On the existence and smoothness of radially symmetric solutions of a BVP for a class of nonlinear, non-Lipschitz perturbations of the Laplace equation.

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Abstract. The existence of radially symmetric solutions $u(x;a)$ to the Dirichlet problems

$$\Delta u(x) + f(|x|, u(x), |\nabla u(x)|) = 0 \qquad x \in B,\ u|_\Gamma = a \in \mathbb{R}\ (\Gamma := \partial B)$$

is proved, where B is the unit ball in $\mathbb{R}^n$ centered at the origin ($n \geq 2$), a is arbitrary ($a > a_0 \geq -\infty$); f is positive, continuous and bounded. It is shown that these solutions belong to $C^2(\overline{B})$. Moreover, in the case $f \in C^1$ a sufficient condition (near necessary) for the smoothness property $u(x;a) \in C^3(\overline{B}) \quad \forall a > a_0$ is also obtained.

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Introduction

The radially symmetric solutions of homogeneous Dirichlet problems or problems in the whole space $\mathbb{R}^n$ (with a condition at infinity) for the nonlinearly perturbed Laplace operator or m-Laplacian were investigated by many authors (see e.g. [1]-[3],[7] and their references). Some related ODE-BVP-s were considered e.g. in [4],[5]. The perturbation-terms $f(u), f(x,u)$ in [1]-[5],[7] were of the class C^1 or locally Lipschitz in u. In [6] a multiplicative perturbation $f(u, |\nabla u|) = \exp(\lambda u + \kappa|\nabla u|) \quad \lambda, \kappa \leq 0$ of the Laplacian was considered.

The preceding perturbations do not cover the general case of perturbation Δu with $f(|x|, u, |\nabla u|)$ if f is non-Lipschitz in u and $|\nabla u|$. We shall consider such type of the perturbations under the restriction when f is continuous, positive and bounded.

In fact the present paper serves as supplement to the paper [8] where the following problem (Problem A) was considered:

$<1>$ $\Delta u(x) + f(|x|, u(x), |\nabla u(x)|) = 0 \quad x \in B,$

$<2>$ $u \in C^2(B) \cap C(\overline{B}), \exists v : [0,1] \to \mathbb{R} : v(\rho) \equiv v(|x|) = u(x) \quad x \in \overline{B},$

$<3>$ $u|_\Gamma = a \in \mathbb{R}.$

This paper is in final form and no version of it will be submitted for publication elsewhere.

Here $f \in C(G_a;(0,\infty))$, $\quad a \in \mathbb{R}$ is arbitrarily fixed; $G_a := [0,1] \times [a,\infty) \times [0,\infty)$, B is the unit ball centered at the origin, $\Gamma := \partial B$, and $\rho := |x|$, $x \in \overline{B}$. The uniqueness of the solution was proved under the assumption that $f(\alpha_1,\alpha_2,\alpha_3)$ strongly decreases in α_2 (or for the case when f is nonincreasing both in α_2 and α_3); moreover, a generalization of the comparison result of [6] and a concavity result were also obtained.

Now our purpose is to prove the existence of solutions $u(x;a)$ to Problem A for any $a > a_0 \geq -\infty \quad (a_0 < +\infty)$, such that $u(x;a) > a \quad x \in B$ (especially positive solutions if $a \geq 0$) without any assumptions on the Lipschitz or Hölder continuity of f in α_2 or α_3, but we suppose firstly that $f \in C(G_{a_0};(0,\infty))$, where $G_{a_0} := [0,1] \times (a_0,\infty) \times [0,\infty)$ and secondly that for any $a > a_0$ there exists a positive constant K_a such that $f \in C(G_a;\ (0,K_a])$, where $G_a := [0,1] \times [a,\infty) \times [0,\infty)$. Moreover we wish to prove that all solutions $u(x;a)$ belong to $C^2(\overline{B})$; and what is more that $u(x;a) \in C^3(\overline{B})$ under additional restrictions on the function f.

To prepare our results we formulate some of them in simplified versions:

Lemma. Let $a > a_0$ appearing in Problem A be arbitrarily fixed, and suppose, that for any $b > a$ there exists a constant $K_{a,b} > 0$ such that

$$< 4 > \qquad f \in C(G_{a,b};(0,K_{a,b}]) \qquad G_{a,b} := [0,1] \times [a,b] \times [0,\infty).$$

Then if $u(x;a)$ is a solution of Problem A with

$$v(\rho) \equiv v(|x|) = u(x;a) \quad x \in \overline{B},$$

then instead of the original smoothness condition in $< 2 >$ we have the additional smoothness:

$$< 5 > \qquad u(x;a) \in C^2(\overline{B}).$$

Theorem I. Considering Problem A with arbitrarily fixed $a > a_0$, suppose that assumption $< 4 >$ holds and

$$< 6 > \quad f \in C([0,1] \times [a,\infty) \times [0,\infty);(0,\infty)) \cap C^m((0,1] \times [a,\infty) \times [0,\infty)) \quad 1 \leq m \leq \infty.$$

Then any solution $u(x;a)$ of Problem A has automatically the additional smoothness property:

$$< 7 > \qquad u \in C^2(\overline{B}) \cap C^{m+2}(\overline{B}\backslash\{0\}).$$

Theorem II. Let for any $a > a_0$ exist a constant $K_a > 0$ such, that

$$< 8 > \qquad f \in C(G_a;(0,K_a])$$

then for all $a > a_0$ Problem A has a solution

$$u(x;a) \equiv v(|x|;a) \equiv v(\rho;a) \quad x \in \overline{B}.$$

Theorem III. Suppose that for any $a > a_0$ function f (with the arguments $|x| \sim \alpha_1, u \sim \alpha_2$, $|\nabla u| \sim \alpha_3$) satisfies the conditions:
$< 9 >$ $f \in C(G_a; (0, K_a])$ $\quad 0 < K_a \in \mathbb{R}$,
$< 10 >$ $f \in C^1(G_a)$,
$< 11 >$ $f_{\alpha_1}(0, \alpha_2, 0) = -\frac{1}{n} f_{\alpha_3}(0, \alpha_2, 0) f(0, \alpha_2, 0) \quad \alpha_2 > a_0$.
Then Problem A for all $a > a_0$ has a solution $u(x;a)$ and any of the solutions belongs to $C^3(\overline{B})$.

Finally, we remark that the proofs of these results are based upon the techniques communicated in [8]-[10].

This paper is divided into three sections: 1) A priori smoothness of the solutions, 2) The existence of the solution 3) Higher order smoothness at the origin.

Notations. Throughout the paper a_0 will be fixed, $-\infty \leq a_0 < \infty$. The real parameter a will be arbitrary, such that $a > a_0$. For these values of a_0, a (and $\infty > b > a$) let

$$G_{a_0} := [0,1] \times (a_0, \infty) \times [0, \infty), \; G_a := [0,1] \times [a, \infty) \times [0, \infty),$$

$$G_{a,b} := [0,1] \times [a,b] \times [0, \infty).$$

Finally, B will denote the unit ball centered at the origin, and

$$\Gamma := \partial B \equiv \{x \in \mathbb{R}^n \| x| = 1\}, \; n \in \mathbb{N}, \; n \geq 2.$$

1. A priori smoothness of the solutions.

For the sake of the completness we shall begin with some, maybe trivial Lemmae (1.1.-1.4.).

Consequences from the equation.

First, suppose that

$$g \in C(\overline{B} \times (a_0, \infty) \times [0, \infty); \; (0, \infty)),$$

and consider **Problem 1**:
(1.1) $\Delta u(x) + g(x, u(x), |\nabla u(x)|) = 0 \quad x \in B$,
(1.2) $u \in C^2(B) \cap C(\overline{B})$,
(1.3) $u|_\Gamma = a$.

Lemma 1.1. If u is a solution of **Problem 1**, then

(1.4) $u(x) > a \quad x \in B.$

Proof. Function u is strongly superharmonic on B, therefore

$$u(x) > \min_{\Gamma} u = a \quad x \in B.$$

Now consider **Problem 2:**

$$\Delta u(x) + g(x, u(x), |\nabla u(x)|) = 0 \quad x \in B, \tag{1.5}$$

$$u \in C^2(B) \cap C(\overline{B}), \exists v : [0,1] \to \mathbb{R} : \tag{1.6}$$

$$v(\rho) \equiv v(|x|) = u(x) \quad \forall x \in \overline{B},$$

$$u|_{\Gamma} = a. \tag{1.7}$$

Lemma 1.2. If u is a solution of **Problem 2** with

$$v(\rho) \equiv v(|x|) = u(x) \quad x \in \overline{B},$$

then the following statements hold:

$$u(x_1) > u(x_2) \quad \forall x_1, x_2 \in \overline{B} : \; |x_1| \equiv \rho_1 < \rho_2 \equiv |x_2|, \tag{1.8}$$

i.e.

$$v(\rho_1) > v(\rho_2) \quad \rho_1, \rho_2 \in [0,1], \; \rho_1 < \rho_2, \tag{1.9}$$

$$(grad\ u)(0) = 0, \tag{1.10}$$

$$v \in C[0,1], \tag{1.11}$$

$$\exists v'(0), \; v'(0) = 0, \; v \in C^1[0,1), \tag{1.12}$$

$$\exists v''(0),\ v''(0) < 0,\ v \in C^2[0,1). \tag{1.13}$$

Proof. Inequalities (1.8), (1.9) are consequences of the assumption that u as a solution of **Problem 2** is radially symmetric and strongly superharmonic on the ball $|x| < \rho_2 \in (0,1]$. Equality (1.10) follows from (1.8) and assumption $u \in C^2(B)$ (see (1.6)). Further let $\rho_0, \rho \in [0,1)$ be arbitrary and let $e = (e_1, \ldots, e_n)$ be any unit vector. Consider the difference and its representation:

$$\begin{aligned} v(\rho) - v(\rho_0) := u(\rho e) - u(\rho_0 e) =< \ (grad\ u)(\rho_0 e), \rho e - \rho_0 e > + \\ + \frac{1}{2}(\rho e - \rho_0 e)\{u_{x_i x_j}(\rho_0 e)\}(\rho e - \rho_0 e)^T + \alpha(\rho_0, \rho, e)(\rho - \rho_0)^2, \end{aligned} \tag{1.14}$$

where $\{u_{x_i x_j}\}$ denotes the Hessian and T is the symbol of the transposition and finally $\alpha \to 0$ as $\rho \to \rho_0$. Relations (1.11)-(1.13) are consequences of (1.14), especially

$$v'(\rho_0) =< (grad\ u)(\rho_0 e), e > \quad \rho_0 \in [0,1), \tag{1.15}$$

$$v''(\rho_0) = e\{u_{x_i x_j}(\rho_0 e)\}e^T \quad \rho_0 \in [0,1), \tag{1.16}$$

where derivatives of v do not depend on e, and therefore they may be computed with help of suitable choice of e: e.g. in (1.16) with $e_i = 1,\ e_j = 0\ j \neq i, i = \overline{1,n}$ we have equalities:

$$v''(0) = u_{x_i x_i}(0) = -\frac{1}{n} g(0, u(0), 0) < 0 \quad i = \overline{1,n}, \tag{1.17}$$

using also equality (1.5) at $x = 0$.

Consequences of the radial symmetry.

The next two lemmae deal with the connection of the smoothness properties of a general function $u(x)$ of n real variables and of the function $v(|x|)$ of one variable if $u(x) = v(|x|)$, $x \in \overline{B}$, without using, that $u(x)$ is a solution of any differential equation. The tools are different from the ones above. Moreover expressions for the partial derivatives of u are presented too, which are necessary for the sequel.

Lemma 1.3. If

$$u(x) \in C^2(B) \cap C(\overline{B}),\ u(x) = v(|x|) \quad x \in \overline{B}, \tag{1.18}$$

then

$$v \in C^2([0,1)) \cap\ C([0,1]), \tag{1.19}$$

and using notation $y = (y_1, \ldots, y_n)$

$$u_{x_i}(y) = 0 \quad y \in B\backslash\{0\}, y_i = 0 \quad i = \overline{1,n},\ (grad\ u)(0) = 0,\ v'(0) = 0, \tag{1.20}$$

$$u_{x_ix_j}(y) = 0 \quad y \in B\backslash\{0\} \quad y_iy_j = 0\ i \neq j,\ i,j \in \overline{1,n}, \quad u_{x_ix_j}(0) = 0\ i \neq j,\ i,j \in \overline{1,n}, \tag{1.21}$$

$$u_{x_ix_i}(0) = c_i = c = v''(0) \quad i = \overline{1,n}. \tag{1.22}$$

Proof. From the assumption (1.18) we get, that $u(x_1, 0, \ldots, 0) \in C^2(-1,1) \cap C[-1,1]$, therefore

$$v(\rho) = u(\rho, 0, \ldots, 0) \in C^2([0,1)) \cap C([0,1]),\ v'(0) = 0. \tag{1.23}$$

Further

$$u_{x_i}(y) = v'(\rho)\frac{y_i}{\rho} \quad i = \overline{1,n},\ y \in B\backslash\{0\},\ \rho = |y| \in (0,1), \tag{1.24}$$

$$\begin{aligned} u_{x_ix_j}(y) &= v''(\rho)\frac{y_i}{\rho}\frac{y_j}{\rho} - v'(\rho)y_i\frac{y_j}{\rho^3} = \\ &= (v''(\rho) - \frac{v'(\rho)}{\rho})\frac{y_iy_j}{\rho^2} \quad i \neq j; i,j \in \overline{1,n},\ y \in B\backslash\{0\}, \rho = |y| \in (0,1), \end{aligned} \tag{1.25}$$

$$\begin{aligned} u_{x_ix_i}(y) &= v''(\rho)\left(\frac{y_i}{\rho}\right)^2 + \frac{v'(\rho)}{\rho} - v'(\rho)y_i\frac{y_i}{\rho^3} = \\ &= (v''(\rho) - \frac{v'(\rho)}{\rho})(\frac{y_i}{\rho})^2 + \frac{v'(\rho)}{\rho} \quad i = \overline{1,n},\ y \in B\backslash\{0\},\ \rho = |y| \in (0,1), \\ u_{x_ix_i}(y) &= \frac{v'(\rho)}{\rho} \qquad y_i = 0, \quad i = \overline{1,n} \quad y \in B\backslash\{0\}. \end{aligned} \tag{1.26}$$

Let in the right-hand side of (1.24) $y_i = 0,\ y \in B\backslash\{0\}$, then we get the first equality in (1.20), moreover if for the points y of $B\backslash\{0\}$ of previous choice $y \to 0$ then

$$u_{x_i}(y) = 0 \to u_{x_i}(0) = 0 \quad i = \overline{1,n},$$

consequently $(grad\ u)(0) = 0$. From this equality – the already proved equality: $v'(0) = 0$ follows immediately; (it follows of course also from (1.24) putting $y_i = \rho \in (0,1)$ and letting $\rho \to 0+0$). The first of the formulae in (1.21) follows from the equality (1.25) putting in the right-hand side $y \in B\backslash\{0\}$ such, that $y_i y_j = 0$, the second of them follows from the first one putting $y \to 0$:

$$u_{x_i x_j}(y) = 0 \to u_{x_i x_j}(0) = 0.$$

Finally, (1.22) follows from equality (1.26) putting $y_i = \rho$ and letting $\rho \to 0+0$. Lemma is proved.

Similar computations give an inverse result.

Lemma 1.4. If

$$v \in C^2[0,1) \cap C[0,1],\ v'(0) = 0, \tag{1.27}$$

then for the function

$$u(x) := v(|x|) \equiv v(\rho) \quad x \in \overline{B},\ |x| \equiv \rho \in [0,1]$$

the following statements hold: $\exists u_{x_i}(y), u_{x_i x_j}(y) \quad y \in B\backslash\{0\} \quad i,j = \overline{1,n}$,; representations (1.24)-(1.26) and statements (1.20)-(1.22) are valid, and

$$u \in C^2(B) \cap C(\overline{B}) \tag{1.28}$$

Proof. The last of the relations in (1.28): $u \in C(\overline{B})$ is trivial. The existence and continuity of the partial derivatives $u_{x_i}, u_{x_i x_j}$ on $B\backslash\{0\}$ may be proven by their definitions as limits, using condition (1.27), the identity $u(x) = v(\rho(x))\ x \in B\backslash\{0\}$, the property $\rho \in C^\infty(B\backslash\{0\})$ and the chain rule. By such a way we also get representations (1.24)-(1.26) from which we can derive relations for $y \in B\backslash\{0\}$ formulated in (1.20), (1.21). It remains to prove the relation $u \in C^2(B)$ and equalities for the derivatives $u_{x_i}, u_{x_i x_j}, u_{x_i x_i}$ at the origin formulated in (1.20)-(1.22). To prove them, first, we remark that the factors $v'(\rho),\quad \frac{v'(\rho)}{\rho},\quad v''(\rho) - \frac{v'(\rho)}{\rho}$ appearing in the right hand sides of (1.24)-(1.26) have limits as $y \in B\backslash\{0\},\quad y \to 0$. In fact, using condition (1.27) and the Lagrange's theorem we get:

$$v'(\rho) \to v'(0) = 0 \quad y \to 0, \tag{1.29}$$

$$(1.30) \qquad \frac{v'(\rho)}{\rho} = \frac{v'(\rho)-v'(0)}{\rho-0} = v''(\xi) \to v''(0) \quad y \to 0,$$

$$(1.31) \qquad v''(\rho) - \frac{v'(\rho)}{\rho} = v''(\rho) - v''(\xi) \to v''(0) - v''(0) = 0 \quad y \to 0.$$

The second argument is the boundedness of the factors $\frac{y_i}{\rho}, \frac{y_j}{\rho}$ appearing in (1.24)-(1.26), that combined with (1.29)-(1.31) guarantees the existence of the limits as $y \in B\backslash\{0\}, y \to 0$ of the right hand sides of (1.24)-(1.26). We get the existence and the concrete value of the partial derivatives $u_{x_i}, u_{x_i x_j}, u_{x_i x_i} \quad i,j = \overline{1,n}$ at the origin as well as the property $u \in C^2(B)$.

Now introduce the really radially symmetric case of the equation (1.5), more precisely of the Problem 2; when there exists a function f:

$$f \in C([0,1] \times (a_0,\infty) \times [0,\infty))$$

such, that

$$g(x,\alpha,\beta) = f(|x|,\alpha,\beta) \quad (x,\alpha,\beta) \in \overline{B} \times (a_0,\infty) \times [0,\infty).$$

So, consider **Problem 3:**

$$(1.32) \qquad \Delta u(x) + f(|x|,\ u(x),\ |\nabla u(x)|) = 0 \quad x \in B,$$

$$(1.33) \qquad u \in C^2(B) \cap C(\overline{B}),\ \exists v : [0,1] \to \mathbb{R} : v(\rho) \equiv v(|x|) = u(x) \quad \forall x \in \overline{B},$$

$$(1.34) \qquad u|_\Gamma = a.$$

An auxiliary integral equation.

Lemma 1.5. If u is a solution of **Problem 3** with $v(\rho) \equiv v(|x|) = u(x) \quad x \in \overline{B}$, then all of the statements of Lemma 1.2. and Lemma 1.3. are valid ((1.8) - (1.13), (1.19) - (1.22)), moreover

$$(1.35) \qquad v''(\rho) + \frac{n-1}{\rho} v'(\rho) + f(\rho, v(\rho), |v'(\rho)|) = 0 \quad \rho \in [0+0,1),$$

in $\rho = 0+0$ in the limit sense,

$$(1.36) \qquad \frac{v'(\rho)}{\rho} \to v''(0) \quad \rho \to 0+0,$$

(1.37) $$v'(\rho) < 0 \quad \rho \in (0,1),\ v'(0) = 0,$$

(1.38) $$v'(t) = -\int_0^t t^{1-n}\rho^{n-1} f(\rho, v(\rho), -v'(\rho))d\rho \quad t \in [0+0, 1)$$

in $t = 0+0$ in the limit sense.

Proof. The statements (1.8) - (1.13), (1.19) - (1.22) of Lemmae 1.2, 1.3 hold trivially. They imply relations (1.37) and - using the Lagrange's theorem - (1.36) (see also (1.29) - (1.31)). For the proof of (1.35) using the radial symmetry of u we get:

(1.39) $$\Delta u(x) = v''(\rho) + \frac{n-1}{\rho} v'(\rho) \quad \rho \in (0,1),$$

(1.40) $$|\nabla u(x)| = \left[\sum_{i=1}^n (v'(\rho)\rho_{x_i})^2\right]^{1/2} \equiv |v'(\rho)| \left(\sum_{i=1}^n \left(\frac{x_i}{\rho}\right)^2\right)^{1/2} = |v'(\rho)| \quad \rho \in (0,1).$$

So (1.35) is proved for $\rho \in (0,1)$:

(1.41) $$v''(\rho) + \frac{n-1}{\rho} v'(\rho) + f(\rho, v(\rho),\ |v'(\rho)|) = 0 \quad \rho \in (0,1).$$

Here - using the continuity of function f - all the terms have limits as $\rho \to 0+0$ and we come to

(1.42) $$v''(0) + (n-1)v''(0) + f(0, v(0),\ 0) = 0.$$

Especially we get the equality

(1.43) $$v''(0) = -\frac{1}{n} f(0, v(0),\ 0).$$

It remains to prove equality (1.38). First: it is clear, that $|v'(\rho)| = -v'(\rho)$. Second: prove (1.38) for $t \in (0,1)$. It is well known, that(1.41) implies:

(1.44) $$(\rho^{n-1}v'(\rho))' = -\rho^{n-1} f(\rho, v(\rho), -v'(\rho)) \quad \rho \in (0,1).$$

Let $t \in (0,1)$ be arbitrary and $\rho_0 \in (0,t)$, then integrating (1.44) over the interval $[\rho_0, t]$ we get

(1.45) $$t^{n-1}v'(t) - \rho_0^{n-1}v'(\rho_0) = -\int_{\rho_0}^t \rho^{n-1} f(\rho, v(\rho), -v'(\rho))\ d\rho.$$

Putting $\rho_0 \to 0+0$, using the property $v \in C^1[0,1)$ and continuity of function f we get

$$v'(t) = -\int_0^t t^{n-1}\rho^{n-1} f(\rho, v(\rho), -v'(\rho))\, d\rho \quad t \in (0,1) \tag{1.46}$$

i.e. (1.38) is proved for $t \in (0,1)$. Finally, if $t \to 0+0$ in (1.46), then $v'(t) \to v'(0) = 0$; consequently

$$-\int_0^t t^{1-n}\rho^{n-1} f(\rho, v(\rho), -v'(\rho))\, d\rho \to 0 \quad t \to 0+0 \tag{1.47}$$

and (1.38) is proven for $t = 0+0$. Lemma is proved.

Of course, relation (1.47) may be proven also autonomously (the proof will be useful later). Namely: using the continuity of the functions

$$v(\rho),\ -v'(\rho),\ f(\rho, v(\rho), -v'(\rho))$$

on the interval $[0,\delta]$ for any $\delta : 0 < \delta < 1$ and especially in $\rho = 0$ and using inequality

$$0 \le t^{1-n}\rho^{n-1} = \left(\frac{\rho}{t}\right)^{n-1} \le 1 \quad 0 \le \rho \le t \quad 0 < t \le \delta \tag{1.48}$$

we get

(1.49)
$$0 < \int_0^t \left(\frac{\rho}{t}\right)^{n-1} f(\rho, v(\rho), -v'(\rho))\, d\rho < t \cdot M_\delta \le \delta \cdot M_\delta \le \delta M_{1/2} \quad \delta > 0,\ 0 < t \le \delta \le \frac{1}{2},$$

where

$$M_\delta := \max_{\rho \in [0,\delta]} f(\rho, v(\rho), -v'(\rho)).$$

Now, if $\delta \to 0+0$ in (1.49), then we get

$$\int_0^t \left(\frac{\rho}{t}\right)^{n-1} f(\rho, v(\rho), -v'(\rho))\, d\rho \to 0 \quad t \to 0+0. \tag{1.50}$$

A result inverse to Lemma 1.5 is expressed in the next Lemma.

Lemma 1.6. Let $a > a_0$ be arbitrary and let

$$v \in C^2[0,1) \cap C[0,1], v'(\rho) \le 0 \quad \rho \in (0,1),\ v'(0) = 0, v(1) = a. \tag{1.51}$$

Suppose, that function $v(\rho)$ satisfies equation

$$v''(\rho) + \frac{n-1}{\rho} v'(\rho) + f(\rho, v(\rho), -v'(\rho)) = 0 \quad \rho \in [0+0, 1], \tag{1.52}$$

in $\rho = 0 + 0$ in the limit sense. Assume, finally that there exists a constant $K_a > 0$ such, that

$$(1.53) \qquad 0 < f(\rho, \alpha, \beta) \leq K_a \quad (\rho, \alpha, \beta) \in [0, 1] \times [a, \infty) \times [0, \infty).$$

Then function $u(x) := v(|x|) \equiv v(\rho) \quad x \in \overline{B}$ is a solution of the **Problem 3**.

Proof. The smoothness properties of u expressed by (1.33) are guaranteed by assumptions (1.51) and Lemma 1.4. Assumptions (1.51), (1.52) combined with (1.39) - (1.43) give (1.32). Lemma is proven.

C^2 smoothness at the boundary.

Lemma 1.7. Let $a \in \mathbb{R}, a > a_0$ appearing in **Problem 3** be fixed, and suppose, that for any $b > a$ there exists a constant $K_{a,b} > 0$ such, that

$$(1.54) \qquad 0 < f(\rho, \alpha, \beta) \leq K_{a,b} \quad (\rho, \alpha, \beta) \in [0, 1] \times [a, b] \times [0, \infty).$$

Then if u is a solution of **Problem 3** with

$$v(\rho) \equiv v(|x|) = u(x) \quad x \in \overline{B},$$

then

$$(1.55) \qquad \exists v'(1) = v'(1-0) \equiv \lim_{t \to 1-0} v'(t); v' \in C[0, 1],$$

$$(1.56) \qquad \exists v''(1) = v''(1-0) \equiv \lim_{t \to 1-0} v''(t); v'' \in C[0, 1],$$

and instead of the original condition in (1.33):

$$u \in C^2(B) \cap C(\overline{B})$$

we have the additional smoothness:

$$(1.57) \qquad u \in C^2(\overline{B}).$$

Proof. Let u be the solution in consideration. So

$$(1.58) \qquad u \in C^2(B) \cap C(\overline{B}),\ u|_\Gamma = a; \exists v : [0, 1] \to \mathbb{R} : v(\rho) \equiv v(|x|) = u(x) \quad x \in \overline{B},$$

therefore in virtue of Lemmae 1.1, 1.3

$$(1.59) \qquad v(\rho) \in C^2[0, 1) \cap C[0, 1],\ v(1) = a, a \leq v(\rho) \leq b := u(0) \quad \rho \in [0, 1]$$

and using assumption (1.54)we get

$$0<\gamma(\rho):=f(\rho,v(\rho),-v'(\rho))\le K_{a,b}\quad \rho\in[0,1]. \tag{1.60}$$

Using Lemma 1.5 we get

$$\begin{aligned}0>v'(t)=-t^{1-n}\int_0^t\rho^{n-1}\gamma(\rho)\,d\rho\ge -t^{1-n}K_{a,b}\frac{\rho^n}{n}\Big|_{\rho=0}^{\rho=t} &=\\ =-\frac{K_{a,b}\cdot t}{n}\ge-\frac{K_{a,b}}{n}\quad t\in(0,1).\end{aligned} \tag{1.61}$$

To begin the proof of (1.55) we first prove the existence of the limit

$$\lim_{t<1,t\to1}\ v'(t)\equiv d_1\in\mathbb{R}. \tag{1.62}$$

The last statement follows from the Cauchy criteria. Namely, let $\epsilon>0$ be arbitrary. We will find a number $\delta\in(0,1/2)$ such, that

$$|v'(t_1)-v'(t_2)|<\epsilon\quad \forall t_1,t_2:\frac{1}{2}<1-\delta<t_1,t_2<1. \tag{1.63}$$

Let us use equality (1.38) for $\frac{1}{2}<t_1<t_2<1$,then we get

$$\begin{aligned}|v'(t_2)-v'(t_1)|&=|\int_0^{t_2}t_2^{1-n}\rho^{n-1}\gamma(\rho)\,d\rho-\int_0^{t_1}t_1^{1-n}\rho^{n-1}\gamma(\rho)\,d\rho|=\\ &=|\int_0^{t_2}(t_2^{1-n}-t_1^{1-n}+t_1^{1-n})\rho^{n-1}\gamma(\rho)\,d\rho-\int_0^{t_1}t_1^{1-n}\rho^{n-1}\gamma(\rho)\,d\rho|=\\ &=|\int_0^{t_2}(t_2^{1-n}-t_1^{1-n})\rho^{n-1}\gamma(\rho)\,d\rho+\int_{t_1}^{t_2}t_1^{1-n}\rho^{n-1}\gamma(\rho)\,d\rho|<\\ &<\int_0^{t_2}(t_1^{1-n}-t_2^{1-n})\rho^{n-1}\gamma(\rho)\,d\rho+\int_{t_1}^{t_2}t_1^{1-n}\rho^{n-1}\gamma(\rho)\,d\rho\equiv I_1+I_2.\end{aligned} \tag{1.64}$$

Estimating the integrals $I_1,\ I_2$ with help of (1.61), and the Lagrange's theorem applied to t^{1-n} on $[t_1,t_2]$ we have

$$\begin{aligned}0<I_1&\le(t_1^{1-n}-t_2^{1-n})K_{a,b}\frac{t_2^n}{n}=\\ &=K_{a,b}\frac{t_2^n}{n}(t_1-t_2)(1-n)t^{-n}|_{t=\xi\in(t_1,t_2)}=\\ &=K_{a,b}\frac{t_2^n}{n}(n-1)(t_2-t_1)\xi^{-n}=K_{a,b}(1-\frac{1}{n})(\frac{t_2}{\xi})^n(t_2-t_1)<\\ &<K_{a,b}\cdot2^n(t_2-t_1)<\frac{\varepsilon}{2}\end{aligned} \tag{1.65}$$

if

$$\frac{1}{2} < t_1 < t_2 < 1,\ t_2 - t_1 < \frac{\epsilon}{K_{a,b}\cdot 2^{n+1}} \equiv \delta_1. \tag{1.66}$$

For the integral I_2 we get (simpler):

$$0 < I_2 < 2^{n-1}(t_2 - t_1)K_{a,b} < \frac{\epsilon}{2} \tag{1.67}$$

if

$$\frac{1}{2} < t_1 < t_2 < 1, \quad t_2 - t_1 < \frac{\epsilon}{K_{a,b}\cdot 2^{n}} \equiv \delta_2. \tag{1.68}$$

Obviously $\delta_2 > \delta_1$, therefore

$$|v'(t_2) - v'(t_1)| < \epsilon \quad \forall t_1, t_2 \in (\frac{1}{2}, 1) :\ t_1 < t_2,\ t_2 - t_1 < \delta_1. \tag{1.69}$$

The existence of the limit in (1.62) is proved. The existence of the derivative $v'(1) \sim v'(1-0)$ follows from the Lagrange's theorem and (1.62), because for any $t \in (\frac{1}{2}, 1)$ there exists a $\xi \in (t, 1)$ such, that

$$\frac{v(t) - v(1)}{t - 1} = v'(\xi)\ (v'(\xi) \to d_1 \quad t \to 1 - 0). \tag{1.70}$$

The first of the relations in (1.59) combined with (1.62), (1.70) implies the statement (1.55). Finally - using (1.55) and formulae

$$u_{x_i}(y) = v'(\rho)\frac{y_i}{\rho} \quad y \in B\backslash\{0\},\ i = \overline{1,n}$$

from (1.24) in virtue of the continuity of the factors $\frac{y_i}{\rho}$ in $\overline{B}\backslash\{0\}$ and using (1.58), (1.59) we get the property:

$$u_{x_i} \in C(\overline{B}) \quad i = \overline{1,n},\ u \in C^1(\overline{B}). \tag{1.71}$$

For the proof of the statements in (1.56) remark, that using (1.35), (1.37) we have the representation

$$v''(\rho) = -\frac{n-1}{\rho}v'(\rho) - f(\rho, v(\rho), -v'(\rho)) \quad \rho \in (0,1), \tag{1.72}$$

where (1.55) and the continuity of function f guarantee the existence of the limits as $\rho \to 1-0$ of both terms in the right hand side of (1.72), consequently there exists also the limit of the left hand side, and

$$\lim_{\rho<1, \rho\to 1} v''(\rho) =: d_2 = -\frac{n-1}{1}\ d_1 - f(1, a, -d_1). \tag{1.73}$$

The existence of the derivative $v''(1)$ follows from the Lagrange's theorem because for any $t \in (0,1)$ and suitable $\xi \in (t,1)$

$$(1.74) \qquad \frac{v'(t)-v'(1)}{t-1} = \frac{v'(t)-d_1}{t-1} = v''(\xi) \to d_2 \quad t \in (0,1),\ t \to 1.$$

The first of the relations in (1.59) combined with (1.73), (1.74) implies the statement (1.56). Finally, using (1.55), (1.56) and formulae (from (1.25), (1.26)):

$$u_{x_i x_j}(y) = (v''(\rho) - \frac{v'(\rho)}{\rho})\frac{y_i}{\rho}\frac{y_j}{\rho} \quad y \in B\backslash\{0\}; \rho := |y|;\ i \neq j; i,j \in \overline{1,n},$$

$$u_{x_i x_i}(y) = (v''(\rho) - \frac{v'(\rho)}{\rho})(\frac{y_i}{\rho})^2 + \frac{v'(\rho)}{\rho} \quad y \in B\backslash\{0\},\ \rho := |y|,\ i = \overline{1,n}$$

with help of (1.71), and properties

$$\frac{1}{\rho},\ \frac{y_i}{\rho} \in C(\overline{B}\backslash\{0\}) \quad i = \overline{1,n}$$

we get, that

$$(1.75) \qquad u_{x_i x_j},\ u_{x_i x_i} \in C(\overline{B}\backslash\{0\}) \quad i \neq j;\ i,j \in \overline{1,n}.$$

Relations(1.75) combined with (1.58) guarantee, that

$$u_{x_i x_j}, u_{x_i x_i} \in C(\overline{B}),$$

the latter together with (1.58), (1.71) give $u \in C^2(\overline{B})$ i.e. relation (1.57). Lemma is proven.

Summarising the results of Lemmae 1.1 - 1.7 we get:

Theorem 1.1. Let f be a function, that satisfies conditions:

$$(1.76) \qquad f \in C(G_{a_0}; (0,\infty)),$$

and for any $a, b \in \mathbb{R}$ such that $(a_0 < a < b)$ there exists a constant $K_{a,b}$ such, that

$$(1.77) \qquad 0 < f(|x|, \alpha, \beta) < K_{a,b} \qquad (|x|, \alpha, \beta) \in [0,1] \times [a,b] \times [0,\infty) \equiv G_{a,b}.$$

Consider **Problem 3** (with function f satisfying (1.76), (1.77), and with arbitrary $a \in \mathbb{R}, a > a_0$ in the boundary condition (1.34)). If $u(x)$ is a solution of **Problem 3**, then function $u(x)$ has automatically the additional smoothness property

$$(1.78) \qquad u \in C^2(\overline{B}).$$

A trivial consequence of Theorem 1.1 and relation (1.73) is the following:

Corollary 1.1. Equalities (1.35),(1.38) appearing in Lemma 1.5 (under the assumptions of Theorem 1.1) are fulfilled even on the whole interval $\rho \in [0,1],\ t \in [0,1]$ respectively (if $u(x) = v(|x|)$ is a solution of **Problem 3**).

Another (more sharp) result on the additional smoothness of solution $u(x)$ is expressed in the next subsection.

Higher order smoothness at the boundary.

Theorem 1.2. Let $a \in \mathbb{R}, a > a_0$ be arbitrarily fixed. Suppose that function f satisfies conditions:

(1.79) $f \in C([0,1] \times [a,\infty) \times [0,\infty); (0,\infty)) \cap C^m((0,1] \times [a,\infty) \times [0,\infty)) \quad 1 \le m \le \infty,$

and for any $b \in \mathbb{R}, b > a$ there exists a constant $K_{a,b}$ such, that

$$0 < f(|x|, \alpha, \beta) < K_{a,b} \quad (|x|, \alpha, \beta) \in G_{a,b}). \tag{1.80}$$

Consider **Problem 3** (with $a \in \mathbb{R}, a > a_0$ fixed above and with function f satisfying conditions (1.79), (1.80)), and let $u(x)$ be a solution of this Problem i.e.:

$$\Delta u(x) + f(|x|,\ u(x),\ |\nabla u(x)|) = 0 \quad x \in B = \{x \in \mathbb{R}^n |\ |x| < 1\}, \tag{1.81}$$

$$u \in C^2(B) \cap C(\overline{B}), \quad \exists v : [0,1] \to \mathbb{R} : v(\rho) \equiv v(|x|) = u(x) \quad x \in \overline{B}, \tag{1.82}$$

$$u|_{\Gamma = \partial B} = a, \tag{1.83}$$

then function $u(x)$ has automatically the additional smoothness property:

$$u \in C^2(\overline{B}) \cap C^{m+2}\ (\overline{B} \backslash \{0\}). \tag{1.84}$$

Proof. Comparing (1.78) and (1.84) one can see, that it is necessary to prove, that for the solution $u(x) = v(|x|) \equiv v(\rho)$ of the **Problem 3**

$$u \in C^k(\overline{B} \backslash \{0\}) \quad k = 3, 4, \ldots, m+2. \tag{1.85}$$

Relations (1.85) may be proven by induction. First, let $k = 3$. Using representation (1.72) and relations (1.55), (1.56), (1.73) we get:

$$v''(\rho) = -\frac{n-1}{\rho} v'(\rho) - f(\rho, v(\rho), -v'(\rho)) \quad \rho \in (0,1],$$

where both terms on the right hand side allow differentiation with respect to ρ (on $(0,1]$) in virtue of condition (1.79), consequently (using the notation $f \sim f(\alpha_1, \alpha_2, \alpha_3)$)

(1.86)
$$\exists v'''(\rho) = \frac{(n-1)v'(\rho)}{\rho^2} - \frac{n-1}{\rho} v''(\rho) - f_{\alpha_1}(\rho, v(\rho), -v'(\rho)) -$$
$$- f_{\alpha_2}(\rho, v(\rho), -v'(\rho))v'(\rho) + f_{\alpha_3}(\rho, v(\rho), -v'(\rho))v''(\rho) \quad \rho \in (0,1].$$

All of the terms on the right hand side of (1.86) are continuous functions of variable ρ on $(0,1]$, consequently

(1.87)
$$v''' \in C(0,1], \quad v \in C^3(0,1].$$

Using relation (1.87) it is easy to show that all terms on the right hand sides of representations (1.25), (1.26) allow differentiation with respect to ρ on $(0,1]$, and we come to the existence of the derivatives $u_{x_i x_j x_k}(x) \quad x \in \overline{B}\backslash\{0\}$ and to representations:

(1.88)
$$u_{x_i x_j x_k}(y) = v'''(\rho)\frac{y_i y_j y_k}{\rho^3} - v''(\rho)\frac{y_i y_j y_k}{\rho^4} + v''(\rho)\left(\frac{y_i y_j}{\rho^2}\right)_{y_k} -$$
$$- v'(\rho)\left(\frac{y_i y_j}{\rho^3}\right)_{y_k} = \left(v'''(\rho) - 3\frac{v''(\rho)}{\rho} + 3\frac{v'(\rho)}{\rho^2}\right)\frac{y_i y_j y_k}{\rho^3}$$
$$y \in \overline{B}\backslash\{0\},\ \rho := |y|; \quad i \neq j \neq k \neq i; \quad i,j,k = \overline{1,n} \quad (n > 2),$$

(1.89)
$$u_{x_i x_i x_j}(y) = v'''(\rho)\frac{y_i^2 y_j}{\rho^3} + \left[1 - 3\left(\frac{y_i}{\rho}\right)^2\right]\frac{y_j}{\rho^2}\left[v''(\rho) - \frac{v'(\rho)}{\rho}\right]$$
$$y \in \overline{B}\backslash\{0\}, \quad \rho := |y|, \quad i \neq j; \quad i,j = \overline{1,n},$$

(1.90)
$$u_{x_i x_i x_i}(y) = v'''(\rho)\left(\frac{y_i}{\rho}\right)^3 + 3\frac{y_i}{\rho^2}\left[1 - \left(\frac{y_i}{\rho}\right)^2\right]\left[v''(\rho) - \frac{v'(\rho)}{\rho}\right]$$
$$y \in \overline{B}\backslash\{0\}, \quad \rho := |y|, \quad i = \overline{1,n}.$$

All the right hand side terms in (1.88) - (1.90) are continuous functions on $\overline{B}\backslash\{0\}$, consequently

(1.91)
$$u_{x_i x_j x_k}, \quad u_{x_i x_i x_j}, \quad u_{x_i x_i x_i} \quad \in C(\overline{B}\backslash\{0\})$$
$$i \neq j \neq k \neq i; \quad i,j,k = \overline{1,n}; \quad u \in C^3(\overline{B}\backslash\{0\}).$$

Further, if $f \in C^m,\ m > 1$, then the right hand side of (1.86) allows differentation with respect to ρ on $\rho \in\ (0,1]$ and we come to relations:

(1.92)
$$\exists v^{(IV)} \text{ on } (0,1],\ v^{(IV)} \in C(0,1];\ v \in C^4\ (0,1]$$

that imply the existence of the derivatives $u_{x_ix_jx_kx_l}(y) \quad y\in \overline{B}\backslash\{0\}$, and property

$$u_{x_ix_jx_kx_l}\in C(\overline{B}\backslash\{0\};\ u\in C^4(\overline{B}\backslash\{0\}) \tag{1.93}$$

in virtue of representations for $u_{x_ix_jx_kx_l}$, similar to ones in (1.88)-(1.90).

The general step of the induction may be made by the following scheme:

1) Suppose, that $k\in\mathbb{N}, \quad 2<k\le m+1 \quad (f\in C^m)$ and also, that the properties

$$v, v', \ldots, v^{(k)}\in C(0,1];\ v\in C^k(0,1], \tag{1.94}$$

$$u, u_{x_i}, \ldots, \partial_x^\alpha u\in C(\overline{B}\backslash\{0\}) \quad |\alpha|\le k;\ u\in C^k(\overline{B}\backslash\{0\}) \tag{1.95}$$

are already proved.

2) Observe, that the right hand side terms in equality

$$v''(\rho) = -\frac{n-1}{\rho}v'(\rho) - f(\rho, v(\rho), -v'(\rho)) \tag{1.96}$$

allow differentiations with respect to ρ (when $\rho\in(0,1]$) $k-1$-times, and the resulted terms belong to $C(0,1]$-we come to the relations

$$\begin{aligned}&\exists\ v^{(k+1)}(\rho) \quad \rho\in(0,1],\\ &v^{(k+1)}(\rho) = P_k(v', v'', \ldots, v^{(k)}; \frac{1}{\rho}; \ldots, \partial^\beta f(\rho, v(\rho), -v'(\rho)), \ldots)\\ &|\beta| := \beta_1+\beta_2+\beta_3\le k-1, \quad \rho\in(0,1],\end{aligned} \tag{1.97}$$

where P_k is a polynomial of all of the variables

$$v', \ldots,, v^{(k)}, \frac{1}{\rho}, \ldots, \partial^\beta f \equiv \frac{\partial^{|\beta|}}{\partial\alpha_1^{\beta_1}\partial\alpha_2^{\beta_2}\partial\alpha_3^{\beta_3}} f, \ldots\ . \tag{1.98}$$

It remains to remark that all of these variables are continuous in $\rho\in(0,1]$, consequently using also relations in (1.94) we get

$$v^{(k+1)}(\rho)\in C(0,1]; \quad v\in C^{k+1}(0,1]. \tag{1.99}$$

3) The existence and continuity of derivatives $\partial_x^\alpha u \quad |\alpha|\le k+1$ may be given starting with formulae (1.88) - (1.90), differentiating them necessary-times and using the chain-rule. The resulted equality has the following form:

$$\begin{aligned}&\partial_x^\alpha u(y) = R_k(v'(\rho), \ldots, v^{(k+1)}(\rho); \frac{1}{\rho}; y_1, \ldots, y_n)\\ &\alpha = (\alpha_1, \ldots, \alpha_n) \quad \alpha_i\in\mathbb{Z}_+ \quad i=\overline{1,n};\ |\alpha| := \sum_{i=1}^n \alpha_i;\\ &\rho\in\ (0,1], \quad y\in\overline{B}\backslash\{0\},\end{aligned} \tag{1.100}$$

where R_k is a polynomial of all of the variables $v'(\rho), \ldots, y_n$; consequently

$$(1.101) \qquad \exists\ \partial_x^\alpha u(y) \quad y \in \overline{B}\backslash\{0\};\ \partial_x^\alpha u(y) \in C(\overline{B}\backslash\{0\}) \quad |\alpha| = k+1$$

that combined with (1.95) serves:

$$(1.102) \qquad u \in C^{k+1}(\overline{B}\backslash\{0\}).$$

Theorem is proved.

2. The existence of the solution.

Throughout this section we shall suppose that the parameter $a > a_0$ appearing in **Problem 3** is arbitrarily fixed and

$$(2.1) \qquad \forall a \in \mathbb{R}, a > a_0 \quad \exists K_a \in \mathbb{R}: \quad 0 < f(\rho, \alpha, \beta) \leq K_a \quad \forall (\rho, \alpha, \beta) \in G_a,$$

and that

$$(2.2) \qquad f \in C(G_{a_0})$$

Lemma 2.1. If $u(x)$ is a solution of **Problem 3** with $v(\rho) := u(|x|) \quad x \in \overline{B} \quad (\rho \in [0,1])$ then vector function $(v(t), -v'(t)) \equiv (v(t), \nu(t)) \quad t \in [0,1]$ satisfies (2.3) - (2.6):
(2.3) All of the statements of Lemmae 1.2, 1.3, 1.5 are valid,

$$(2.4) \qquad v \in C^2(\overline{B}), \quad \nu \in C^1(\overline{B}),$$

$$(2.5) \qquad \nu(t) = \int_0^t \left(\frac{\rho}{t}\right)^{n-1} \cdot f(\rho, v(\rho),\ \nu(\rho))\ d\rho \quad t \in [0+0, 1]$$

in $t = 0+0$ in the limit sense ($\nu(0) = -v'(0) = 0$)

$$(2.6) \qquad v(t) = a + \int_t^1 \nu(s)\ ds \quad t \in [0,1].$$

Proof. Conditions of Lemmae 1.2, 1.3, 1.5 are fulfilled, consequently (2.3) holds. Relations in (2.4) follow from Theorem 1.1 (see (1.78)). Finally, (2.3), (2.4) and Corollary 1.1 imply (2.5), (2.6).

An inverse result is expressed in the following

Lemma 2.2. Let $\nu, v : [0,1] \to \mathbb{R}$ be given functions,

$$\nu \in C(0,1],\ \nu(0)=0, \quad \nu(\rho) \geq 0 \quad \rho \in (0,1], \tag{2.7}$$

that satisfy system (2.8), (2.9):

$$\nu(t) = \int_0^t \left(\frac{\rho}{t}\right)^{n-1} f(\rho, v(\rho),\ \nu(\rho)\ d\rho \quad t \in (0,1], \tag{2.8}$$

$$v(t) = a + \int_t^1 \nu(s)\ ds \quad t \in [0,1], \tag{2.9}$$

where $a \in \mathbb{R}, a > a_0$ is fixed, $f \in C(G_a;\ (0, K_a])$, then function

$$u(x) := v(|x|) \equiv v(\rho) \quad x \in \overline{B},\ \rho \in [0,1]$$

is a solution of **Problem 3**.

Proof. Condition (1.34) is fulfilled (see (2.9) for $t=1$). The property $u \in C(\overline{B}) \cap C^2(B)$ (the first relation of (1.33)) may be proved by the following way. First, using boundedness of f, from (2.7), (2.8) we get

$$0 \leq \nu(t) \leq \frac{1}{t^{n-1}} \left.\frac{\rho^n}{n}\right|_{\rho=0}^{\rho=t} \cdot K_a = \frac{K_a \cdot t}{n} \quad t \in (0,1],\ \nu(0) := 0, \tag{2.10}$$

consequently

$$\nu \in C[0,1], \quad v \in C^1[0,1] \quad \text{(see (2.9))}. \tag{2.11}$$

Second, using (2.11) continuity of the integrand in (2.8) for $t \in (0,1]$: i.e.:

$$\left(\frac{\rho}{t}\right)^{n-1} f(\rho, a + \int_\rho^1 \nu(s)\ ds,\ \nu(\rho)) \quad \in C[0,1], \quad \forall\, t \in (0,1] \tag{2.12}$$

we get

$$\nu' \in C(0,1], \quad \nu \in C^1(0,1] \cap C[0,1],$$

$$\begin{aligned} \nu'(t) &= f(t, v(t),\ \nu(t)) - \frac{n-1}{t} \int_0^t \left(\frac{\rho}{t}\right)^{n-1} f(\rho, v(\rho),\ \nu(\rho))\ d\rho \equiv \\ &\equiv \alpha_1(t) + \alpha_2(t) \quad t \in (0,1] \end{aligned} \tag{2.13}$$

with the terms

$$\alpha_1, \alpha_2 \in C(0,1]. \tag{2.14}$$

Further

$$\exists \lim_{t\to 0+0} \alpha_1(t) = f(0, v(0), \nu(0)) = f(0, v(0), 0) \tag{2.15}$$

and

$$\exists \lim_{t\to 0+0} \alpha_2(t) \tag{2.16}$$

that follows from the ℓ' Hospital's rule because

$$\alpha_2(t) = \frac{-(n-1)\int_0^t \rho^{n-1} f(\rho, v(\rho), \nu(\rho))\, d\rho}{t^n} \quad t \in (0,1], \tag{2.17}$$

where the denominator: $t^n \to 0$ as $t \to 0+0$; and the same is true for the numerator using (2.10)

$$0 \le \int_0^t \rho^{n-1} f(\rho, v(\rho),\ \nu(\rho))\, d\rho \le \frac{t^n}{n} K_a \to 0 \quad t \to 0+0; \tag{2.18}$$

and finally there exists the limit

$$\lim_{t\to 0+0} \frac{-(n-1)\ t^{n-1} f(t, v(t),\ \nu(t))}{n\ t^{n-1}} = -\left(1 - \frac{1}{n}\right) f(0, v(0), \nu(0)) = \tag{2.19}$$

$$= \left(\frac{1}{n} - 1\right) f(0, v(0),\ 0).$$

Combining these calculations with (2.13), (2.15) we come to relations

$$\exists \lim_{t\to 0+0} \nu'(t) \equiv -\lim_{t\to 0+0} v''(t) = \frac{1}{n} f(0, v(0),\ 0) \tag{2.20}$$

and (using also the Lagrange's theorem) to the existence of the derivatives

$$\nu'(0+0) \equiv -v''(0+0) = \frac{1}{n}\ f(0, v(0),\ 0) \tag{2.21.}$$

and finally to the properties

$$\nu' \in C[0,1],\ \nu \in C^1[0,1] \quad (v'' \in C[0,1],\ v \in C^2[0,1]). \tag{2.22}$$

Relations (2.22) combined with (2.7) - (2.9), (2.13) guarantee the validity of assumptions (1.51), (1.52) of Lemma 1.6. So, Lemma 1.6 may be applied, and conditions (1.32), (1.33) of **Problem 3** are also fulfilled. Lemma 2.2 is proven.

Next we shall prove the existence of functions ν, v satisfying assumptions of Lemma 2.2.

In fact we need (only) function ν because function v is defined uniquely with help of ν by the formula (2.9). We will use the Schauder's fixed point theorem. For this goal consider the Banach space $\mathcal{H} := C[0,1]$ equipped with the usual norm

$$||w|| := \max_{t\in[0,1]} |w(t)| \quad \forall w \in \mathcal{H}, \tag{2.23}$$

and introduce the subset $\mathcal{H}_0$ of $\mathcal{H}$:

$$\mathcal{H}_0 := \{\nu \in \mathcal{H} | \nu(t) \geq 0 \quad t \in (0,1],\ \nu(0) = 0,\ \nu(t) \leq \frac{K_a}{n} t \quad t \in [0,1]\}. \tag{2.24}$$

It is clear that $\mathcal{H}_0$ is a closed subset of $\mathcal{H}$. Moreover $\mathcal{H}_0$ is bounded, because

$$||\nu|| \leq \frac{K_a}{n} \quad \forall \nu \in \mathcal{H}_0, \tag{2.25}$$

and $\mathcal{H}_0$ is a convex set. The last statement follows from the considerations in below:

$$\nu_i \in \mathcal{H}, \quad \nu_i(t) \geq 0 \quad t \in (0,1],\ \nu_i(0) = 0,\ \nu_i(t) \leq \frac{K_a}{n} t \quad t \in [0,1];\ i = 1,2$$

imply

$$\lambda_1\nu_1 + \lambda_2\nu_2 \in C[0,1],\ \lambda_1\nu_1(t) + \lambda_2\nu_2(t) \geq 0 \quad t \in (0,1],$$

$$\lambda_1\nu_1(0) + \lambda_2\nu_2(0) = 0 + 0 = 0, \quad \lambda_1\nu_1(t) + \lambda_2\nu_2(t) \leq \lambda_1 \frac{K_a}{n} t + \lambda_2 \frac{K_a}{n} t =$$

$$= (\lambda_1 + \lambda_2)\frac{K_a}{n} t \equiv \frac{K_a}{n} t \quad \forall \lambda_1, \lambda_2 \in [0,1] : \quad \lambda_1 + \lambda_2 = 1.$$

Let $\mathcal{A}$ be the operator defined by the formula:

$$(\mathcal{A}\nu)(t) := \begin{cases} 0 & t = 0 \\ \int_0^t \left(\frac{\rho}{t}\right)^{n-1} f(\rho,\ a + \int_\rho^1 \nu(s)\ ds,\ \nu(\rho))\ d\rho & t \in (0,1], \end{cases} \tag{2.26}$$

where $\nu \in \mathcal{H}$. The definition is correct because

$$\left(\frac{\rho}{t}\right)^{n-1} \in C[0,1],\ \gamma(\rho) := f(\rho, a + \int_\rho^1 \nu(s)\ ds,\ \nu(\rho)) \in C[0,1]$$

for any $t \in (0,1]$ and any $\nu \in \mathcal{H}$, therefore the integrand appearing in (2.26) is a continuous function of ρ, when $\rho \in [0,1]$ which guarantees the existence of the integral standing in (2.26). Show that $\mathcal{A}$ maps $\mathcal{H}_0$ into $\mathcal{H}_0$. In fact, if $\nu \in \mathcal{H}_0$, then $(\mathcal{A}\nu)(0) = 0$ by the definition in (2.26). Instead of the property $\mathcal{A}\nu \in C[0,1]$ for any $\nu \in \mathcal{H}_0$ let us prove the uniform equicontinuity of the family $\{\mathcal{A}\nu\} \quad (\nu \in \mathcal{H}_0)$ on $[0,1]$. Let $\epsilon > 0$ be arbitrary, and suppose first that $t_1=0,\ t_2\in(0,1]$, then using the inequality $0<f\le K_a<\infty$ on G_a we get

$$
\begin{aligned}
|(\mathcal{A}\nu)(t_2)-(\mathcal{A}\nu)(0)| = |(\mathcal{A}\nu)(t_2)| = \int_0^{t_2} t_2^{1-n}\rho^{n-1}\gamma(\rho)\,d\rho \le \\
\le t_2^{1-n}K_a\frac{\rho^n}{n}|_{\rho=0}^{\rho=t_2} = \frac{K_a}{n}t_2<\epsilon
\end{aligned}
\tag{2.27}
$$

if

$$
t_2<\frac{\epsilon n}{K_a}=:\delta_1 \quad (t_2-t_1<\delta_1),\ \forall\nu\in\mathcal{H}_0.
$$

For the case $0<t_1<t_2\le 1$ we have

$$
\begin{aligned}
|(\mathcal{A}\nu)(t_2)-(\mathcal{A}\nu)(t_1)| &= |\int_0^{t_2}(\rho/t_2)^{n-1}\ \gamma(\rho)\ d\rho-\int_0^{t_1}(\rho/t_1)^{n-1}\ \gamma(\rho)\ d\rho| = \\
&= \left|\int_0^{t_1}\left[\left(\frac{\rho}{t_2}\right)^{n-1}-\left(\frac{\rho}{t_1}\right)^{n-1}\right]\gamma(\rho)\ d\rho+\int_{t_1}^{t_2}\left(\frac{\rho}{t_2}\right)^{n-1}\gamma(\rho)\ d\rho\right| \le \\
&\le \int_0^{t_1}\left[\left(\frac{\rho}{t_1}\right)^{n-1}-\left(\frac{\rho}{t_2}\right)^{n-1}\right]\gamma(\rho)\ d\rho+\int_{t_1}^{t_2}\left(\frac{\rho}{t_2}\right)^{n-1}\gamma(\rho)\ d\rho = \\
&= (t_1^{1-n}-t_2^{1-n})\int_0^{t_1}\rho^{n-1}\gamma(\rho)\ d\rho)+\int_{t_1}^{t_2}\left(\frac{\rho}{t_2}\right)^{n-1}\gamma(\rho)\ d\rho).
\end{aligned}
\tag{2.28}
$$

Here estimating the first term we get:

$$
\begin{aligned}
0 &\le \left(t_1^{1-n}-t_2^{1-n}\right)\int_0^{t_1}\rho^{n-1}\gamma(\rho)\ d\rho \le \left(t_1^{1-n}-t_2^{1-n}\right)K_a\frac{t_1^n}{n} = \\
&= \frac{K_a}{n}\left(\frac{1}{t_1^{n-1}}-\frac{1}{t_2^{n-1}}\right)t_1^n = \frac{K_a}{n}t_1^n\frac{t_2^{n-1}-t_1^{n-1}}{t_1^{n-1}t_2^{n-1}} = \\
&= \frac{K_a}{n}t_1^n(t_2-t_1)(t_2^{n-2}+t_2^{n-3}t_1+\ldots+t_1^{n-2})\frac{1}{t_1^{n-1}t_2^{n-1}} \le \\
&\le \frac{K_a}{n}\frac{t_1^n(t_2-t_1)(n-1)t_2^{n-2}}{t_1^{n-1}t_2^{n-1}} = \frac{K_a}{n}(t_2-t_1)(n-1)\frac{t_1}{t_2} = \\
&= K_a\frac{n-1}{n}\frac{t_1}{t_2}(t_2-t_1)<K_a(t_2-t_1)<\frac{\epsilon}{2}.
\end{aligned}
$$

if

$$
t_2-t_1<\frac{\epsilon}{2K_a}=:\delta_2.
$$

Using the inequality

$$
0<\frac{\rho}{t_2}\le 1 \quad \rho\in[t_1,t_2],
$$

for the second term we have

$$0 \le \int_{t_1}^{t_2} \left(\frac{\rho}{t_2}\right)^{n-1} \gamma(\rho)\, d\rho \le \int_{t_1}^{t_2} K_a\, d\rho = K_a(t_2 - t_1) < \frac{\epsilon}{2} \tag{2.29}$$

if

$$t_2 - t_1 < \frac{\epsilon}{2K_a} = \delta_2.$$

So (2.28) turnes into inequality

$$|(\mathcal{A}\nu)(t_2) - (\mathcal{A}\nu)(t_1)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall \nu \in \mathcal{H}_0,\ \forall t_1, t_2 \in (0,1] \quad (t_1 < t_2) \tag{2.30}$$

if

$$t_2 - t_1 < \delta_2 := \frac{\epsilon}{2K_a}.$$

Inequality (2.30) combined with (2.27) gives:

$$|(\mathcal{A}\nu)(t_2) - (\mathcal{A}\nu)(t_1)| < \epsilon \quad \forall \nu \in \mathcal{H}_0,\ \forall t_1, t_2 \in [0,1] \quad (t_1 < t_2) \tag{2.31}$$

if

$$t_2 - t_1 < \delta := \min\,(\delta_1, \delta_2) \equiv \min\left(\frac{\epsilon n}{K_a},\ \frac{\epsilon}{2K_a}\right) = \frac{\epsilon}{2K_a} \equiv \delta_2.$$

So, the uniform equicontinuity of the family $\{\mathcal{A}\nu\} \quad \nu \in \mathcal{H}_0$ is proved. Especially for any $\nu \in \mathcal{H}_0$ we have $\mathcal{A}\nu \in C[0,1]$. To complete the proof of the property $\mathcal{A}\mathcal{H}_0 \subseteq \mathcal{H}_0$ it remains to observe that

$$\begin{aligned} 0 \le (\mathcal{A}\nu)(t) &= \int_0^t \left(\frac{\rho}{t}\right)^{n-1} \gamma(\rho)\, d\rho \le \int_0^t t^{1-n}\rho^{n-1} K_a\, d\rho = \\ &= t^{1-n} K_a \frac{\rho^n}{n}\Big|_{\rho=0}^{\rho=t} = \frac{K_a}{n} t \quad \forall t \in (0,1],\ \forall \nu \in \mathcal{H}_0. \end{aligned} \tag{2.32}$$

Inequality (2.32) guarantees also the boundedness of family $\{\mathcal{A}\nu\} \quad \nu \in \mathcal{H}_0$.

The considerations above show, that all of the conditions of Schauder's fixed point theorem are fulfilled for the case of space $\mathcal{H}$, subset $\mathcal{H}_0$ and operator $\mathcal{A}$. Consequently we have proved the following

Lemma 2.3. There exists a fixed point: $\nu \in \mathcal{H}_0$ of the operator $\mathcal{A}$ i.e $\nu = \mathcal{A}\nu$ for some $\nu \in \mathcal{H}_0$. Taking any of the fixed points ν of the operator $\mathcal{A}$, and defining $v(t)$ by (2.9) we get a pair ν, v satisfying conditions of Lemma 2.2. As a consequence we get

Theorem 2.1. Let $a > a_0$ appearing in **Problem 3** be arbitrarily fixed and suppose that there exists a constant $K_a > 0$ such, that $f \in C(G_a; (0, K_a])$, then **Problem 3** has a solution: namely $u(x) := v(|x|)\ x \in \overline{B}$, where v is defined by the formula (2.9):

$$v(t) := a + \int_t^1 \nu(s)\, ds \quad t \in [0,1],$$

and ν is any of the fixed points in $\mathcal{H}_0$ of the operator $\mathcal{A}$.

Higher order smoothness at the origin.

Consider **Problem 4** ((3.1)-(3.4)):

$$\Delta u(x) + f(|x|, u(x), |\nabla u(x)|) = 0 \qquad x \in B, \tag{3.1}$$

$$u \in C^2(B) \cap C(\overline{B}), \exists v : [0,1] \to \mathbb{R} : v(\rho) \equiv v(|x|) = u(x) \ \forall x \in \overline{B}, \tag{3.2}$$

$$u|_\Gamma = a, \tag{3.3}$$

where B is the unit ball centered at the origin, $\Gamma := \partial B$, and $a \in \mathbb{R}$ is arbitrary, $a > a_0 \geq -\infty$; a_0 is fixed and function $f = f(\alpha_1, \alpha_2, \alpha_3)$ satisfies condition

$$f \in C^1([0,\delta) \times (a_0, \infty) \times [0,\infty)) \cap C\big(G_{a_0}; (0,\infty)\big) \tag{3.4}$$

for some $0 < \delta \leq 1$.

Lemma 3.1. Condition

$$f_{\alpha_1}(0, \alpha_2, 0) = -\frac{1}{n} f(0, \alpha_2, 0) f_{\alpha_3}(0, \alpha_2, 0) \qquad \forall \alpha_2 > a_0 \tag{3.5}$$

is sufficient for the property:

$$u(x) \equiv u(x; a) \in C^3(B_\delta^0) \tag{3.6}$$

where $B_\delta^0 := \{x \in \mathbb{R}^n | \quad |x| < \delta \leq 1\}$ and $u(x;a)$ is any of the solutions of **Problem 4** with arbitrary $a > a_0$.

Proof. From the proof (for $0 \leq \rho_0 < \delta \leq 1, \ 0 \leq \rho \equiv |x| < \delta \leq 1$) of Lemma 1.2 it follows that

$$\begin{aligned} v(\rho; a) = v(|x|; a) := u(x; a) \in C^2[0, \delta); \quad a_0 < a < v(0; a) < \infty, \\ v'(0; a) = 0, \ v''(0; a) = -\frac{1}{n} f(0, v(0; a), 0)(< 0). \end{aligned} \tag{3.7}$$

Similarly the proof of Theorem 1.2 shows, that

$$v(\rho; a) \in C^3(0, \delta), \ u(x; a) \in C^3(B_\delta^0 \backslash \{0\}), \tag{3.8}$$

therefore it remains to prove, that the derivatives

$$
\begin{aligned}
&u_{x_ix_jx_k}(x) \quad i\neq j\neq k\neq i \quad i,j,k=\overline{1,n} \quad n>2,\\
&u_{x_ix_ix_j}(x) \quad i\neq j \quad i,j=\overline{1,n}, \quad u_{x_ix_ix_i}(x) \quad i=\overline{1,n}
\end{aligned}
$$

have limits as $x \to 0$. For this reason rewrite representations (1.88)-(1.90) into more suitable forms:

$$
u_{x_ix_jx_k}(x) = \{v'''(\rho) - 3\frac{\rho v''(\rho)-v'(\rho)}{\rho^2}\}\frac{x_ix_jx_k}{\rho^3} \quad x\in B_\delta^0\backslash\{0\} \quad n>2, \tag{3.9}
$$

$$
u_{x_ix_ix_j}(x) = v'''(\rho)\frac{x_i^2x_j}{\rho^3} + \frac{x_j}{\rho}[1-3(\frac{x_i}{\rho})^2]\frac{\rho v''(\rho)-v'(\rho)}{\rho^2} \quad x\in B_\delta^0\backslash\{0\} \quad n\geq 2. \tag{3.10}
$$

$$
u_{x_ix_ix_i}(x) = v'''(\rho)\left(\frac{x_i}{\rho}\right)^3 + 3\frac{x_i}{\rho}[1-\left(\frac{x_i}{\rho}\right)^2]\frac{\rho v''(\rho)-v'(\rho)}{\rho^2} \quad x\in B_\delta^0\backslash\{0\} \quad n\geq 2. \tag{3.11}
$$

Now, prove for $v(t;a)\sim v(t)$ that

$$
\exists \lim_{t\to 0+0} v'''(t) = A \in \mathbb{R}. \tag{3.12}
$$

Since $u(x;a)=v(|x|;a)=v(\rho;a)$ is a solution of **Problem 4**, then for any $a>a_0$ fixed

$$
\alpha(\rho) := f(\rho, v(\rho), -v'(\rho)) \in C^1([0,\delta));\ |\alpha(\rho)|, |\alpha'(\rho)| \leq C < \infty \quad \rho\in[0,\frac{\delta}{2}]. \tag{3.13}
$$

Further, from (1.38),(3.8),(3.13) we have

$$
v''(t) = -f(t,v(t),-v'(t)) + [\int_0^t \rho^{n-1}\alpha(\rho)d\rho](n-1)t^{-n} \quad 0<t\equiv|x|<\delta, \tag{3.14}
$$

from which

$$
v'''(t) = -\alpha'(t) + (n-1)\frac{\alpha(t)t^n - n\int_0^t \rho^{n-1}\alpha(\rho)d\rho}{t^{n+1}} \quad t\in(0,\delta/2]. \tag{3.15}
$$

The second term on the right hand side has limit as $t\to 0+0$, because using the ℓ' Hospital's rule

$$
(n-1)\frac{\alpha'(t)t^n + \alpha(t)nt^{n-1} - nt^{n-1}\alpha(t)}{(n+1)t^n} \to \frac{n-1}{n+1}\alpha'(0) \quad t\to 0+0; \tag{3.16}
$$

therefore from (3.15) we get

$$
v'''(t) \to -\alpha'(0) + \frac{n-1}{n+1}\alpha'(0) = -\frac{2}{n+1}\alpha'(0) \quad t\to 0+0. \tag{3.17}
$$

From (3.4), (3.13) we get

$$\begin{aligned}\alpha'(t) = f_{\alpha_1}(t, v(t), -v'(t)) + f_{\alpha_2}(t, v(t), -v'(t))v'(t) - \\ - f_{\alpha_3}(t, v(t), -v'(t))v''(t) \quad t \in [0, \delta).\end{aligned} \tag{3.18}$$

Using equalities in (3.7), and substituting $\alpha'(0)$ from (3.18) into (3.17) we conclude, that

$$v'''(t) \to -\frac{2}{n+1}[f_{\alpha_1}(0, v(0), 0) + \frac{1}{n} f_{\alpha_3}(0, v(0), 0) f(0, v(0), 0)] = v'''(0) = B \in \mathbb{R} \tag{3.19}$$

as $t \to 0+0$.

Representation (3.10) holds for any $n \geq 2$, therefore setting in (3.10) $x_j = 0$ we get that if there exists the limit

$$\lim_{x \to 0} u_{x_i x_i x_j}(x) \tag{3.20}$$

then it must be zero. Now setting $x_i = x_j = \frac{\rho}{\sqrt{3}} \quad \rho > 0$ we get

$$u_{x_i x_i x_j}(x) = v'''(\rho) \frac{1}{3\sqrt{3}}, \tag{3.21}$$

Consequently, the following must be true :

$$\exists \lim_{\rho \to 0+0} v'''(\rho) = 0 \quad (= B). \tag{3.22}$$

If (3.22) holds, then $u_{x_i x_i x_j}(x) \to 0 \quad x \to 0$, because the factors $\frac{x_j}{\rho}$, $1 - 3\left(\frac{x_i}{\rho}\right)^2$ of the second term on the right hand side of (3.10) are bounded as $x \to 0$, and for the last factor $[\rho v''(\rho) - v'(\rho)]/\rho^2$ in virtue of the ℓ' Hospital's rule

$$\frac{v''(\rho) + \rho v'''(\rho) - v''(\rho)}{2\rho} = \frac{v'''(\rho)}{2} \to 0 \quad \rho \to 0+0. \tag{3.23}$$

The same arguments provide (see (3.9), (3.11)), that

$$\exists \lim_{x \to 0} u_{x_i x_j x_k}(x) = 0 \quad n > 2, \quad \exists \lim_{x \to 0} u_{x_i x_i x_i}(x) = 0 \quad n \geq 2. \tag{3.24}$$

It remains to observe, that if (3.5) is fulfilled, then relations (3.19), (3.22) hold. Lemma is proved.

Remark 3.1. If $u(x; a) = v(|x|; a)$ is any of the solutions of **Problem 4**, then condition

$$f_{\alpha_1}(0, v(0; a), 0) = -\frac{1}{n} f(0, v(0; a), 0) f_{\alpha_3}(0, v(0; a), 0) \tag{3.25}$$

is necessary and sufficient for the property $u \in C^3(B_\delta^0)$.

Theorem 3.1. Let $a > a_0$ appearing in **Problem 3** be arbitrarily fixed. Suppose, that there exists a constant $K_a > 0$ such, that function f satisfies condition

$$f \in C([0,1] \times [a,\infty) \times [0,\infty); (0, K_a]) \cap C^1(G_a). \tag{3.26}$$

Then **Problem 3** has a solution $u(x;a) = v(|x|;a)$ for any $a > a_0$, and every solution has the property

$$u(x;a) \in C^3(\overline{B}) \quad \forall a > a_0, \tag{3.27}$$

if condition (3.5) is fulfilled.

Proof. The existence of solutions follows from **Theorem 2.1**; the smoothness property (3.27) is a consequence of **Lemma 3.1** and **Theorem 1.2**. Theorem is proved.

An analysis of conditions (3.4), (3.5), (3.24), (3.25), and C^4, C^5, C^6 -smoothness results (in B_δ^0) will be given in a subsequent paper.

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