Nonresonance Impulsive Higher Order Functional Nonconvex-Valued Differential Inclusions

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Abstract

In this paper, the authors investigate the existence of solutions for nonresonance impulsive higher order functional differential inclusions in Banach spaces with nonconvex valued right hand side. They present two results. In the first one, they rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler, and for the second one, they use Schaefer's fixed point theorem combined with lower semi-continuous multivalued operators with decomposable values.

Key words and phrases: Impulsive functional differential inclusions, nonresonance problem, selection, fixed point, decomposable values.
2000 AMS Subject Classifications: 34A37, 34A60, 34G20, 34K45

1 Introduction

In the interval $J = [0, T]$, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be fixed. In this paper, we are concerned with the existence of solutions for a nonresonance problem for the functional differential inclusion,

$$y^{(n)}(t) - \lambda y(t) \in F(t, y_t), \quad t \in J \setminus \{t_1, \ldots, t_m\}, \tag{1}$$

subject to the impulse effects,

$$\Delta y^{(i)}(t_k) = I_k^i(y(t_k^-)),\ 0 \leq i \leq n-1,\ \ 1 \leq k \leq m, \tag{2}$$

satisfying the initial condition,

$$y(t) = \phi(t),\ \ t \in [-r, 0], \tag{3}$$

and satisfying the boundary conditions,

$$y^{(i)}(0) - y^{(i)}(T) = \mu_i,\ \ 0 \leq i \leq n-1, \tag{4}$$

where $\lambda \in \mathbb{R},\ 0 < r < \infty$, E is a real, separable Banach space with norm $|\cdot|$, $\mathcal{P}(E)$ is the family of all subsets of E, $F : J \times D \to \mathcal{P}(E)$ is a multivalued map,

$$\begin{aligned} D = \{\psi : [-r, 0] \to E \mid \psi & \text{ is continuous everywhere except for a finite number} \\ & \text{of points } \tilde{t} \text{ at which } \psi(\tilde{t}^-) \text{ and } \psi(\tilde{t}^+) \text{ exist and } \psi(\tilde{t}^-) = \psi(\tilde{t})\}, \end{aligned}$$

$\phi \in D$, $\mu_i \in E$, $0 \leq i \leq n-1$, $I_k^i \in C(E, E)$, $0 \leq i \leq n-1$, $1 \leq k \leq m$, and

$$\Delta y^{(i)}(t_k)) = y^{(i)}(t_k^+) - y^{(i)}(t_k^-),\ \ \ 0 \leq i \leq n-1.$$

As usual, for any continuous function y from $[-r, T] \setminus \{t_1, \ldots, t_m\}$ to E, and any $t \in J$, we define $y_t \in D$ by

$$y_t(\theta) = y(t + \theta),\ \ \ \theta \in [-r, 0].$$

We observe, in addition, that when $\mu_i = 0$, $0 \leq i \leq n-1$, the conditions (4) are periodic boundary conditions.

Impulsive differential equations have been used for some time to model evolution processes subject to abrupt changes in their state. The books by Bainov and Simeonov [1], Lakshmikantham, *et al.* [14], and Samoilenko and Perestyuk [15] give such models for space-craft control, inspection processes in operations research, drug administration, and threshold theory in biology.

More recently, there have been extensions concerning impulsive problems made to functional differential equations and inclusions. Some of these extensions for classes of nonresonance problems with convex nonlinearity can be found in Benchohra, *et al.* [2, 3], and Dong [9], in which coincidence degree theory or a Martelli fixed point theorem for multivalued maps were applied.

We consider the case when $\lambda \neq 0$. We observe that, if the impulses were absent (i.e., $I_k^i \equiv 0,\ 0 \leq i \leq n-1,\ 1 \leq k \leq m$), then the problem (1)–(4) is a *nonresonance problem* since the linear part in the equation (1) is invertible. In that light, this paper constitutes a generalization of Benchohra, Henderson, and Ntouyas [4], which dealt with (1)–(4) for $n = 1$ and $n = 2$. As in [4], the approach used to obtain our first result (Theorem 3.3 below) is based on a Covitz-Nadler [7] fixed point theorem for contraction multivalued maps. Our second main result (Theorem 3.8 below) makes use of a selection result of Bressan and Colombo [5] and Schaefer's fixed point theorem. The results in this paper allow the nonlinearity F to be nonconvex-valued.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Given a compact interval $I \subset \mathbb{R}$, $C(I, E)$ is the Banach space of all continuous functions from I to E with the norm

$$\|\phi\| = \sup\{|\phi(t)| : t \in I\},$$

and given an interval $H \subset \mathbb{R}$, we will let $AC^i(H, E)$ denote the space of i-times differentiable functions $y : H \to E$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous. Also, $L^1(H, E)$ denotes the Banach space of Bochner integrable functions $y : H \to E$. Let (X, d) be a metric space. We use the notations:

$$P(X) = \{Y \subset X : Y \neq \emptyset\}, \qquad P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\},$$

$$P_b(X) = \{Y \in P(X) : Y \text{ bounded }\}, \qquad P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}.$$

We define $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b),\ d(a, B) = \inf_{b\in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space (see [13]).

Definition 2.1 *A multivalued operator $N : X \to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \textit{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

A multivalued operator $N : X \to P_{cl}(X)$ has *a fixed point* if there is an $x \in X$ such that $x \in N(x)$. The set of fixed points of the multivalued operator N will be denoted by $FixN$. For more detailed works on multivalued maps, we cite the books of Deimling [8], Gorniewicz [11], Hu and Papageorgiou [12], and Smirnov [17].

Our first existence result for (1)–(4) will arise as an application of a Covitz-Nadler [7] fixed point theorem for multivalued mappings (see also Theorem 11.1 in Deimling [8]).

Theorem 2.2 *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

To set the framework for our second existence result, we need to introduce the following concepts. Let A be a subset of $J \times E$. We say that A is $\mathcal{L} \otimes \mathcal{B}$ *measurable* if A belongs to the σ-algebra generated by all sets of the form $\mathcal{N} \times \mathcal{D}$ where $\mathcal{N}$ is Lebesgue measurable in J and $\mathcal{D}$ is Borel measurable in E. A subset B of $L^1(J, E)$ is *decomposable* if, for all $u, v \in B$ and all measurable subsets $\mathcal{N}$ of J, the function $u\chi_{\mathcal{N}} + v\chi_{J-\mathcal{N}} \in B$, where χ denotes the characteristic function.

Let E be a Banach space, X be a nonempty closed subset of E, and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. Then G is *lower semi-continuous* (l.s.c.) if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set C in E.

Definition 2.3 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multivalued operator. We say N has property (BC) if*

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : J \times D \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : \Omega \to \mathcal{P}(L^1(J, E))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1(J, E) : w(t) \in F(t, y_t) \text{ for a.e. } t \in J\}$$

where Ω is an appropriately chosen Banach space. The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.4 *Let $F : J \times D \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continous and has nonempty closed and decomposable values.*

Finally, we state a selection theorem due to Bressan and Colombo.

Theorem 2.5 *[5]. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, T], E))$ be a multivalued operator which has property (BC). Then N has a continuous selection; i.e., there exists a continuous function (single-valued) $g : Y \to L^1(J, E)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

3 Nonresonance Higher Order Impulsive FDIs

In this section, we provide constraints on F and the impulse operators I_k^i so that (1)–(4) has a solution. This will be done by an application of Theorem 2.2.

Let $J_k = [t_k, t_{k+1}]$, $0 \leq k \leq m$, and given a function $y : [-r, T] \to E$, let y_k denote the restriction of y to J_k. We will seek a solution of (1)–(4) from a subset of the space,

$$\begin{aligned}\Omega : &= \Omega([-r, T]) \\ &= \{y : [-r, T] \longrightarrow E : y_k \in C(J_k, E), 0 \leq k \leq m, \text{ and both} \\ &\qquad y(t_k^-) \text{ and } y(t_k^+) \text{ exist, with } y(t_k^-) = y(t_k), 1 \leq k \leq m\},\end{aligned}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|,\ 0 \leq k \leq m\}.$$

In addition, for each $y \in \Omega$ we define the set

$$S_{F,y} = \Big\{v \in L^1(J, E) : v(t) \in F(t, y_t) \text{ for a.e. } t \in J\Big\}.$$

We next define what we mean by a solution of (1)–(4).

Definition 3.1 *A function $y \in \Omega \cap AC^{n-1}((t_k, t_{k+1}), E)$, $k = 0, \ldots, m$, is said to be a solution of (1)–(4), if y satisfies the conditions (1) to (4).*

In applying Theorem 2.2, we will define an operator whose resolvent kernel is the Green's function, $G(t, s)$, for the periodic boundary value problem,

$$y^{(n)}(t) - \lambda y(t) = 0,\ y^{(i)}(0) - y^{(i)}(T) = 0,\ 0 \leq i \leq n-1. \tag{5}$$

Among various properties of $G(t, s)$, we recall that

$$\frac{\partial^i}{\partial t^i} G(0, 0) - \frac{\partial^i}{\partial t^i} G(T, 0) = \begin{cases} 0, & 0 \leq i \leq n-2, \\ 1, & i = n-1. \end{cases}$$

The following result is fundamental is establishing solutions of (1)–(4). The proof is simply an extension of the result for second order problems given in [3], and so we omit the proof.

Lemma 3.2 *[3]. A function $y \in \Omega \cap AC^{n-1}((t_k, t_{k+1}), E)$, $k = 0, \ldots, m$, is a solution to the problem (1)–(4) if and only if $y \in \Omega$ and there exists $v \in S_{F,y}$ such that y satisfies the impulsive integral equation,*

$$y(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \displaystyle\int_0^T G(t, s) v(s) ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t, 0) \mu_{n-i-1} & \\ \displaystyle+ \sum_{k=1}^{m} \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t, t_k) I_k^i(y(t_k)), & t \in J. \end{cases} \tag{6}$$

We now establish the existence of solutions of (1)–(4).

Theorem 3.3 *Assume the following conditions are satisfied:*

(H1) $F : [0,T] \times D \longrightarrow P_{cp}(E)$ *has the property that, for each* $u \in D$, $F(\cdot, u) : [0,T] \to P_{cp}(E)$ *is measurable.*

(H2) There exists $l \in L^1([0,T], \mathbb{R})$ *such that* $H_d(F(t,u), F(t,\overline{u})) \leq l(t)\|u - \overline{u}\|$, *for each* $t \in [0,T]$ *and* $u, \overline{u} \in D$ *and* $d(0, F(t,0)) \leq l(t)$, *for all* $t \in J$.

(H3) For each $0 \leq i \leq n-1$, $1 \leq k \leq m$, *there exist constants* $d_k^i \geq 0$, *such that* $|I_k^i(y) - I_k^i(\overline{y})| \leq d_k^i |y - \overline{y}|$, *for each* $y, \overline{y} \in E$.

Let $l^* = \int_0^T l(t)dt$ *and* $M_i = \sup_{(t,s)\in J\times J} \left|\frac{\partial^i}{\partial t^i} G(t,s)\right|$, $0 \leq i \leq n-1$. *If*

$$\left[M_0 l^* + \sum_{i=0}^{n-1} M_i \left(\sum_{k=1}^{m} d_k^i\right)\right] < 1,$$

then the problem (1)-(4) has at least one solution on $[-r, T]$.

Proof. In order to apply the Covitz-Nadler fixed point theorem, that is, Theorem 2.2, we define a multivalued operator $N : \Omega \to P(\Omega)$ by

$$N(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \displaystyle\int_0^T G(t,s)v(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} & \\ \displaystyle +\sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)), & t \in J, \end{cases} \right\}$$

where $v \in S_{F,y}$. It is straightforward that fixed points of N are solutions of (1)–(4). In addition, by (H1), F has a measurable selection from which Castaing and Valadier (see Theorem III in [6]) have proved that, for each $y \in \Omega$, the set $S_{F,y}$ is nonempty.

We will now verify that N satisfies the conditions of Theorem 2.2; this will be done in a couple of steps.

Our first step is to show that, for each $y \in \Omega$, we have $N(y) \in P_{cl}(\Omega)$. Indeed, let $(y_n)_{n\geq 0} \in N(y)$ be such that $y_n \longrightarrow \tilde{y}$ in Ω. Then $\tilde{y} \in \Omega$, and there exists $g_n \in S_{F,y}$ such that for each $t \in J$,

$$y_n(t) \in \int_0^T G(t,s)g_n(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k^-)).$$

Using the fact that F has compact values and (H2) holds, we may pass to a subsequence if necessary to obtain that g_n converges to g in $L^1(J,E)$, and hence $g \in S_{F,y}$. Then, for each $t \in [0,b]$,

$$y_n(t) \to \tilde{y}(t) \in \int_0^T G(t,s)g(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k^-)).$$

So $\tilde{y} \in N(y)$, and in particular, $N(y) \in P_{cl}(\Omega)$.

Our second step is to show there exists a $0 \le \gamma < 1$ such that $H_d(N(y), N(\overline{y})) \le \gamma\|y - \overline{y}\|$ for each $y, \overline{y} \in \Omega$. To this end, let $y, \overline{y} \in \Omega$ and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t, y_t)$ such that for each $t \in J$,

$$h_1(t) = \int_0^T G(t,s)v_1(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k^-)).$$

From (H2) it follows that, for $t \in J$,

$$H_d(F(t,y_t), F(t,\overline{y}_t)) \le l(t)\|y_t - \overline{y}_t\|.$$

Hence, there is $w \in F(t, \overline{y}_t)$ such that

$$|v_1(t) - w| \le l(t)\|y_t - \overline{y}_t\|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(E)$ defined by

$$U(t) = \{w \in E : |v_1(t) - w| \le l(t)\|y_t - \overline{y}_t\|\}.$$

By a result in Castaing and Valadier (see Proposition III.4 in [6]), the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable, and hence there exists a measurable selection for V, call it $v_2(t)$. Now $v_2(t) \in F(t, \overline{y}_t)$ and

$$|v_1(t) - v_2(t)| \le l(t)\|y - \overline{y}\|, \quad t \in J.$$

For each $t \in J$, we define

$$h_2(t) = \int_0^T G(t,s)v_2(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(\overline{y}(t_k^-)).$$

Then, for $t \in J$, we have

$$|h_1(t) - h_2(t)| \le \int_0^T |G(t,s)||v_1(s) - v_2(s)|\, ds$$

$$
\begin{aligned}
& +\sum_{k=1}^{m}\sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right| |I_k^i(y(t_k^-)) - I_k^i(\overline{y}(t_k^-))| \\
\leq\ & M_0\int_0^T l(s)\|y_s - \overline{y}_s\|ds \\
& +\sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i |y(t_k^-) - \overline{y}(t_k^-)| \\
\leq\ & \left[M_0 l^* + \sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i\right] \|y-\overline{y}\|_\Omega.
\end{aligned}
$$

Thus,

$$
\|h_1 - h_2\|_\Omega \leq \left[M_0 l^* + \sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i\right] \|y-\overline{y}\|_\Omega.
$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$
H_d(N(y), N(\overline{y})) \leq \left[M_0 l^* + \sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i\right] \|y-\overline{y}\|_\Omega.
$$

Therefore, N is a contraction and so by Theorem 2.2, N has a fixed point y, which is a solution to (1)-(4). This completes the proof of the theorem.

Using Schaefer's fixed point theorem combined with the selection theorem of Bressan and Colombo for lower semi-continuous maps with decomposable values, we will next present our second existence result for the problem (1)-(4). We will make use of the following conditions.

(A1) $F : [0,T]\times D \longrightarrow \mathcal{P}(E)$ is a nonempty, compact-valued, multivalued map such that:
a) $(t,u)\mapsto F(t,u)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable;
b) $u \mapsto F(t,u)$ is lower semi-continuous for a.e. $t\in[0,T]$.

(A2) For each $q>0$, there exists a function $h_q \in L^1([0,T], \mathbb{R}^+)$ such that

$$
\|F(t,u)\| := \sup\{|v| : v\in F(t,u)\} \leq h_q(t)
$$

for a.e. $t\in[0,T]$ and $u\in D$ with $\|u\|\leq q$.

The following lemma is crucial in the proof of our main theorem:

Lemma 3.4 *[10]. Let $F : [0,T]\times D \to \mathcal{P}(E)$ be a multivalued map with nonempty, compact values. Assume (A1) and (A2) hold. Then F is of l.s.c. type.*

The following result gives sufficient conditions for the existence of solutions to the problem (1)-(4).

Theorem 3.5 *Suppose that hypotheses (A1) and (A2) and the following are satisfied:*

(A3) For each $0 \leq i \leq n-1,\ 1 \leq k \leq m$, *there exist constants* $d_k^i \geq 0$, *such that* $|I_k^i(y)| \leq d_k^i$, *for each* $y \in \mathbb{R}$;

(A4) There exists $M \in L^1(J, \mathbb{R})$ *such that, for all* $y \in \Omega$ *and almost all* $t \in J$,

$$\|F(t, y_t)\| = \sup\{|v| : v \in F(t, y_t)\} \leq M(t).$$

(A5) For each $t \in J$, *the multivalued map* $F(t, \cdot) : D \to \mathcal{P}(E)$ *maps bounded sets into relatively compact sets.*

Then the problem (1)–(4) has at least one solution on $[-r, T]$.

Proof. First note that (A1), (A2), and Lemma 3.4 imply that F is of lower semi-continuous type. Then, from Theorem 2.5, there exists a continuous function $f : \Omega \to L^1([0, T], E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. Consider the problem,

$$y'(t) = f(y_t), \quad t \in [0, T],\ t \neq t_k, \quad k = 1, \ldots, m, \tag{7}$$

$$\Delta y^{(i)}(t_k) = I_k^i(y(t_k^-)),\ 0 \leq i \leq n-1, \quad 1 \leq k \leq m, \tag{8}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{9}$$

$$y^{(i)}(0) - y^{(i)}(T) = \mu_i, \quad 0 \leq i \leq n-1. \tag{10}$$

It is clear that if $y \in \Omega$ is a solution of the problem (7)–(10), then y is a solution to the problem (1)-(4).

We transform the problem (7)–(10) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by:

$$N(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \displaystyle\int_0^T G(t, s) f(y_s) ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t, 0) \mu_{n-i-1} & \\ \displaystyle + \sum_{k=1}^{m} \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t, t_k) I_k^i(y(t_k)), & t \in J. \end{cases}$$

We will show that N is a completely continuous, that is, it is continuous and sends bounded sets into relatively compact sets.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in Ω. Then,

$$\begin{aligned}|N(y_n(t)) - N(y(t))| &\leq \int_0^T |G(t,s)||f(y_{ns}) - f(y_s)|ds \\ &+ \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i}|G(t,t_k)||I_k^i(y_n(t_k)) - I_k^i(y(t_k))|.\end{aligned}$$

Since the functions f and $I_k,\ k = 1,\ldots,m$, are continuous,

$$\|N(y_n) - N(y)\|_\Omega \to 0 \ \ \textit{as } n \to \infty.$$

Step 2: *N maps bounded sets into bounded sets in Ω.*

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$, we have $\|N(y)\|_\Omega \leq \ell$. From (A1)-(A2), for each $t \in J$, we have

$$\begin{aligned}|h(t)| &\leq \int_0^T |G(t,s)||f(y_s)|ds + \sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right||\mu_{n-i-1}| \\ &+ \sum_{k=1}^{m}\sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y(t_k))\right| \\ &\leq \int_0^T |G(t,s)|h_q(s)ds + \sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right||\mu_{n-i-1}| \\ &+ \sum_{k=1}^{m}\sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right|\sup\{|I_k^i(|y|)| : \|y\|_\Omega \leq q\}.\end{aligned}$$

Then, for each $h \in N(B_q)$, we have

$$\begin{aligned}\|h\|_\Omega &\leq \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^T h_q(s)ds + \sum_{i=0}^{n-1}|\mu_{n-i-1}|\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right| \\ &+ \sum_{k=1}^{m}\sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right|\sup\{|I_k^i(|y|)| : \|y\|_\Omega \leq q\} \\ &:= \ell.\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets in Ω.*

Let $\tau_1, \tau_2 \in J,\ \tau_1 < \tau_2$ and B_q be a bounded set (as described above) in Ω. Let

$y \in B_q$. Then,

$$\begin{aligned} |h(\tau_2)-h(\tau_1)| &\leq \int_0^T |G(\tau_2,s)-G(\tau_1,s)|h_q(s)ds \\ &+ \sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(\tau_2,0)-\frac{\partial^i}{\partial t^i}G(\tau_1,0)\right| |\mu_{n-i-1}| \\ &+ \sum_{k=1}^{m}\sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(\tau_2,t_k)-\frac{\partial^i}{\partial t^i}G(\tau_1,t_k)\right| d_k^i. \end{aligned}$$

If we let $\tau_2 \longrightarrow \tau_1$ in the above inequality, the right hand side tends to zero. Also, the equicontinuity for the other cases, $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$, are straightforward.

As a consequence of Steps 1 to 3, condition (A5), and the Arzela-Ascoli theorem, we conclude that $N : \Omega \longrightarrow \Omega$ is completely continuous.

Step 4: *It remains to show that the set*

$$\mathcal{E}(N) := \{y \in \Omega : y = \beta N(y), \quad \textit{for some} \quad 0 < \beta < 1\}$$

is bounded.

Choose $y \in \mathcal{E}(N)$; then $y = \beta N(y)$, for some $0 < \beta < 1$, and thus, for each $t \in J$,

$$y(t) = \beta\left[\int_0^T G(t,s)f(y_s)ds + \sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y(t_k))\right].$$

By (A3) and (A4), we have

$$\begin{aligned} |y(t)| &\leq \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^T M(s)ds + \sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right| |\mu_{n-i-1}| \\ &+ \sum_{k=1}^{m}\sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right| d_k^i, \\ &:= b, \end{aligned}$$

where b depends only on T and the function M. In particular, $\|y\|_\Omega \leq b$, and $\mathcal{E}(N)$ is bounded.

With $X := \Omega$, we conclude by Schaefer's theorem (see [16], p. 29) that N has a fixed point which in turn is a solution of (1)–(4). This completes the proof of the theorem.

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