Non-existence Criteria for Laurent Polynomial First Integrals

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Abstract

In this paper we derived some simple criteria for non-existence and partial non-existence Laurent polynomial first integrals for a general nonlinear systems of ordinary differential equations $\dot{x} = f(x), x \in \mathbb{R}^n$ with $f(0) = 0$. We show that if the eigenvalues of the Jacobi matrix of the vector field $f(x)$ are $\mathbb{Z}$-independent, then the system has no nontrivial Laurent polynomial integrals.

Keywords. First integrals, integrability, Laurent polynomials, non-integrability, partial integrabiltiy.

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1 Introduction

A system of differential equations

$$\dot{x} = f(x) \tag{1}$$

is called completely integrable, if it has a sufficiently rich set of first integrals such that its solutions can be expressed by these integrals. According to the famous Liouville-Arnold theorem, a Hamiltonian system with n degrees of freedom is integrable if it possesses n independent integrals of motion in involution. Here, a single-valued function $\phi(x)$ is called a first integral of system (1) if it is constant along any solution curve of system (1). If $\phi(x)$ is differentiable, then this definition can be written as the condition

$$\langle \frac{d\phi}{dx}, f(x) \rangle = 0.$$

Obviously, if no such a nontrivial integral exists, then system (1) is called non-integrable. So finding some simple test for the existence or non-existence of nontrivial first integrals(in given function spaces, such as those of polynomials, rational,

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or analytic functions) is an important problem in considering integrability and non-integrability, see Costin[2], Kozlov[6] and Kruskal and Clarkson [7].

The theory of Ziglin[17] has been proved to be one of the most successful approach for proving non-integrability of the n degrees of freedom Hamiltonian systems. This theory has been shown to be useful for many systems, such as the motion of rigid body around a fixed point[17], homogeneous potentials[16], and a reduced Yang-Mills potentials[5], etc. The technique consists of linearizing a system around particular solutions(forming linear manifolds). The linearized equation is then studied. If its monodromy group is too large(i.e., the branches of solutions have too many values) then the linearized equation has no meromorphic first integrals. It is then shown that this implies that the original system has no meromorphic first integrals.

One of the pioneering authors in considering the non-integrability problems for non-Hamiltonian systems is Yoshida. Using a singularity analysis type method, he was able to derive necessary conditions for algebraic integrability for similarity-invariant system[15]. In [3], Furta suggested a simple and easily verifiable criterion of non-existence of nontrivial analytic integrals for general analytic autonomous systems. He proved that if the eigenvalues of the Jacobi matrix of the vector field $f(x)$ at some fixed point are $\mathbb{N}$-independent, then the system $\dot{x} = f(x)$ has no nontrivial integral analytic in a neighborhood of this fixed point. Based on this key criterion, he also made some further study on non-integrability of general semi-quasihomogeneous systems. Some related results can be found in [4, 9, 10, 11, 12, 13].

However, there are still many systems encountered in physics which do not fall in the set of completely integrable or completely non-integrable systems. Indeed, if a system admits a certain number of first integrals less than the number required for the complete integration, then non-integrability cannot be proved in general. Such systems will be called partially integrable[4]. Some works have been done in this direction, see [1, 8, 14].

The function space we are interested in this paper is the Laurent polynomial ring $C[x_1^{\pm}, \cdots, x_n^{\pm}]$. A *Laurent polynomial P(x)* in the n variables $x = (x_1, x_2, \cdots, x_n)$ is given by

$$P(x) = \sum_{(k_1,\cdots,k_n)\in\mathcal{A}} P_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

where $P_{k_1\cdots k_n} \in \mathbb{C}$ and $\mathcal{A}$, the support of $P(x)$, is a finite subset of integer group $\mathbb{Z}^n$. We will give a simple criterion for non-existence of Laurent polynomial first integrals for general nonlinear analytic systems.

The outline of this paper is as follows. We will first give a simple criterion of non-existence of Laurent polynomial first integral for general analytic systems of differential equations in Section 2. Then in Section 3, we consider the partial integrability for general nonlinear systems. Some examples are presented in Section 4 to illustrate our results.

2 A criterion for non-integrability

Consider an analytic system of differential equations

$$\dot{x} = f(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{C}^n \tag{2}$$

in a neighbourhood of the trivial stationary solution $x = 0$. Let A denote the Jacobi matrix of the vector field $f(x)$ at $x = 0$. System (2) can be rewritten as

$$\dot{x} = Ax + \tilde{f}(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{C}^n \tag{3}$$

near some neighborhood of the origin $x = 0$, where $\tilde{f}(x) = o(x)$.

Theorem 2.1. *If the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A are $\mathbb{Z}$-independent, i.e., they do not satisfy any resonant equality of the following type*

$$\sum_{j=1}^{n} k_j \lambda_j = 0, \quad k_j \in \mathbb{Z}, \quad \sum_{j=1}^{n} |k_j| > 0, \tag{4}$$

then system (3) *does not have any nontrivial Laurent polynomial integral.*

Proof. Since after a nonsingular linear transformation, A can be changed to a Jordan canonical form, for simplicity, we can assume A is a Jordan canonical form, i.e.,

$$A = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_{\tilde{m}} \end{pmatrix}, \quad J_r = \begin{pmatrix} \lambda_r & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_r \end{pmatrix},$$

where $J_r (r = 1, \cdots, \tilde{m})$ is a Jordan block with degree equal to i_r, $i_1 + \cdots + i_{\tilde{m}} = n$.

Suppose system (3) has a Laurent polynomial integral

$$P(x) = \sum_{(k_1, \cdots, k_n) \in \mathcal{A}} P_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

where $\mathcal{A}$ is the support of $P(x)$, then $P(x)$ has to satisfy the following partial differential equation

$$\langle \frac{dP}{dx}(x), Ax + \tilde{f}(x) \rangle \equiv 0, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{C}^n$.

Note that $P(x)$ can be rewritten as

$$P(x) = P_l(x) + P_{l+1}(x) + \cdots + P_p(x), \quad l \leq p, \quad l, p \in \mathbb{Z}, \tag{6}$$

where $P_k(x)$ are homogeneous Laurent polynomials of degree k in x and $P_l(x) \not\equiv 0, P_p(x) \not\equiv 0$.

Substitute (6) into (5) and equate all the terms in (5) of the same order with respect to x with zero and consider the first nonzero term in (6), we get

$$\langle \frac{dP_l}{dx}(x), Ax \rangle \equiv 0.$$

This means that $P_l(x)$ is an integral of the linear system

$$\dot{x} = Ax. \tag{7}$$

Make the following transformation of variables

$$x = Cy,$$

where

$$C = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_{\tilde{m}} \end{pmatrix}, \quad C_r = \begin{pmatrix} 1 & & & \\ & \epsilon & & \\ & & \ddots & \\ & & & \epsilon^{i_r - 1} \end{pmatrix},$$

$\epsilon > 0$ is a constant.

Under the transformation, system (7) can be rewritten as

$$\dot{y} = (B + \epsilon \tilde{B})y, \tag{8}$$

where

$$B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_{\tilde{m}} \end{pmatrix}, \quad B_r = \begin{pmatrix} \lambda_r & & & \\ & \lambda_r & & \\ & & \ddots & \\ & & & \lambda_r \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} \tilde{B}_1 & & & \\ & \tilde{B}_2 & & \\ & & \ddots & \\ & & & \tilde{B}_{\tilde{m}} \end{pmatrix}, \quad \tilde{B}_r = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

So $Q(y,\epsilon) = P_l(Cy)$ is an integral of linear system (8), i.e.,

$$\langle \frac{dQ}{dy}(\epsilon, y), (B + \epsilon\tilde{B})y\rangle \equiv 0. \tag{9}$$

Since

$$P_l(x) = \sum_{k_1+\cdots+k_n=l} P_{k_1\cdots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n},$$

$Q(y,\epsilon)$ has the form

$$Q(y,\epsilon) = P_l(Cy) = \epsilon^L Q_L(y) + \epsilon^{L+1} Q_{L+1}(y) + \cdots + \epsilon^M Q_M(y), \tag{10}$$

where $L, M \in \mathbb{Z}$ are certain integers, $L \leq M$, $Q_i(v)$ are homogeneous form of degree l and $Q_L(y) \not\equiv 0, Q_M(y) \not\equiv 0$.

By (9) and (10), $Q_L(y)$ has to satisfy the following equation

$$\langle \frac{dQ_L}{dy}(y), By\rangle \equiv 0. \tag{11}$$

Since

$$Q_L(y) = \sum_{k_1+\cdots+k_n=l} Q^L_{k_1\cdots k_n} y_1^{k_1} \cdots y_n^{k_n} \not\equiv 0,$$

by (11)

$$\begin{aligned}\sum_{k_1+\cdots+k_n=l} &[(k_1+\cdots+k_{i_1})\lambda_1 + (k_{i_1+1}+\cdots+k_{i_1+i_2})\lambda_2 + \cdots \\ &+ (k_{i_1+\cdots+i_{\tilde{m}-1}+1} + \cdots + k_n)\lambda_{\tilde{m}}] Q^L_{k_1\cdots k_n} y_1^{k_1} \cdots y_n^{k_n} \equiv 0.\end{aligned}$$

Thus, a resonant condition of (4) type has to be fulfilled for some nonzero coefficient $Q^L_{k_1\cdots k_n}$, which contradicts the conditions of Theorem 2.1. The proof is now complete.

3 A criterion for partial integrability

By the proof of Theorem 2.1, we can see that if system (3) has a nontrivial Laurent polynomial integral $P(x)$, then linear system (7) must have a nontrivial homogeneous Laurent polynomial integral $P_l(x)$. So in general at least one resonant relationship of type (4) must be satisfied, and the set

$$G = \{\mathbf{k} = \{k_1, \cdots, k_n\} \in \mathbb{Z}^n : \sum_{j=1}^{n} k_j\lambda_j = 0\}$$

is a nonempty subgroup of $\mathbb{Z}^n$.

Lemma 1. *Let system* (3) *have* $s(s < n)$ *nontrivial Laurent polynomial integrals* $P^1(x)$, $\cdots$, $P^s(x)$. *If any nontrivial homogeneous Laurent polynomial integral* $Q_q(x)$ *of linear system* (7) *is a smooth function of* $P^1_{l_1}(x)$, $\cdots$, $P^s_{l_s}(x)$, *i.e.*, $Q_q(x) = \mathcal{H}(P^1_{l_1}(x), \cdots, P^s_{l_s}(x))$, *then any nontrivial Laurent polynomial integral* $Q(x)$ *of system* (3) *is a smooth function of* $P^1(x)$, $\cdots$, $P^s(x)$.

Proof: Under the transformation $x = \varepsilon y$, system (3) can be rewritten as

$$\dot{y} = Ay + \varepsilon \tilde{f}(y, \varepsilon). \tag{12}$$

We can also rewrite the integrals $P^i(x)$ and $Q(x)$ of system (3) as follows

$$\tilde{P}^i(y, \varepsilon) = P^i(\varepsilon y) = \varepsilon^{l_i}(P^i_{l_i}(y) + \varepsilon P^i_{l_i+1}(y) + \cdots + \varepsilon^{p_i - l_i} P^i_{p_i}(y)), \tag{13}$$
$$\tilde{Q}(y, \varepsilon) = Q(\varepsilon y) = \varepsilon^q(Q_q(y) + \varepsilon Q_{q+1}(y) + \cdots + \varepsilon^{p-q} Q_p(y)), \tag{14}$$

where $P^i_{l_i+j}(x)$ and Q_{q+j} are homogeneous Laurent polynomials. Therefore $\tilde{P}^1(y, \varepsilon)$, $\cdots$, $\tilde{P}^s(y, \varepsilon)$ and $\tilde{Q}(y, \varepsilon)$ are integrals of system (12).

Since

$$Q_q(x) = \mathcal{H}(P^1_{l_1}(x), \cdots, P^s_{l_s}(x)), \tag{15}$$

under the transformation $x = \varepsilon y$, we have

$$\varepsilon^q Q_q(y) = Q_q(\varepsilon y) = \mathcal{H}(P^1_{l_1}(\varepsilon y), \cdots, P^s_{l_s}(\varepsilon y)) = \mathcal{H}(\varepsilon^{l_1} P^1_{l_1}(y), \cdots, \varepsilon^{l_s} P^s_{l_s}(y)). \tag{16}$$

By (15) and (16) we obtain

$$\mathcal{H}(\varepsilon^{l_1} P^1_{l_1}(y), \cdots, \varepsilon^{l_s} P^s_{l_s}(y)) = \varepsilon^q \mathcal{H}(P^1_{l_1}(y), \cdots, P^s_{l_s}(y)). \tag{17}$$

Let $\mathcal{H}^{(0)} = \mathcal{H}$. Then the function

$$\tilde{Q}^{(1)}(y, \varepsilon) = \tilde{Q}(y, \varepsilon) - \mathcal{H}^{(0)}(\tilde{P}^1(y, \varepsilon), \cdots, \tilde{P}^s(y, \varepsilon))$$

is an integral of system (12), since $\tilde{Q}(y, \varepsilon)$ and $\tilde{P}^1(y, \varepsilon)$, $\cdots$, $\tilde{P}^s(y, \varepsilon)$ are all integrals of system (12).

By (13), (14), (15) and (17), it is not difficult to see that the function $\tilde{Q}^{(1)}(y, \varepsilon)$ is at least of $q + 1$ order with respect to ε and can be rewritten as

$$\tilde{Q}^{(1)}(y, \varepsilon) = \varepsilon^{q_1}(Q^{(1)}_{q_1}(y) + \sum_{j=1}^{\infty} \varepsilon^j Q^{(1)}_{q_1+j}(y)),$$

where $q_1 \geq q + 1$ is an integer, $Q^{(1)}_{q_1+j}(y)$ is a homogeneous form of degree $q_1 + j$.

Now $Q^{(1)}_{q_1}(y)$ is also an integral of linear system (7). According to the assumptions of the lemma, $Q^{(1)}_{q_1} = \mathcal{H}^{(1)}(P^1_{l_1}, \cdots, P^s_{l_s})$. So the function

$$\tilde{Q}^{(2)}(y, \varepsilon) = \tilde{Q}^{(1)}(y, \varepsilon) - \mathcal{H}^{(1)}(\tilde{P}^1(y, \varepsilon), \cdots, \tilde{P}^s(y, \varepsilon))$$

is also an integral of system (12) which is at least of q_1+1 degree with respect to ε.

By repeating infinitely this process, we obtain that

$$\tilde{Q}(y,\varepsilon)=\sum_{j=0}^{\infty}\mathcal{H}^{(j)}(\tilde{P}^1(y,\varepsilon),\cdots,\tilde{P}^s(y,\varepsilon)),$$

which is equivalent to the fact that

$$Q(x)=\sum_{j=0}^{\infty}\mathcal{H}^{(j)}(P^1(x),\cdots,P^s(x))=\mathcal{F}(P^1(x),\cdots,P^s(x)),$$

for a certain smooth function $\mathcal{F}$.

Lemma 2. *Assume A is diagonalizable and* rank $G=s$. *If linear system* (7) *has s homogeneous Laurent polynomial integrals $P^1_{l_1}(x),\cdots,P^s_{l_s}(x)$ which are functionally independent, then any other nontrivial homogeneous Laurent polynomial integrals $Q_q(x)$ of system* (7) *is a function of $P^1_{l_1}(x),\cdots,P^s_{l_s}(x)$.*

Proof. For simplicity, we assume A has already a diagonal form $\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$.

Let $\tau_1=(\tau_{11},\cdots,\tau_{1n}),\cdots,\tau_s=(\tau_{s1},\cdots,\tau_{sn})$ are generating elements of group G. It is clearly that $\omega^1(x)=x_1^{\tau_{11}}\cdots x_n^{\tau_{1n}},\cdots,\omega^s(x)=x_1^{\tau_{s1}}\cdots x_n^{\tau_{sn}}$ are s rational integrals of linear system (7). Furthermore, $\omega^1(x),\cdots,\omega^s(x)$ are functionally independent.

In fact, since $\tau_1=(\tau_{11},\cdots,\tau_{1n}),\cdots,\tau_s=(\tau_{s1},\cdots,\tau_{sn})$ are generating elements of group G,

$$\begin{pmatrix}\tau_{11} & \tau_{21} & \cdots & \tau_{s1}\\ \tau_{12} & \tau_{22} & \cdots & \tau_{s2}\\ \cdots & \cdots & \cdots & \cdots\\ \tau_{1n} & \tau_{2n} & \cdots & \tau_{sn}\end{pmatrix}$$

is full-ranked. Therefore, it must have a subdeterminant of degree s which is nonzero, without loss of generality, we can assume

$$\begin{vmatrix}\tau_{11} & \tau_{21} & \cdots & \tau_{s1}\\ \tau_{12} & \tau_{22} & \cdots & \tau_{s2}\\ \cdots & \cdots & \cdots & \cdots\\ \tau_{1s} & \tau_{2s} & \cdots & \tau_{ss}\end{vmatrix}\neq 0.$$

Then

$$\begin{vmatrix} \frac{\partial \omega^1}{\partial x_1} & \frac{\partial \omega^1}{\partial x_2} & \cdots & \frac{\partial \omega^1}{\partial x_s} \\ \frac{\partial \omega^2}{\partial x_1} & \frac{\partial \omega^2}{\partial x_2} & \cdots & \frac{\partial \omega^2}{\partial x_s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \omega^s}{\partial x_1} & \frac{\partial \omega^s}{\partial x_2} & \cdots & \frac{\partial \omega^s}{\partial x_s} \end{vmatrix} = \prod_{i=1}^{n} \omega^i(x) x_i^{-1} \begin{vmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1s} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{s1} & \tau_{s2} & \cdots & \tau_{ss} \end{vmatrix} \neq 0.$$

And hence

$$\begin{pmatrix} \frac{\partial \omega^1}{\partial x_1} & \frac{\partial \omega^1}{\partial x_2} & \cdots & \frac{\partial \omega^1}{\partial x_n} \\ \frac{\partial \omega^2}{\partial x_1} & \frac{\partial \omega^2}{\partial x_2} & \cdots & \frac{\partial \omega^2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \omega^s}{\partial x_1} & \frac{\partial \omega^s}{\partial x_2} & \cdots & \frac{\partial \omega^s}{\partial x_n} \end{pmatrix}$$

is full-ranked, i.e., $\omega^1(x), \cdots, \omega^s(x)$ are functionally independent.

Now assume

$$Q_q(x) = \sum_{k_1+\cdots+k_n=q} Q_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

is a nontrivial homogeneous Laurent polynomial integral of linear system (7). Then

$$\langle \frac{dQ_q}{dx}(x), Ax \rangle = \sum_{k_1+\cdots+k_n=q} (k_1\lambda_1 + \cdots + k_n\lambda_n) Q_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n} \equiv 0.$$

Hence for every nontrivial monomial $x_1^{k_1} \cdots x_n^{k_n}$ of above expansion, geometric point $\mathbf{k} = (k_1, \cdots, k_n) \in G$. Therefore there exist $a_1, \cdots, a_s \in \mathbb{Z}$ such that

$$\mathrm{k} = a_1\tau_1 + \cdots + a_s\tau_s,$$

or equivalently

$$k_i = a_1\tau_{1i} + \cdots + a_s\tau_{si}, \quad i = 1, 2, \cdots, n,$$

and we have

$$\begin{aligned} x_1^{k_1} \cdots x_n^{k_n} &= (x_1^{\tau_{11}} \cdots x_n^{\tau_{1n}})^{a_1} \cdots (x_1^{\tau_{s1}} \cdots x_n^{\tau_{sn}})^{a_s} \\ &= (\omega^1(x))^{a_1} \cdots (\omega^s(x))^{a_s}. \end{aligned}$$

So there exist a Laurent polynomial $\mathcal{G}$ such that

$$Q_q(x) = \mathcal{G}(\omega^1(x), \cdots, \omega^s(x)). \tag{18}$$

Similarly, there also exist Laurent polynomials $\mathcal{F}_i$ such that

$$P^i_{l_i}(x) = \mathcal{F}_i(\omega^1(x), \cdots, \omega^s(x)), \quad i = 1, 2, \cdots, s. \tag{19}$$

Since $P^1_{l_1}(x), \cdots, P^s_{l_s}(x)$ are functionally independent, the matrix

$$\frac{\partial(P^1_{l_1}, \cdots, P^s_{l_s})}{\partial(x_1, \cdots, x_n)} = \frac{\partial(\mathcal{F}_1, \cdots, \mathcal{F}_s)}{\partial(\omega^1, \cdots, \omega^s)} \cdot \frac{\partial(\omega^1, \cdots, \omega^s)}{\partial(x_1, \cdots, x_n)}$$

is full-ranked. On the other hand, $\omega^1(x), \cdots, \omega^s(x)$ are functionally independent, thus the matrix

$$\frac{\partial(\omega^1, \cdots, \omega^s)}{\partial(x_1, \cdots, x_n)}$$

is also full-ranked. Therefore the matrix

$$\frac{\partial(\mathcal{F}_1, \cdots, \mathcal{F}_s)}{\partial(\omega^1, \cdots, \omega^s)}$$

is full-ranked(nondegenerated). By Inverse Function Theorem and (19)

$$\omega^i(x) = \mathcal{G}_i(P^1_{l_1}(x), \cdots, P^s_{l_s}(x)), \quad i = 1, 2, \cdots, s, \tag{20}$$

where $\mathcal{G}_i$ is a smooth function. Combining (18) and (20) we have

$$\begin{aligned} Q_q(x) &= \mathcal{G}(\omega^1(x), \cdots, \omega^s(x)) \\ &= \mathcal{G}(\mathcal{G}_1(P^1_{l_1}(x), \cdots, P^s_{l_s}(x)), \cdots \mathcal{G}_m(P^1_{l_1}(x), \cdots, P^s_{l_s}(x))) \\ &= \mathcal{F}(P^1_{l_1}(x), \cdots, P^s_{l_s}(x)). \end{aligned}$$

The proof is complete.

The following theorem is the direct result of Lemma 1 and Lemma 2.

Theorem 3.1. *Assume system* (3) *has* $s (s < n)$ *nontrivial Laurent polynomial integrals* $P^1(x), \cdots, P^s(x)$ *and matrix* A *is diagonalizable. If* $P^1_{l_1}(x), \cdots, P^s_{l_s}(x)$ *are functionally independent and* rank $G = s$*, then any other nontrivial Laurent polynomial integral* $Q(x)$ *of system* (3) *must be a function of* $P^1(x), \cdots, P^s(x)$.

4 Examples

Example 1. Consider the following system

$$\dot{x}_j = \alpha_j x_j + \beta_j x_{j+1} + \sum_{k=1}^{n} a_{jk} x_j x_k, \quad j = 1, 2, \cdots, n, \tag{21}$$

where α_j, β_j and a_{jk} are real constants, $\beta_n = 0$. Some ecological systems, such as Lotka-Volterra systems, can be reduced to such forms by a linear transformation near an equilibrium.

According to Theorem 2.1, we know that if $\alpha_1, \cdots, \alpha_n$ are $\mathbb{Z}$-independent, i.e., they do not satisfy any resonant equality

$$\sum_{i=1}^{n} k_i \alpha_i = 0, \qquad k_i \in \mathbb{Z}, \quad \sum_{i=1}^{n} |k_i| \geq 1,$$

then system (21) does not have any Laurent polynomial first integral.

Example 2. We give another artificial example illustrating Theorem 3.1. Consider following system

$$\begin{aligned}
\dot{x}_1 &= x_1, \\
\dot{x}_2 &= x_2, \\
\dot{x}_3 &= x_3, \\
\dot{x}_4 &= \alpha x_4 + f(x_1, x_2, x_3, x_4, x_5), \\
\dot{x}_5 &= \beta x_5 + g(x_1, x_2, x_3, x_4, x_5),
\end{aligned} \tag{22}$$

where $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5, f(x) = o(x), g(x) = o(x)$.

System (22) has two Laurent polynomial integrals $P_1(x) = x_1 x_2^{-1}$ and $P_2(x) = x_1 x_3^{-1}$ which are functionally independent. According to Theorem 3.1, we can conclude that any other nontrivial Laurent polynomial integral of system (22) is a smooth function of $P_1(x)$ and $P_2(x)$ if the rank of the group

$$G = \{\mathrm{k} = (k_1, k_2, k_3, k_4, k_5) \in \mathbb{Z}^5 : k_1 + k_2 + k_3 + k_4\alpha + k_5\beta = 0\}$$

is equal to 2. This condition is equivalent to that for any $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3 \in \mathbb{Z}$,

$$\tilde{k}_1 + \tilde{k}_2\alpha + \tilde{k}_3\beta \neq 0,$$

since $(1, -1, 0, 0, 0)$ and $(1, 0, -1, 0, 0)$ are two generating elements of G.

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