STABILITY OF SIMPLE PERIODIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study the stability property of a simple periodic solution of an autonomous neutral functional differential equation (NFDE) of the form $\frac{\mathrm{d}}{\mathrm{d}t}D(x_t) = f(x_t)$. A new proof based on local integral manifold theory and the implicit function theorem is given for the classical result that a simple periodic orbit of the equation above is asymptotically orbitally stable with asymptotic phase. The technique used overcomes the difficulty that the solution operator of a NFDE does not smooth as t increases.

1. Introduction

Suppose that $r \geq 0$ is a given real number, and $\mathbb{R}^n$ is the n-dimensional Euclidean space with norm $|\cdot|$. Let $C = C([-r, 0], \mathbb{R}^n)$ be the Banach space of all the continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with norm $|\phi|_C = \max\limits_{-r\leq\theta\leq 0} |\phi(\theta)|$, $\phi \in C$. If $t_0 \in \mathbb{R}$, $\sigma \geq 0$, and $x : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is a continuous function, for any $t \in [t_0, t_0 + \sigma]$, $x_t \in C$ is defined by $x_t(\theta) = x\,(t + \theta)$, $\theta \in [-r, 0]$.

We consider an autonomous neutral functional differential equation of the form

$$\frac{d}{dt}D(x_t) = f(x_t). \tag{1.1}$$

Throughout this paper, the functional f is assumed to be twice continuously Fréchet differentiable on C and the linear operator D is assumed to be stable and to have an integral representation given by

$$D(\varphi) = \varphi(0) - \int_{-r}^{0} [d\mu(\theta)]\varphi(\theta),$$

where μ is an $n \times n$ matrix function on $(-\infty, \infty)$ of bounded variation which is left continuous on $(-r, 0]$ and satisfies the following

(1). $\mu(\theta) = \mu(-r)$ for $\theta \leq -r$, and $\mu(\theta) = \mu(0) = 0$ for $\theta \geq 0$.

(2). There is a continuous nonnegative scalar function γ such that $\gamma(0) = 0$ and

$$|\int_{-\delta}^{0} [d\mu(\theta)]\varphi(\theta)| \leq \gamma(\delta)|\varphi|_C.$$

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It is noted that the representation for D given above are equivalent to saying that D is atomic at 0. For more details on stable D-operators, see Hale [8] (Sections 12.4 and 12.10) or Hale and Lunel [9] (Sections 9.2 and 9.3).

We assume that equation (1.1) has a nontrivial periodic solution $p(t)$ with period ω and corresponding orbit $W = \{p_t : t \in [0, \omega]\}$ in C. We are interested in the stability property of $p(t)$. For any $\phi \in C$, we will let $x(\phi)(t)$ or $x_t(\phi)$ represent the solution of (1.1) through $(0, \phi)$. For completeness, we state the following definitions, which follow naturally from the corresponding definitions for ordinary and delay functional differential equations.

Definition 1.1. *A periodic solution $p(t)$ of (1.1) is said to be orbitally asymptotically stable if*

(1). *For any given $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any $\phi \in C$ with* $\mathrm{dist}(\phi, W) < \delta$, $\mathrm{dist}(x_t(\phi), W) < \varepsilon$, *for all* $t \geq 0$.

(2). *There is a $\delta_0 > 0$ such that, for any $\phi \in C$ with* $\mathrm{dist}(\phi, W) < \delta_0$,

$$\mathrm{dist}(x_t(\phi), W) \to 0 \qquad as \quad t \to \infty.$$

Definition 1.2. *A periodic solution $p(t)$ of (1.1) is said to be orbitally asymptotically stable with asymptotic phase if $p(t)$ is orbitally asymptotically stable and the δ_0 in Definition 1.1 can be chosen so that, for any $\phi \in C$ with* $\mathrm{dist}(\phi, W) < \delta_0$, *there is a constant $c = c(\phi)$ such that*

$$|x_{t+c}(\phi) - p_t|_C \to 0 \qquad \text{as} \quad t \to \infty.$$

For ordinary differential equations, it is proved that any simple periodic orbit is orbitally asymptotically stable with asymptotic phase. By a simple periodic orbit, we mean that the periodic orbit has 1 as a simple characteristic multiplier and all the other characteristic multipliers have norm less than 1. This result was extended to the case of a family of periodic solutions by Hale and Stokes [10], and to delay functional differential equations by Hale [5] and Stokes [13,14]. For neutral functional differential equations, Hale [6,7] studied the behavior of solutions near constant solutions. See Hale and Lunel [9] for other related results. As to the behavior of solutions near a periodic orbit of a neutral functional differential equation, one can derive the following result from Hale and Lunel [9] (Theorem 3 in Chapter 10).

Main Theorem: *Assume that the linear operator $D : C \to \mathbb{R}^n$ in (1.1) is stable and $p(t)$ is a nontrivial periodic solution of (1.1). If 1 is a simple characteristic multiplier of $p(t)$ and all the other characteristic multipliers of $p(t)$ have norm less than 1, then $p(t)$ is orbitally asymptotically stable with asymptotic phase.*

The result is a direct extension of the corresponding results for ordinary and delay functional differential equations. However, the techniques used in the proof of the corresponding results for ordinary and delay functional differential equations do not extend directly to the case of neutral functional differential equations. The major difficulty lies in the fact that the associated solution operator does not smooth with time t. Actually, this is the main reason that the qualitative theory of neutral functional differential equations is less satisfactory than that for ordinary and delay functional differential equations. The proof of above result in Hale and Lunel [9] uses the theory of synchronized stable and unstable manifolds. Here we provide a proof that uses local integral manifolds and the implicit function theorem, and is similar to that used in Hale [6, 7] and Stokes [13, 14]. To overcome the difficulty that the solutions are not smooth enough as t increases, we first prove that the desired property holds for smooth solutions and then use continuity argument to show that the property is actually valid for all solutions. We need a result on the differentiability of the solutions of neutral functional differential equations, which states that the set of initial functions such that the corresponding solutions of (1.1) is differentiable in time is a dense set in the phase space C, see Shao [11]. The idea used here extends to other problems in the study of the asymptotic behavior of solutions of NFDE. For example, similar technique can be used to show the stability result of an integral manifold.

This paper is organized as follows. In Section 2, we present some preliminary results related to the linear variational equation of (1.1) associated with $p(t)$. In Section 3, we prove the existence of local stable manifolds, and Section 4 is devoted to the proof of the Main Theorem stated above.

2. Preliminary Results

Performing the change of variable $y_t = x_t - p_t$ on (1.1), we have

$$\frac{d}{dt}D(y_t) = L(t, y_t) + F(t, y_t), \tag{2.1}$$

where L is the linear operator defined by $L(t, \phi) = f'_\phi(p_t)\phi$, and F is the functional defined by $F(t, \phi) = f(p_t+\phi) - f(p_t) - f'_\phi(p_t)\phi$. One can easily see that F is continuously Fréchet differentiable and satisfies

$$F(t, 0) = 0, \qquad F'_\phi(t, 0) = 0. \tag{2.2}$$

The compactness of W and (2.2) imply that

$$|F(t, \phi_1) - F(t, \phi_2)| \le l(\max\{|\phi_1|_C, |\phi_2|_C\})|\phi_1 - \phi_2|_C \tag{2.3}$$

for all $t \in \mathbb{R}$ and ϕ_1, ϕ_2 in a neighborhood of 0 in C, where $l : [0, \infty) \to [0, \infty)$ is a continuous function with $l(0) = 0$. Because of (2.2) and (2.3), we can regard (2.1) as a perturbation of the equation

$$\frac{d}{dt}D(y_t) = L(t, y_t) \tag{2.4}$$

near the solution $y \equiv 0$. Equation (2.4) is a periodic linear equation and $\dot{p}(t) = \dfrac{d}{dt}p(t)$ is a periodic solution.

For any $s \in \mathbb{R}$, $t \ge s$, let $T(t, s)$ be the solution operator of equation (2.4), that is, $T(t, s)\phi = y_t(s, \phi)$, where $y_t(s, \phi)$ is the solution of (2.4) through (s,ϕ), and $T(s) =$

$T(s, s+\omega)$ for any $s \in \mathbb{R}$. Let $\sigma(T(s))$ denote the spectrum of $T(s)$, and $\sigma_n(T(s))$ the set of all the normal eigenvalues of $T(s)$. It has been shown that $\sigma_n(T(s))$ and the dimension of the generalized eigenspace corresponding to each element in $\sigma_n(T(s))$ are independent of s. If $\mu \in \sigma_n(T(s))$, μ is called a characteristic multiplier of (2.4) or the periodic solution $p(t)$, and the dimension of the generalized eigenspace of μ is defined to be the multiplicity of μ. Since $\dot{p}(t)$ is a periodic solution of (2.4), 1 must be a characteristic multiplier. As mentioned earlier, if 1 has multiplicity 1, we say that the periodic orbit $p(t)$ is simple. For more details on related definitions and results, see Hale [8], Hale and Lunel [9], and Stokes [12].

The following two lemmas provide some preliminary results on (2.4), most of which are corollaries of proven results tailored in our special case for the convenience of later reference. See Cruz and Hale [1], Hale [8](Sections 12.8 and 12.10), and Hale and Lunel [9] (Section 10.3) for proofs and more general results.

Lemma 2.1. *Assume that the linear operator $D : C \to \mathbb{R}^n$ in (2.4) is stable. If 1 is a simple characteristic multiplier of (2.4), and all the other characteristic multipliers of (2.4) have absolute value less than 1, then for any $s \in \mathbb{R}$, there exist a one dimensional subspace $P(s)$ of C and a closed subspace $Q(s)$ of C such that*

(1). $C = P(s) \oplus Q(s)$.
(2). $T(s)P(s) \subset P(s)$, $T(s)Q(s) \subset Q(s)$.
(3). $\sigma(T(s)\mid_{P(s)}) = \{1\}$, $\sigma(T(s)\mid_{Q(s)}) = \sigma(T(s)) - \{1\}$.
(4). *For any $\phi \in P(s), \psi \in Q(s)$, the solution $y_t(s,\phi)$ of (2.4) through (s,ϕ) is periodic with period ω and there exist constants $M, \alpha > 0$ independent of s such that*

$$|y_t(s,\phi)|_C \le M\,|\phi| \qquad \textit{for all} \quad t,s \in \mathbb{R}$$
$$|y_t(s,\psi)|_C \le M\,|\psi|_C\, e^{-\alpha(t-s)} \qquad \textit{for all} \quad t \ge s \in \mathbb{R}$$

For any $s \in \mathbb{R}$, let $X(t,s)$ denote the matrix solution of

$$D(X_t(\cdot,s)) = I + \int_s^t L(\tau, X_\tau(\cdot,s))ds$$

defined on the interval $[s-r,\infty)$ with initial value

$$X_s(\cdot,s) = X_0,$$

where X_0 is the matrix function defined by $X_0(\theta) = 0$, for $\theta \in [-r,0)$ and $X_0(0) = I$. Then, by the variation-of-constant formula, the solution $y_t = y_t(s,\phi)$ of (2.1) through (s,ϕ) can be written as

$$\begin{aligned} y_t &= T(t,s)\phi + \int_0^t X(t,\tau)F(\tau,y_\tau)d\tau \\ &= T(t,s)\phi + \int_0^t d[K(t,\tau)]F(\tau,y_\tau), \end{aligned}$$

where $K(t,\cdot):[s,t]\to C$ is defined by

$$K(t,\tau)=\int_s^\tau X(t+\theta,\alpha)d\alpha.$$

We will let $\pi_{P(s)}:C\to P(s)$, and $\pi_{Q(s)}:C\to Q(s)$ represent the canonical projections of C onto $P(s)$ and $Q(s)$, respectively.

Lemma 2.2. *Under the assumptions of Lemma 2.1, the solution y_t of (2.1) through (s,ϕ) can be written as $y_t=y_t^{P(t)}+y_t^{Q(t)}$, where $y_t^{P(t)}=\pi_{P(t)}y_t\in P(t)$ and $y_t^{Q(t)}=\pi_{Q(t)}y_t\in Q(t)$ are given by the following formulas*

$$\begin{aligned} y_t^{P(t)} &= T(t,s)\pi_{P(s)}\phi+\int_0^t d[\pi_{P(\tau)}K(t,\tau)]F(\tau,y_\tau)\\ y_t^{Q(t)} &= T(t,s)\pi_{Q(s)}\phi+\int_0^t d[\pi_{Q(\tau)}K(t,\tau)]F(\tau,y_\tau). \end{aligned}$$

Moreover, we have

(1). *$Var_{[s,t)}[\pi_{P(\cdot)}K(t,\cdot)]\le M$ for all s, $t\in(-\infty,\infty)$,*
(2). *$Var_{[s,t)}[\pi_{Q(\cdot)}K(t,\cdot)]\le Me^{-\alpha(t-s)}$ for all $t\ge s$,* where M and α are constants determined in Lemma 2.1.

3. Local Stable Manifolds

In this section, we will consider the nonlinear equation (2.1). Based on the results presented in the previous section, we can establish the existence of local stable manifolds for equation (2.1) near the solution $y(t)\equiv 0$.

Lemma 3.1. *Assume that the assumptions of Lemma 2.2 hold. Then, for any $\phi\in C$, if $z^*(\phi)\in C([-r,\infty),\mathbb{R}^n)$ is a solution of the integral equation*

$$z_t=T(t,0)\phi+\int_0^t d[\pi_{Q(s)}K(t,s)]F(s,z_s)-\int_t^\infty d[\pi_{P(s)}K(t,s)]F(s,z_s),,\quad t\ge 0 \tag{3.1}$$

then $z^*(\phi)(t)$ is also a solution of (2.1).

The proof of this Lemma uses standard arguments used by Hale and Lunel [9] (Lemma 1.1 in Section 10.1)) or Stokes [13] (Lemma 3.3).

Lemma 3.2. *Under the assumptions of Lemma 2.2, for any given constant β, $0<\beta<\alpha$, there exists a constant $\eta_0>0$ such that, for any $\phi\in\mathcal{U}_Q(\eta_0)\overset{\text{def}}{=}\{\phi\in Q(0):|\phi|_C\le\eta_0\}$, there exists a solution $z^*(\phi)\in C([-r,\infty),\mathbb{R}^n)$ of (3.1) satisfying*

$$|z^*(\phi)(t)|\le\rho(\eta_0)e^{-\beta t},\qquad t\ge 0,$$

where $\rho:[0,\infty)\to[0,\infty)$ is a continuous function satisfying $\rho(0)=0$. Furthermore, $H:\mathcal{U}_Q(\eta_0)\to C$ defined by

$$H(\phi)(\cdot)=\int_0^\infty d[\pi_{P(s)}K(\cdot,s)]F(s,z_s^*(\phi))$$

is continuously Fréchet differentiable and satisfies

$$H'_\phi(0) = 0 : \quad Q(0) \to C.$$

Proof. For any $\delta > 0$, we define

$$\begin{aligned} Z(\delta) &= \{z \in C([-r,\infty),\mathbb{R}^n) : \ |z(t)| \le \delta e^{-\beta t}, \ t \ge 0, \ |z_0| \le \delta\} \\ \bar{Z}(\delta) &= \{z \in C([-r,\infty),\mathbb{R}^n) : \ |\bar{z}(t)| \le \delta, \ \ t \ge -r\}. \end{aligned}$$

Then $Z(\delta)$ and $\bar{Z}(\delta)$ are bounded and convex in the Banach space $C([-r,\infty),\mathbb{R}^n)$ with the uniform topology. Moreover, there exists a one-to-one correspondence h: $Z(\delta) \to \bar{Z}(\delta)$, defined by, for any $z \in Z(\delta)$,

$$h(z) = \bar{z} \in \bar{Z}(\delta) : \quad \bar{z}(t) = z(t)e^{\beta t}, \ t \ge 0; \ \bar{z}_0 = z_0.$$

For any $\phi \in C$, let $A(\phi,\cdot) : Z(\delta) \to C([-r,\infty),\mathbb{R}^n)$ be the operator defined by the right-hand side of (3.1), that is, for each $z \in Z(\delta)$ and $\phi \in C$, $A(\phi,z) \in C([-r,\infty),\mathbb{R}^n)$ is the function defined by

$$A(\phi,z)(t) \ = \ \text{the righthand side of (3.1)} \quad \text{for } t \ge 0$$

and let $B(\phi,\cdot) : \bar{Z}(\delta) \to C([-r,\infty),\mathbb{R}^n)$ be the operator defined as $B = h \circ A(\phi,\cdot) \circ h^{-1}$, i.e.,

$$\begin{aligned} B(\phi,\bar{z})(t) &= e^{\beta t}A(\phi,z)(t), \quad \text{for } \ t \ge 0 \\ B(\phi,\bar{z})_0 &= A(\phi,z)_0, \end{aligned}$$

where $z = h^{-1}(\bar{z})$.

Similarly as in the proof presented in Stokes [13,14] (see Lemma 3.4 in Stokes [13], for example), using (2.2), (2.3), and Lemma 2.2, we can prove that there exist constants $\eta_0 > 0$ and $\delta_0 > 0$ such that $B(\phi,\cdot) : \bar{Z}(\delta_0) \to \bar{Z}(\delta_0)$ is a uniform contraction operator with respect to $\phi \in \mathcal{U}_Q(\eta_0)$. This implies that $B(\phi,\cdot) : \bar{Z}(\delta_0) \to \bar{Z}(\delta_0)$ has a unique fixed point $\bar{z}^*(\phi) \in \bar{Z}(\delta_0)$. Therefore, $z^*(\phi) = h^{-1}(\bar{z}^*(\phi)) \in Z(\delta_0)$ is a unique fixed point of $A(\phi,\cdot) : Z(\delta_0) \to Z(\delta_0)$. Hence $z^*(\phi)$ is a solution of (3.1) and (3.2). Also, from (2.2), it is clear that $z^*(0) = 0$.

The existence of function ρ with required properties can be obtained from estimating the right-hand side of (3.1) directly using (2.2) and (2.3). Furthermore, we can choose $\delta_0 = \rho(\eta_0)$.

In order to show that H is continuously Fréchet differentiable, we first show that $\bar{z}^*(\phi)$ is continuously Fréchet differentiable on $\mathcal{U}_Q(\eta_0)$. By Theorem 3.2 in Chapter 0 of Hale [4], we only need to show that $B(\phi,\bar{z})$ is continuously Fréchet differentiable with respect to both ϕ and $\bar{z}$ on $\mathcal{U}_Q(\eta_0) \times \bar{Z}(\rho(\eta_0))$. It is quite easy to see that $B'_\phi(\phi,\bar{z})$ exists and is continuous. To show that $B'_{\bar{z}}(\phi,\bar{z})$ exists on $\mathcal{U}_Q(\eta_0) \times \bar{Z}(\rho(\eta_0))$, we first show that $B'_{\bar{z}}(\phi,\bar{z})$

is the linear operator from $C([-r,\infty),\mathbb{R}^n)$ to itself defined by

$$
\begin{aligned}
B'_{\bar z}(\phi,\bar z)\bar y(t) &= e^{\beta t}[\int_0^t d[\pi_{Q(s)}K(t,s)]F'_\phi(s,z_s)y_s \\
&\qquad - \int_t^\infty d[\pi_{P(s)}K(t,s)]F'_\phi(s,z_s)y_s, \quad t\ge 0 \qquad (3.2)\\
&= \int_0^\infty d[\pi_{Q(s)}K(t,s)]F'_\phi(s,z_s)y_s, \quad -r\le t\le 0,
\end{aligned}
$$

where $y=h^{-1}(\bar y)$, $z=h^{-1}(\bar z)$. In fact, if we let

$$w(\phi,\bar z,\bar y,t)=B(\phi,\bar z+\bar y)(t)-B(\phi,\bar z)(t)-B'_\phi(\phi,\bar z)\bar y(t),$$

we have, for all $t\ge 0$,

$$
\begin{aligned}
&|w(\phi,\bar z,\bar y,t)|\\
\le\ & e^{\beta t}[|\int_0^t d[\pi_{Q(s)}K(t,s)](F(s,z_s+y_s)-F(s,z_s)-F'_\phi(s,z_s)y_s)|\\
&+|\int_t^\infty d[\pi_{P(s)}K(t,s)](F(s,z_s+y_s)-F(s,z_s)-F'_\phi(s,z_s)y_s)|]\\
\le\ & e^{\beta t}[\int_0^t Me^{-\alpha(t-s)}\|f'_\phi(p_s+z_s+\tau y_s)-f'_\phi(p_s+z_s)\||y_s|_C\\
&+\int_t^\infty M\parallel f'_\phi(p_s+z_s+\tau y_s)-f'_\phi(p_s+z_s)\parallel |y_s|_C\,].
\end{aligned}
$$

Since for any fixed $z\in Z(\rho(\eta_0))$, $\{p_s+z_s : s\in[0,\infty)\}$ is a compact set in C, by the continuity of f'_ϕ, for any given $\epsilon>0$, we can choose $\eta>0$ so small that

$$\parallel f'_\phi(p_s+z_s+\psi)-f'_\phi(p_s+z_s)\parallel<\epsilon,\quad 0\le s<\infty$$

provided that $|\psi|_C<\eta$. Therefore, using the fact that $|y_s|_C=\sup_{s-r\le\theta\le s}|y(s+\theta)|=\sup_{s-r\le\theta\le s}e^{-\beta(s+\theta)}|\bar y(s+\theta)|\le e^{\beta r}\parallel\bar y\parallel$, we have

$$|w(\phi,\bar z,\bar y,t)|\le\epsilon Me^{\beta r}(\frac{1}{\alpha-\beta}+\frac{1}{\beta})\parallel\bar y\parallel \qquad (3.3)$$

provided that $\parallel\bar y\parallel<\eta e^{-\beta r}$, where $\parallel\bar y\parallel=\sup\{|\bar y(t)|,-r\le t<\infty\}$.

Similarly, we can show that (3.3) holds for $-r\le t\le 0$. This says that $B'_{\bar z}(\phi,\bar z)$ exists and is given by (3.2). To show the continuity of $B_{\bar z}(\phi,\bar z)$, we note that $f''_{\phi\phi}$ is assumed to be continuous and W is a compact set in C. Thus we may assume that $f''_{\phi\phi}$ is bounded on the set $\{p_s+z_s :$ for all $s\in[-r,\infty),\ z\in Z(\rho(\eta_0))\}$ (choose a smaller η_0 if necessary). From this fact, the continuity of $B'_{\bar z}(\phi,\bar z)$ on $\mathcal{U}_Q(\eta_0)\times\bar Z(\rho(\eta_0))$ can be proved by using the continuity of $f''_{\phi\phi}$. Details are omitted. Therefore, we have proved that $\bar z^*(\phi)$, and thus $z^*(\phi)$, is continuously Fréchet differentiable on $\mathcal{U}_Q(\eta_0)$. Moreover, one has, for any $\psi \in C([-r,\infty),\mathbb{R}^n)$,

$$z^*_\phi(\phi)\psi(t) = e^{-\beta t}\bar{z}^*_\phi(\phi)\psi(t),\ t \geq 0.$$

From this, one can show that, for any given $\epsilon > 0$, we can choose $\eta > 0$ so that

$$|z^*_{s,\phi}(\phi+\psi) - z^*_{s,\phi}(\phi) - z^*_{z,\phi}(\phi)\psi| \leq e^{-\beta s}\epsilon$$

for all $s \in [0,\infty)$ and $\psi \in C$ with $|\psi|_C \leq \eta$. This implies that

$$\begin{aligned}
& |F(s,z^*_s(\phi+\psi)) - F(s,z^*_s(\phi)) - F'_\phi(s,z^*_s(\phi))z^{*\prime}_{s,\phi}(\phi)\psi| \\
\leq\ & |f(p_s+z^*_s(\phi+\psi)) - f(p_s+z^*_s(\phi)) - f'_\phi(p_s+z^*_s(\phi))z^{*\prime}_{s,\phi}(\phi)\psi| \\
& +|f'_\phi(p_s)[z^*_{s,\phi}(\phi+\psi) - z^*_{s,\phi}(\phi) - z^*_{z,\phi}(\phi)\psi]| \\
\leq\ & |f'_\phi(p_s+z^*_s(\phi))(z^*_{s,\phi}(\phi)\psi + e^{-\beta s}\epsilon]) - f'_\phi(p_s+z^*_s(\phi))z^{*\prime}_{s,\phi}(\phi)\psi| \\
& +||f'_\phi(p_s)||e^{-\beta s}\epsilon. \\
\leq\ & (||f'_\phi(p_s+z^*_s(\phi))|| + ||f'_\phi(p_s)||)e^{-\beta s}\epsilon.
\end{aligned}$$

Therefore one has the following

$$|F(s,z^*_s(\phi+\psi)) - F(s,z^*_s(\phi)) - F'_\phi(s,z^*_s(\phi))z^{*\prime}_{s,\phi}(\phi)\psi| \leq Ne^{-\beta s}\epsilon,$$

where N is constant independent of s. Now by the definition of H, we can easily show that H is Fréchet differentiable on $\mathcal{U}_Q(\eta_0)$ and

$$H'_\phi(\phi)\psi = \int_0^\infty d[\pi_{P(s)}K(t,s)]F'_\phi(s,z^*_s(\phi))z^{*\prime}_{s,\phi}(\phi)\psi.$$

The continuity of H'_ϕ can be proved by standard argument. By (2.2), one can show that $H'_\phi(0) = 0 : Q(0) \to C$.

Corollary 3.3. *Under the assumptions in Lemma 3.2, if we let* $R = I - H(\cdot) : \mathcal{U}_Q(\eta_0) \to C$, *then we have*

(1). $R : \mathcal{U}_Q(\eta_0) \to R(\mathcal{U}_Q(\eta_0))$ *is a homeomorphism.*

(2). R *is continuously Fréchet differentiable and* $R'_\phi(0) = I : Q(0) \to C$;

(3). *If* z *is a solution of (3.1) with* $\phi \in \mathcal{U}_Q(\eta_0)$, *then*

$$|z_t| \leq \rho(\eta_0)e^{-\beta t,} \qquad t \geq 0.$$

This is a direct consequence of the implicit function theorem and Lemma 3.2.

Corollary 3.4. *Let* $\pi = \pi_{Q(0)} : C \to Q(0)$ *be the canonical projection, and* $H_{1,} R_1 : \mathcal{U}_Q(\eta_0) \to Q(0)$ *be defined by* $H_1(\phi) = \pi H(\phi)$, *and* $R_1(\phi) = \phi - H_1(\phi)$. *Then there exist open neighborhoods of 0 in* $Q(0)$, W_1 *and* W_2, *such that* $R_1 : W_1 \to W_2$ *is a diffeomorphism. Moreover,* $R_1(0) = 0$.

Proof. Since $H : \mathcal{U}_Q(\eta_0) \to C$ is continuously Fréchet differentiable, $H'_\phi(0) = 0 : Q(0) \to C$, and $\pi : C \to Q(0)$ is a bounded linear operator, we have $H_1 = \pi H : \mathcal{U}_Q(\eta_0) \to Q(0)$ is continuously Fréchet differentiable and $H'_{1,\phi}(0) = 0 : Q(0) \to C$. Therefore R_1 :

$\mathcal{U}_Q(\eta_0) \to Q(0)$ is continuously Fréchet differentiable and $R'_{1,\phi}(0) = I : Q(0) \to Q(0)$. The conclusion of the corollary then follows from the inverse function theorem.

4. Proof of the Main Theorem

In order to prove our main theorem, we will need the following two Lemmas. The first is a slight extension of the implicit function theorem stated in most introductory functional analysis book, see for example, Dieudonné [3]. The proof is almost identical to that of the standard implicit function theorem and will be omitted. The second concerns the differentiability of solutions of equation (1.1) with respect to time t.

Lemma 4.1. *Let E_1 be a metric space with metric d, E_2 and E_3 be Banach spaces* with norms $|\cdot|_{E_2}$ and $|\cdot|_{E_3}$, *$\Omega \subset E_1 \times E_2$ be an open set, and $(x_0, y_0) \in \Omega$. Assume that $G : \Omega \to E_3$ satisfies*

(1). *$G(x, y)$ is continuous in a neighborhood of (x_0, y_0) contained in Ω, and $G'_y(x, y)$ is continuous at (x_0, y_0).*

(2). *$G(x_0, y_0) = 0$.*

(3). *$G'_y(x_0, y_0) : E_2 \to E_3$ has bounded inverse.*

Then there exist constants $\tau > 0$, and *$r > 0$ such that equation $G(x, y) = 0$ has a unique solution $y = g(x)$ on $\{x : d(x, x_0) < r\}$ satisfying $y_0 = g(x_0)$ and $|\, g(x) - y_0|_{E_2} < \tau$.*

Lemma 4.2. *Consider equation (1.1) and let $S_0 \subset C$ be defined as*

$$S_0 = \{\phi : \ \dot{\phi} \in C,\ D(\dot{\phi}) = D_-\phi(0) - \int_{-r}^{0} [d\mu(\theta)]\dot{\phi}(\theta) = f(\phi)\},$$

where $D_-\phi(0)$ represents the left derivative of ϕ at $\theta = 0$. Then S_0 is dense in C and, for any $\phi \in S_0$, the solution $x_t(\phi)$ of (1.1) through $(0, \phi)$ is continuously differentiable with respect to t on its interval of existence.

This Lemma is a corollary of the more general result proved in Shao [11].

Proof of The Main Theorem. We first show that there exists a $\delta > 0$ such that, if $\phi \in S_0, |\phi - p_0|_c \le \delta$, there exists a $c_0 = c_0(\phi) > 0$ such that

$$x_{c_0}(\phi) \in p_0 + R(\mathcal{U}_Q(\eta_0)), \tag{4.1}$$

where R and $\mathcal{U}_Q(\eta_0)$ are as in Corollary 3.3.

By the continuity of solutions, $|x_t(\phi) - p_t|_C$ can be made arbitrary small over any given finite interval if we choose ϕ sufficiently close to p_0. Since p_t is periodic, $|p_t - p_0|_C$ can be made as small as we want if we choose t sufficiently close to ω. Hence, there exist constants $\delta_1 > 0$ and $\varsigma > 0$ such that if $\phi \in C$, $|\phi - p_0|_C < \delta_1$, $t \in (\omega - \varsigma, \omega + \varsigma)$, the solution $x_t(\phi)$ of (1.1) through $(0, \phi)$ satisfies

$$\pi(x_t(\phi) - p_0) \in W_2, \tag{4.2}$$

where π and W_2 are defined in Corollary 3.4. Let R and R_1 be defined as in Corollary 3.3 and 3.4. By (4.2) and Corollary 3.4, for any $\phi \in C$, $|\phi - p_0|_C < \delta_1$, $t \in (\omega - \varsigma, \omega + \varsigma)$, there exists $\psi^t(\phi) \in W_1$ such that

$$\psi^t(\phi) = R_1^{-1}\pi(x_t(\phi) - p_0),$$

where W_1 is defined in Corollary 3.4. This leads to

$$\pi(x_t(\phi) - p_0) = R_1\psi^t(\phi) = \pi R\psi^t(\phi), \qquad t \in (\omega - \varsigma, \omega + \varsigma).$$

To show (4.1), we need only to show that there exists a $\delta : 0 < \delta < \delta_1$ such that if $\phi \in S_{0,}\ |\phi - p_0|_C \leq \delta$, there exists a $c_0 = c_0(\phi) > 0$ such that

$$(I - \pi)(x_{c_0}(\phi) - p_0) = (I - \pi)R(\psi^t(\phi)).$$

Let us define the function on G on $\{\phi \in C, |\phi - p_0|_C < \delta_1\} \times (\omega - \varsigma, \omega + \varsigma)$ by

$$G(\phi, t) = (I - \pi)(x_t(\phi) - p_0) - (I - \pi)R(\psi^t(\phi)).$$

It is obvious that $G(\phi, t)$ is continuous with respect to $t \in (\omega - \varsigma, \omega + \varsigma)$, $\phi \in C$, $|\phi - p_0|_C < \delta_1$. If we further restrict that $\phi \in S_0$, we also have that $x_t(\phi)$ is continuously differentiable with respect to $t \in (\omega - \varsigma, \omega + \varsigma)$. Since π and $I - \pi$ are both bounded linear operators, by the properties of R and $R_{1,}$ $G_t'(\phi, t)$ is continuous for $t \in (\omega - \varsigma, \omega + \varsigma)$, $\phi \in S_0$, $|\phi - p_0|_C < \delta_1$. By the periodicity of p_t, we have $\psi^\omega(p_{0)} = 0$, and thus

$$G(p_0, \omega) = (I - \pi)(p_\omega - p_0) - (I - \pi)R(\psi^\omega(p_0)) = 0.$$

Further, $G_t'(\phi, t)|_{(p_0,\omega)}$ has a bounded inverse. In fact we have

$$\frac{\partial}{\partial t}(x_t(\phi) - p_0)|_{(p_0,\omega)} = \dot{x}_\omega(p_0) = \dot{p}_0 \in P(0).$$

This implies that

$$\frac{\partial}{\partial t}(I - \pi)(x_t(\phi) - p_0)|_{(p_0,\omega)} = \dot{p}_0 \quad \text{and} \quad \frac{\partial}{\partial t}\pi(x_t(\phi) - p_0)|_{(p_0,\omega)} = 0.$$

Therefore, $\dfrac{\partial}{\partial t}\psi^t(\phi)|_{(p_0,\omega)} = 0$. By the definition of G, we have $G_t'(\phi, t)|_{(p_0,\omega)} = \dot{p}_0$. Since $\dot{p}_0$ is a basis of $P(0)$, we know that $G_t'(\phi, t)|_{(p_0,\omega)}$ is a linear homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Now by Lemma 4.1, there exists $\delta : 0 < \delta < \delta_1$ such that for any $\phi \in S_0, |\phi - p_0| < \delta$, there exists an $c_0(\phi) > 0$ such that

$$G(\phi, c_0(\phi)) = 0.$$

This proves that (4.1) holds provided that $\phi \in S_0$.

Since S_0 is dense in $\{\phi \in C : |\phi|_C < \delta\}$, for any $\phi \in C$ with $|\phi - p_0|_C < \delta$, we can find a sequence $\{\phi_n\} \subset S_0$ satisfying that $|\phi_n - p_0|_C < \delta$ and $\phi_n \to \phi$ as $n \to \infty$. Let $c_n = c_0(\phi_n)$ be such that $G(\phi_n, c_n) = 0$. Since $\{c_n\}$ is bounded, we may assume that $c_n \to c_0$ as $n \to \infty$. By the continuity of G, we know that

$$0 = \lim_{n\to\infty} G(\phi_n, c_n) = G(\phi, c_0),$$

i.e., (4.1) hold for $\phi \in C$, $|\phi|_C < \delta$.

Corollary 3.3 says that the solution $z(t)$ of (3.1), which is identical to the solution of (2.1) through $(0, x_{c_0}(\phi) - p_0)$, satisfies that

$$|z_t|_C \leq \rho(\eta_0)e^{-\beta t} \qquad t \geq 0,$$

Therefore we have

$$\left|x_{t+c_0(\phi)} - p_t\right|_C \leq \rho(\eta_0)e^{-\beta t} \qquad t \geq 0.$$

This completes the proof of the main theorem.

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