \emptyset$. Now

$$\begin{aligned} p_1(E) - p_2(E) &= \int_G f(x, p_1(\overline{S}_x), p_1(\overline{S}_{x_w}))d\mu - \int_G f(x, p_2(\overline{S}_x), p_2(\overline{S}_{x_w}))d\mu \\ &= \int_G [f(x, p_1(\overline{S}_x), p_1(\overline{S}_{x_w})) - f(x, p_2(\overline{S}_x), p_2(\overline{S}_{x_w}))]d\mu. \end{aligned}$$

Therefore,

$$\begin{aligned} |p_1(E) - p_2(E)|_n &\leq \int_G |f(x, p_1(\overline{S}_x), p_1(\overline{S}_{x_w})) - f(x, p_2(\overline{S}_x), p_2(\overline{S}_{x_w}))|_n d\mu \\ &\leq \int_G g(x, |p_1 - p_2|_n(\overline{S}_x), |p_1 - p_2|_n(\overline{S}_{x_w}))d\mu. \end{aligned}$$

Since AMDDE (7) has a identically zero function on M_z, one has $\|p_1 - p_2\| = |p_1 - p_2|_n(S_z) = 0 \Rightarrow p_1 = p_2$.

Therefore AMDDE has at most one solution on M_z. This completes the proof.

5 Special case

In this section it is shown that, in a certain situation, the AMDDE (4) reduces to an ordinary differential-difference equation

$$\left.\begin{aligned} &\frac{dy}{dx} = f(x, y(x), y(x-w)), \quad x \geq x_0, \\ &y(x) = g(x), \;\; x \in [x_0 - w, x_0], \end{aligned}\right\} \tag{8}$$

where g is continuous real function on $[x_0 - w, x_0]$, and f satisfies Carathéodory conditions.

Let $X = \mathbb{R}$, $\mu = m$, the Lebesgue measure on $\mathbb{R}$, $\overline{S}_{x_w} = (-\infty, x]$, $x \in \mathbb{R}$, and q a given real Borel measure on M_0. Then equation (4) takes the form

$$\begin{aligned} &\frac{dp}{dm} = f(x, p((-\infty, x]), p((-\infty, x-w])), \\ &p(E) = q(E), \;\; E \in M_0. \end{aligned} \tag{9}$$

It will now be shown that, the equations (8) and (9) are equivalent in the sense of the following theorem.

Theorem 5.1 *Let $q(\{x\}) = 0$, $x \in [x_0 - w, x_0]$. Then*

(a) *to each solution* $p = p(\overline{S}_{x_0}, q)$ *of (9) existing on* $[x_0, x_1)$*, there corresponds a solution* y *of (8) satisfying*

$$y(x) = g(x), \ \ x \in [x_0 - w, x_0].$$

(b) *Conversely, if* g *is a continuous function of bounded variation on* $[x_0 - w, x_0]$*, then to every solution* $y(x)$ *of (8), there corresponds a solution* $p(\overline{S}_{x_0}, q)$*, of (9) existing on* $[x_0, x_1)$ *with a suitable initial measure* q*.*

Proof. (a) Let $p = p(\overline{S}_{x_0}, q)$ be a solution of (9), existing on $[x_0, x_1)$. Define a real Borel measure p_1 on $\mathbb{R}$ as follows.

$$p_1((-\infty, x)) = \begin{cases} 0, & \text{if } x \le x_0 - w, \\ p((-\infty, x]) - p((-\infty, x_0 - w]), & \text{if } x_0 - w < x < x_1 \\ p((-\infty, x_1)), & \text{if } x \ge x_1, \end{cases} \tag{10}$$

and

$$p_1(E) = p(E), \quad \text{if } \ E \subset [x_0 - w, x_1).$$

Define the functions $y_1(x)$, $y(x)$ and $g(x)$ by

$$\begin{aligned} y_1(x) \ &= p_1((-\infty, x)), & x \in \mathbb{R} \\ y(x) \ &= y_1(x) + p((-\infty, x_0 - w]), & x \in [x_0 - w, x_1) \end{aligned}$$

and

$$g(x) = y(x), \ \ x \in [x_0 - w, x_0].$$

The condition $q(\{x\}) = 0$, $x \in [x_0 - w, x_0]$, the definition of the solution p, and the definitions of $y(x), g(x)$ imply that

$$p_1(\{x\}) = p(\{x\}) = 0, \ \ x \in [x_0 - w, x_0].$$

Hence by [8] (Theorem 8.14, p. 163) g is continuous on $[x_0 - w, x_0]$.

Now for each $x \in [x_0 - w, x_1)$ we obtain from (10) and the definition of $y(x)$

$$\begin{aligned} y(x) \ &= y_1(x) + p((-\infty, x_0 - w]) \\ &= p_1((-\infty, x)) + p((-\infty, x_0 - w]) \\ &= p(\overline{S}_{x_w}). \end{aligned} \tag{11}$$

Since p is a solution of (9) we have $p << m$ on $[x_0, x_1)$. Hence $y(x)$ is absolutely continuous on $[x_0, x_1)$. This shows that $y'(x)$ exists a.e. on $[x_0, x_1)$. Now for each $x \in [x_0, x_1)$, we have, by virtue of (11) and (9)

$$p([x_0, x]) = \int_{[x_0, x]} (dp/dm) dm,$$

that is,

$$p((-\infty, x]) - p((-\infty, x_0]) = \int_{x_0}^{x} (dp/dm) dm.$$

This further implies that

$$p(\overline{S}_{x_w}) = p(\overline{S}_{x_0}) + \int_{x_0}^{x} f(t, p(\overline{S}_x), p(\overline{S}_{x-w})) dt.$$

That is

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t), y(t-w)) dt.$$

Hence

$$y'(x) = f(x, y(x), y(x-w)) \quad \text{a.e on } [x_0, x_1).$$

This proves that $y(x)$ is a solution of (8) on $[x_0, x_1)$ satisfying

$$y(x) = g(x), \quad x \in [x_0 - w, x_0].$$

(b) Let $y(x)$ be a solution of (8) existing on $[x_0, x_1]$, where g is continuous and of bounded variation on $[x_0 - w, x_0]$. Define the function g_1 on $\mathbb{R}$ as follows.

$$g_1(x) = \begin{cases} 0, & \text{if } x < x_0 - w, \\ g(x) - g(x_0 - w), & \text{if } x_0 - w \le x \le x_0 \\ g(x_0) - g(x_0 - w), & \text{if } x > x_0. \end{cases} \tag{12}$$

Clearly $g_1 \in NBV$ (where NBV is the class of left continuous functions ϕ of bounded variation such that $\phi(x) \to 0$ as $x \to \infty$). Hence by [8] [Theorem 8.14, p. 163] there exists a real Borel measure q_1 on $\mathbb{R}$, such that,

$$q_1((-\infty, x)) = g_1(x). \tag{13}$$

Let us now define the initial measure q on M_0 as follows.

$$q((-\infty, x]) = q_1((-\infty, x)) + g(x_0 - w), \;\; x \in [x_0 - w, x_0],$$

$$q(E) = q_1(E), \;\; E \subset [x_0 - w, x_0].$$

From (12), (13) and the definition of q we have

$$q(\overline{S}_{x_w}) = q((-\infty, x]) = g(x), \;\; x \in [x_0 - w, x_0].$$

Similarly corresponding to the function $y(x)$ which is a solution of (8) on $[x_0, x_1)$, we can construct a real Borel measure p on M_{x_1}, such that,

$$\begin{aligned} p(E) \quad &= q(E), \quad \text{if } E \in M_0, \\ p(\overline{S}_{x_w}) \quad &= y(x), \;\; x \in [x_0, x_1). \end{aligned} \tag{14}$$

Since $y(x)$ is a solution of (8) we have for $x \in [x_0, x_1)$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t), y(t-w))dt.$$

Hence by (14) it follows that

$$p(\overline{S}_{x_w}) - p(\overline{S}_{x_0}) = \int_{[x_0,x]} f(t, p(\overline{S}_t), p(\overline{S}_{t_w}))dm.$$

That is

$$p([x_0, x]) = \int_{[x_0,x]} f(t, p(\overline{S}_t), p(\overline{S}_{t_w}))dm.$$

In general, if $E \in M_{x_1}, E \subset \overline{x_0 x_1}$, then

$$p(E) = \int_E f(t, p((-\infty, x], p((-\infty, x-w]))dm.$$

This shows that p is a solution of (9) on $[x_0, x_1)$ and the proof of (b) is complete.

Remark 5.1 *In proving (b) part of the above theorem we required $g \in BV$. That is not surprising, since g_1 is constructed from g, such that, $g_1 \in NBV$.*

Remark 5.2 *Theorem 5.1 shows that our results for the equation (4) are general in the sense that they include the corresponding results for the equation (8).*

Remark 5.3 *If we allow w to be zero then $\overline{S}_{x_w} = \overline{S}_{x_0}$ for each $x \geq x_0$. Hence if we define the initial measure q by*

$$q(\overline{S}_{x_0}) = \alpha, \;\; q(E) = 0 \;\; \textit{if} \;\; E \neq \overline{S}_{x_0},$$

the equation (4) takes the form

$$\frac{dp}{d\mu} = f(x, p(\overline{S}_{x_w})), \;\; p(\overline{S}_{x_0}) = \alpha$$

which is the AMDDE studied in [9], [10]. Thus our results include as particular cases, the results in [9], [10].

6 Examples

Example 1. Let $X = \mathbb{R}$, $\overline{S}_x = (-\infty, x]$, $x_0 = 0, w = 2$ and M_0 be the σ-algebra defined on $(-\infty, 0]$. Define an initial measure q on M_0 as follows

$$\begin{aligned} q(E) \;&= \sum_{n \in E \cap \{-1,-2\}} 2^n, && \text{if } E \cap \{-1,-2\} \neq \emptyset \\ &= 0, && \text{if } E \cap \{-1,-2\} = \emptyset. \end{aligned}$$

Define a real measure μ by

$$
\begin{aligned}
\mu(E) &= \sum_{n\in N\cap(E)} \frac{1}{3^n}, \qquad \text{if } E\subset \mathbb{R}, E\cap N\neq\emptyset \\
&= 0, \qquad\qquad\qquad \text{if } E\cap N=\emptyset.
\end{aligned}
$$

where N is the set of natural numbers. Consider the AMDDE

$$
\begin{aligned}
\frac{dp}{d\mu} &= p(\overline{S}_x)+p(\overline{S}_{x-2}), && (15)\\
p(E) &= q(E),\ \ E\in M_0. && (16)
\end{aligned}
$$

The above AMDDE is equivalent to

$$
p(E)=\begin{cases} \int_E p(\overline{S}_x)d\mu+\int_E p(\overline{S}_{x-2})d\mu, & E\subset[0,\infty),\\ p(E)=q(E), & E\in M_0. \end{cases} \tag{17}
$$

It is not difficult to show that the operator T defined by the right hand side of (17) is a contraction on $ca(R,M)$ with the usual total variation norm. Hence AMDDE (15)-(16) has a unique solution on $[0,\infty)$.

We also observe that

$$
\begin{aligned}
p(\overline{S}_1) &= p(\overline{S}_0)+\int_{(0,1]} p(\overline{S}_x)d\mu+\int_{(0,1]} p(\overline{S}_{x-2})d\mu\\
&= q(\overline{S}_0)+p(\overline{S}_1)\mu(\{1\})+p(\overline{S}_{-1})\mu(\{1\})\\
&= 1+\frac{1}{3}p(\overline{S}_1)(1/2)\\
&= 3/2.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
p(\overline{S}_2) &= p(\overline{S}_1)+\int_{(1,2]} p(\overline{S}_x)d\mu+\int_{(1,2]} p(\overline{S}_{x-2})d\mu\\
&= p(\overline{S}_1)+p(\overline{S}_2)\mu(\{2\})+p(\overline{S}_0)\mu(\{2\})\\
&= 1+\frac{3}{2}+\frac{1}{9}p(\overline{S}_1)+\frac{1}{12}\\
&= 19/12.
\end{aligned}
$$

Thus we have

$$
p(\overline{S}_0)=\frac{3}{4},\ \ p(\overline{S}_1)=\frac{3}{2},\ \ p(\overline{S}_0)=\frac{57}{32},\ \text{and so on.}
$$

It is easy to verify that the sequence $\{p(\overline{S}_n)\},\ n=0,1,2,3,\ldots$ is convergent, showing thereby that the solution p of the above AMDDE is a finite measure.

Example 2. Let $X = \mathbb{R}$, μ the Lebesgue measure on $\mathbb{R}$, $\overline{S}_t = [0,t], t > 0$, and $q(E) = \mu(E), E \subset [0,1]$. Consider the AMDDE

$$\frac{dp}{d\mu} = 6p(\overline{S}_{t-1}),$$

$$p(E) = q(E), \;\; E \subset [0,1].$$

Here $w = 1$. For $0 \leq t \leq 1$, we observe that

$$p(\overline{S}_t) = p([0,t]) = q([0,t]) = t.$$

If $t \in [1,2]$, we have

$$\begin{aligned} p(\overline{S}_t) &= q(\overline{S}_1) + \int_{[1,t]} 6p(\overline{S}_{s-1})ds \\ &= 1 + \int_1^t 6(s-1)ds \\ &= 1 + 3(t-1)^2. \end{aligned}$$

Again, if $2 \leq t \leq 3$, we obtain

$$p(\overline{S}_t) = 6t + 6(t-2)^3 - 8,$$

and so on, the solution p can be found recursively on $[0,\infty)$.

Remark 6.1 *The above examples suggest a method to compute the solution of an AMDDE, in the particular case when $f(x,y,z)$ is linear in y and z.*

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