Some Stability and Boundedness Criteria for a Class of Volterra Integro-differential Systems

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Abstract

Using Lyapunov and Lyapunov-like functionals, we study the stability and boundedness of the solutions of a system of Volterra integro-differential equations. Our results, also extending some of the more well-known criteria, give new sufficient conditions for stability of the zero solution of the nonperturbed system, and prove that the same conditions for the perturbed system yield boundedness when the perturbation is L^2.

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1 Introduction

We consider the stability and boundedness of solutions of systems of Volterra integro-differential equations, with forcing functions, of the form

$$\frac{d}{dt}[\mathbf{x}(t)] = \mathbf{A}(t)\mathbf{f}(\mathbf{x}(t)) + \int_0^t \mathbf{B}(t,s)\mathbf{g}(\mathbf{x}(s))ds + \mathbf{h}(t)\,, \tag{1}$$

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in which $\mathbf{A}(t)$ is an $n \times n$ matrix function continuous on $[0, \infty)$, $\mathbf{B}(t, s)$ is an $n \times n$ matrix continuous for $0 \leq s \leq t < \infty$, $\mathbf{f}$ and $\mathbf{g}$ are $n \times 1$ vector functions continuous on $(-\infty, \infty)$ and $\mathbf{h}$ is an $n \times 1$ vector function defined almost everywhere on $[0, \infty)$. Here, $\mathbf{h}(t)$ represent the forcing functions or external disturbances or inputs into system (1).

The qualitative behaviour of the solutions of systems of Volterra integro-differential equations, especially the case where $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) = \mathbf{x}$ and $\mathbf{h}(t) = \mathbf{0}$, has been thoroughly analyzed by many researchers. Among the contributions in the 1980s, those of Burton are worthy of mention. His work ([1], [2]) laid the foundation for a systematic treatment of the basic structure and stability properties of Volterra integro-differential equations, mainly, via the direct method of Lyapunov. This paper essentially looks into some of the many interesting results established by Burton and proposes ways of utilizing the form of the Lyapunov functionals proposed by Burton to construct new or similar ones for system (1).

Now, if $\mathbf{f}(\mathbf{0}) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{h}(t) = \mathbf{0}$, then system (1) reduces to

$$\frac{d}{dt}[\mathbf{x}(t)] = \mathbf{A}(t)\mathbf{f}(\mathbf{x}(t)) + \int_0^t \mathbf{B}(t, s)\mathbf{g}(\mathbf{x}(s))ds\,, \tag{2}$$

so that $\mathbf{x}(t) \equiv \mathbf{0}$ is a solution of (2) called the *zero solution*. The initial conditions for integral equations such as (1) or (2) involve continuous *initial functions* on an *initial interval*, say, $\mathbf{x}(t) = \phi(t)$ for $0 \leq t \leq t_0$. Hence, $\mathbf{x}(t; t_0, \phi)$, $t \geq t_0 \geq 0$ denotes the solution of (1) or (2), with the initial function $\phi : [0, t_0] \to \mathbf{R}^n$ assumed to be bounded and continuous on $[0, t_0]$.

The definitions of the stability and the boundedness of solutions of (1) are given in Burton [1]. It is assumed that the functions in (1) are well-behaved, that continuous initial functions generate solutions, and that solutions which remain bounded can be continued.

2 The Scalar Equation

2.1 Nonperturbed Case

Consider the scalar equation

$$x'(t) = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds\,. \tag{3}$$

We suppose that

$$A(t) \text{ is continuous for } 0 \le t < \infty; \tag{4}$$

$$B(t,s) \text{ is continuous for } 0 \le s \le t < \infty; \tag{5}$$

$$\int_0^t |B(u,s)|du \text{ is defined and continuous for } 0 \le s \le t < \infty; \tag{6}$$

$$f(x) \text{ and } g(x) \text{ are continuous on } (-\infty,\infty); \tag{7}$$

$$xf(x) > 0 \;\; \forall x \ne 0\,, \text{ and } f(0) = g(0) = 0\,. \tag{8}$$

For comparison sake, we first state Burton's theorem regarding the stability of the zero solution of (3).

Theorem 1 (Burton [3]). *Let (4)–(8) hold and suppose there are constants $m > 0$ and $M > 0$ such that $g^2(x) \le m^2 f^2(x)$ if $| x |\le M$. Define*

$$\beta(t,k) = A(t) + k\int_t^\infty |B(u,t)|du + \frac{1}{2}\int_0^t |B(t,s)|ds$$

If there exists $k > 0$ with $m^2 < 2k$ and $\beta(t,k) \le 0$ for $t \ge 0$, then the zero solution of (3) is stable.

We next state an extension of Theorem 1, which Burton proved via the Lyapunov functional

$$V_1(t,x(\cdot)) = \int_0^x f(s)ds + k\int_0^t \int_t^\infty |B(u,s)|du f^2(x(s))ds\,, \tag{9}$$

the time-derivative along a trajectory of (3) of which is,

$$V'_{1(3)} \le \beta(t,k)f^2(x) - (2k - m^2)\int_0^t |B(t,s)|f^2(x(s))ds \le \beta(t,k)f^2(x)\,.$$

In the process, and motivated by the work of Miyagi et. al in the construction of generalized Lyapunov functions for power systems [4] and single-machine systems [5], we propose a new Lyapunov functional. As a simple example will show, the new stability criterion may be used in situations where Theorem 1 cannot be applied.

Theorem 2. *Let (4)–(8) hold, with $A(t) < 0$, and suppose there are constants*

$$m > 0 \text{ and } M > 0 \text{ such that } g^2(x) \leq m^2 f^2(x) \text{ if } |x| \leq M\,, \tag{10}$$

$$\alpha > 4 \text{ and } N > 0 \text{ such that } 4x^2 \leq (\alpha - 4) f^2(x) \text{ if } |x| \leq N\,, \text{ and} \tag{11}$$

$$J \geq 1 \text{ such that } \quad -\frac{1}{4A(t)} \int_0^t |B(t,s)|ds < \frac{1}{J} \quad \text{for every} \quad t \geq 0\,. \tag{12}$$

Suppose further there is some constant $k > 0$ such that

$$\frac{(1+\alpha)m^2}{J} < k\,, \tag{13}$$

and

$$A(t) + k\int_t^\infty |B(u,t)|du \leq 0 \tag{14}$$

for $t \geq 0$. Then the zero solution of (3) is stable.

Proof. Consider the functional

$$\begin{aligned} V_2(t, x(\cdot)) &= \frac{1}{2}x^2 + \sqrt{\alpha}\int_0^x \sqrt{uf(u)}\,du + \frac{1}{2}\alpha\int_0^x f(u)du \\ &\quad + k\int_0^t \int_t^\infty |B(u,s)|du f^2(x(s))ds\,. \end{aligned}$$

We have, along a trajectory of (3),

$$\begin{aligned} V'_{2(3)} &= x\,x' + \sqrt{\alpha}\,\sqrt{xf(x)}\,x' + \frac{1}{2}\alpha f(x)\,x' \\ &\quad + \frac{d}{dt}\left[k\int_0^t \int_t^\infty |B(u,s)|du f^2(x(s))ds\right]. \end{aligned}$$

Recalling that $A(t) < 0$ for all $t \geq 0$ and noting that the Schwarz inequality yields,

$$\left(\int_0^t B(t,s)g(x(s))ds\right)^2 \leq \int_0^t |B(t,s)|ds \int_0^t |B(t,s)|g^2(x(s))ds,$$

we have,

$$
\begin{aligned}
x\,x' &= A(t)xf(x) + x\int_0^t B(t,s)g(x(s))ds \\
&= A(t)xf(x) - \left(\sqrt{-A(t)}\,x - \frac{1}{2\sqrt{-A(t)}}\int_0^t B(t,s)g(x(s))ds\right)^2 \\
&\quad -A(t)x^2 - \frac{1}{4A(t)}\left(\int_0^t B(t,s)g(x(s))ds\right)^2 \\
&\le A(t)xf(x) - \frac{1}{4}(\alpha-4)A(t)f^2(x) \\
&\quad -\frac{1}{4A(t)}\int_0^t |B(t,s)|ds\int_0^t |B(t,s)|g^2(x(s))ds \\
&\le A(t)xf(x) - \frac{1}{4}(\alpha-4)A(t)f^2(x) + \frac{m^2}{J}\int_0^t |B(t,s)|f^2(x(s))ds \\
&= A(t)xf(x) - \frac{1}{4}\alpha A(t)f^2(x) + A(t)f^2(x) + \frac{m^2}{J}\int_0^t |B(t,s)|f^2(x(s))ds\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{\alpha}\,\sqrt{xf(x)}\,x' &= -\left(\frac{\sqrt{\alpha}}{2\sqrt{-A(t)}}\,x' - \sqrt{-A(t)}\,\sqrt{xf(x)}\right)^2 \\
&\quad -A(t)xf(x) - \frac{\alpha}{4A(t)}\,(x')^2 \\
&\le -A(t)xf(x) - \frac{1}{4}\alpha A(t)f^2(x) - \frac{1}{2}\alpha f(x)\int_0^t B(t,s)g(x(s))ds \\
&\quad -\frac{\alpha}{4A(t)}\left(\int_0^t B(t,s)g(x(s))ds\right)^2 \\
&\le -A(t)xf(x) - \frac{1}{4}\alpha A(t)f^2(x) - \frac{1}{2}\alpha f(x)\int_0^t B(t,s)g(x(s))ds \\
&\quad +\frac{m^2\alpha}{J}\int_0^t |B(t,s)|f^2(x(s))ds\,.
\end{aligned}
$$

The third and fourth terms of $V'_{2(3)}$ yield

$$
\frac{1}{2}\alpha f(x)\,x' = \frac{1}{2}\alpha A(t)f^2(x) + \frac{1}{2}\alpha f(x)\int_0^t B(t,s)g(x(s))ds\,,
$$

and

$$\frac{d}{dt}\left[k\int_0^t\int_t^\infty |B(u,s)|du f^2(x(s))ds\right] = k\int_t^\infty |B(u,t)|du f^2(x) - k\int_0^t |B(t,s)|f^2(x(s))ds\,,$$

respectively.

Thus,

$$V'_{2(3)} \le \left[A(t)+k\int_t^\infty |B(u,t)|du\right]f^2(x) - \left[k-\frac{m^2(1+\alpha)}{J}\right]\int_0^t |B(t,s)|f^2(x(s))ds\,,$$

which will be nonpositive if equations (13) and (14) are satisfied.

Finally, to prove the positive definiteness of V_2, we see that if we define

$$r(u)=\begin{cases}\left(\sqrt{u}+\frac{1}{2}\sqrt{\alpha}\sqrt{f(u)}\right)^2\,, & u\ge 0\,,\\ -\left(\sqrt{-u}-\frac{1}{2}\sqrt{\alpha}\sqrt{-f(u)}\right)^2\,, & u<0\,,\end{cases}$$

then we can rewrite V_2 as

$$V_2(t,x(\cdot)) = \int_0^x r(u)du+\frac{1}{4}\alpha\int_0^x f(u)du + k\int_0^t\int_t^\infty |B(u,s)|du f^2(x(s))ds\,,$$

which is clearly positive definite given that $ur(u)>0$ for $u\ne 0$.

This completes the proof of Theorem 2. □

Thus, we have proposed an alternate stability criterion for the scalar equation (3), and the criterion may be considered for cases where Burton's Theorem 1, though simpler, cannot be applied.

Example 1. For the equation

$$x' = -x+\int_0^t \frac{1}{(1+t-s)^2}\left[x^2(s)+\frac{1}{2}x(s)\right]ds\,,$$

both Theorems 1 and 2 establish the stability of the zero solution. To see this, Theorem 1 yields $M = m - 1/2$ so that $|x| \leq m - 1/2$. Also, we have $\beta(t,k) < -1 + k + 1/2 \leq 0$ so that $0 < k \leq 1/2$. From $m^2 < 2k$, we have, if we choose $k = 1/2$, the inequality $0 < m < 1$. Thus, we may choose $1/2 < m < 1$ to satisfy all conditions of Theorem 1. Theorem 2 yields, from (10), $M = m - 1/2$. From (11), $\alpha \geq 8$ and $N = \infty$. From (12), $4 > J$. From (13), $0 < 9m^2/4 < k$ if we pick $\alpha = 8$. From (14), $0 < k \leq 1$. Choose $k = 1$. Then $0 < m < 2/3$. Thus, we may choose $1/2 < m < 2/3$ to satisfy all conditions of Theorem 2.

□

Example 2. Analysis via Theorem 2 shows that the zero solution of

$$x' = -x + \int_0^t e^{-(t-s)/2}\left[x^2(s) + \frac{1}{4}x(s)\right] ds\,,$$

is stable. That is, Theorem 2 yields, from (10), $M = m - 1/4$. From (11), $\alpha \geq 8$ and $N = \infty$. From (12), $2 > J$. From (13), $0 < 9m^2/2 < k$ if we pick $\alpha = 8$. From (14), $0 < k \leq 1/2$. Choose $k = 1/2$. Then $0 < m < 1/3$. Thus, we may choose $1/4 < m < 1/3$ to satisfy all conditions of the theorem.

Theorem 1 is not applicable.

□

2.2 Perturbed Case

The next two results, which extend Theorem 1 and Theorem 2, give a class of forcing functions that maintains the boundedness of the solutions of the equation

$$x'(t) = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds + h(t)\,, \tag{15}$$

where $h : [0,\infty) \to \mathbf{R}$ is defined almost everywhere on $[0,\infty)$.

Theorem 3. *Let (4)–(8) hold and suppose there is a constant $m > 0$ such that $g^2(x) \leq m^2\, f^2(x)$ for all $x \in \mathbf{R}$. Define*

$$\beta(t,k) = A(t) + k\int_t^\infty |B(u,t)|du + \frac{1}{2}\int_0^t |B(t,s)|ds$$

and let there be constants $\rho > 0$ *and* $k > 0$ *such that* $m^2 < 2k$ *and* $\beta(t,k) \leq -\rho$ *for* $t \geq 0$*. If*

$$\int_0^x f(x)dx \to \infty \ \text{as} \ |x| \to \infty\,, \tag{16}$$

and

$$h(\cdot) \in L^2[0,\infty)\,,$$

then all solutions of (15) are bounded.

Proof. Let $\epsilon > 0$ and consider the functional

$$V_3(t, x(\cdot)) = V_1(t, x(\cdot)) + \frac{1}{4\epsilon}\int_t^\infty h^2(u)du\,.$$

Since

$$\int_0^\infty [h(u)]^2 du < \infty\,,$$

we have,

$$\frac{d}{dt}\left[\int_t^\infty h^2(u)du\right] = \frac{d}{dt}\left[\int_0^\infty h^2(u)du - \int_0^t h^2(u)du\right] = -h^2(t)\,,$$

implying, therefore, the differentiability and hence the existence on $[0,\infty)$ of the second term of the functional V_3. Thus, we have

$$\begin{aligned}
V'_{3(15)} &\leq \beta(t,k)f^2(x) + f(x)h(t) - \frac{1}{4\epsilon}h^2(t) \\
&\leq -\rho f^2(x) + \epsilon f^2(x) + \frac{1}{4\epsilon}h^2(t) - \frac{1}{4\epsilon}h^2(t) \\
&= -(\rho - \epsilon)f^2(x)\,.
\end{aligned}$$

This completes the proof of Theorem 3 since we can always find some $\epsilon > 0$ small enough such that $(\rho - \epsilon) > 0$. Note that (16) ensures the radial unboundedness of V_3. □

In the same fashion, we prove the following extension of Theorem 2.

Theorem 4. *Let (4)–(8) hold, with* $A(t) < 0$*, and suppose there are constants*

$m > 0$ *such that* $g^2(x) \leq m^2\, f^2(x)$ *for all* $x \in \mathbf{R}$,

$\alpha > 4$ *such that* $4x^2 \leq (\alpha - 4)\, f^2(x)$ *for all* $x \in \mathbf{R}$, *and*

$J \geq 1$ *such that* $\quad -\dfrac{1}{4A(t)}\displaystyle\int_0^t |B(t,s)|ds < \frac{1}{J}$ *for every* $t \geq 0$.

Further, suppose there are constants $k > 0$ and $\rho > 0$ such that

$$\frac{(1+\alpha)m^2}{J} < k\,,$$

and

$$A(t) + k\int_t^{\infty} |B(u,t)|du \leq -\rho\,,$$

for all $t \geq 0$. If $h(\cdot) \in L^2[0,\infty)$, then all solutions of (15) are bounded.

Proof. For $\epsilon > 0$, the functional

$$V_4(t,x(\cdot)) = V_2(t,x(\cdot)) + \frac{1}{4\epsilon}\int_t^{\infty} h^2(u)du\,,$$

yields, given the definition,

$$\tau = \left(\sqrt{\frac{\sqrt{\alpha-4}}{2}} + \sqrt{\alpha}\right)^2,$$

the time-derivative,

$$\begin{aligned}
V'_{4(15)} &\leq \left[A(t) + k\int_t^{\infty}|B(u,t)|du\right] f^2(x) \\
&\quad + \left[x + \sqrt{\alpha}\sqrt{xf(x)} + \frac{1}{2}\alpha f(x)\right] h(t) - \frac{1}{4\epsilon}h^2(t) \\
&\leq -\rho f^2(x) + \left[\sqrt{|x|} + \sqrt{\alpha}\sqrt{|f(x)|}\right]^2 h(t) - \frac{1}{4\epsilon}h^2(t) \\
&\leq -\rho f^2(x) + \left[\sqrt{\frac{\sqrt{\alpha-4}}{2}}\sqrt{|f(x)|} + \sqrt{\alpha}\sqrt{|f(x)|}\right]^2 h(t) - \frac{1}{4\epsilon}h^2(t) \\
&= -\rho f^2(x) + \tau|f(x)|h(t) - \frac{1}{4\epsilon}h^2(t) \\
&\leq -\rho f^2(x) + \tau^2\epsilon f^2(x) + \frac{1}{4\epsilon}h^2(t) - \frac{1}{4\epsilon}h^2(t) \\
&= -\left[\rho - \tau^2\epsilon\right] f^2(x)\,.
\end{aligned}$$

This completes the proof of Theorem 4 since we can always find some $\epsilon > 0$ small enough such that $(\rho - \tau^2\epsilon) > 0$. We note that $V_4 \to \infty$ if $|x| \to \infty$. $\square$

3 The Vector Equation

3.1 Nonperturbed Case

First, we consider the nonperturbed system (2). If we suppose that $\mathbf{f}, \mathbf{g} \in C^1[\mathbf{R}^n, \mathbf{R}^n]$, then we can define

$$\mathbf{D}(\mathbf{x}) = [d_{ij}(\mathbf{x})]_{n\times n} \quad \text{with} \quad d_{ij}(\mathbf{x}) = \int_0^1 \frac{\partial f_i(u\mathbf{x})}{\partial(ux_j)} du\,,$$

and

$$\mathbf{E}(\mathbf{x}) = [e_{ij}(\mathbf{x})]_{n\times n} \quad \text{with} \quad e_{ij}(\mathbf{x}) = \int_0^1 \frac{\partial g_i(u\mathbf{x})}{\partial(ux_j)} du\,,$$

which are defined for all $\mathbf{x} \in \mathbf{R}^n$. Hence, assuming $\mathbf{f}(\mathbf{0}) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$, system (2) can be written as

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{D}(\mathbf{x}(t))\mathbf{x}(t) + \int_0^t \mathbf{B}(t,s)\mathbf{E}(\mathbf{x}(s))\mathbf{x}(s)\, ds\,, \tag{17}$$

the i-th component of which is

$$\begin{aligned} x_i'(t) \;&=\; a_{ii}(t)\left[d_{ii}(\mathbf{x})x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}(\mathbf{x})x_j\right] \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)\left[d_{ji}(\mathbf{x})x_i + \sum_{\substack{k=1\\ k\neq i}}^{n} d_{jk}(\mathbf{x})x_k\right] \\ &\quad + \sum_{k=1}^{n}\int_0^t \left[b_{ii}(t,s)e_{ik}(\mathbf{x}(s)) + \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}(t,s)e_{jk}(\mathbf{x}(s))\right] x_k(s)ds\,, \end{aligned}$$

noting that in the above equation $d_{ij}(\mathbf{x})x_j$ and $e_{ij}(\mathbf{x})x_j$, for $i,j = 1,\ldots,n$, are continuously differentiable with respect to $\mathbf{x} \in \mathbf{R}^n$ simply for the reason that $\mathbf{D}(\mathbf{x})\mathbf{x} = \mathbf{f}(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{g}(\mathbf{x})$ with $\mathbf{f}, \mathbf{g} \in C^1[\mathbf{R}^n, \mathbf{R}^n]$.

The next result is new.

Theorem 5. *Let* $\mathbf{f}, \mathbf{g} \in C^1[\mathbf{R}^n, \mathbf{R}^n]$, $\mathbf{f}(\mathbf{0}) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$ *and*

$$\beta_i(t,\mathbf{x}) = a_{ii}(t)d_{ii}(\mathbf{x}) + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)d_{ji}(\mathbf{x})$$

$$+\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[|a_{ii}(t)d_{ij}(\mathbf{x})| + |a_{jj}(t)d_{ji}(\mathbf{x})| + |a_{ji}(t)d_{ii}(\mathbf{x})| + |a_{ij}(t)d_{jj}(\mathbf{x})|\right]$$

$$+\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\left[|a_{ik}(t)d_{kj}(\mathbf{x})| + |a_{jk}(t)d_{ki}(\mathbf{x})|\right]$$

$$+\frac{1}{2}\sum_{k=1}^{n}\int_t^{\infty}\left[|b_{kk}(u,t)|e_{ki}^2(\mathbf{x}) + \sum_{\substack{j=1\\ j\neq i}}^{n}|b_{kj}(u,t)|e_{ji}^2(\mathbf{x})\right]du$$

$$+\frac{n}{2}\int_0^t\left[|b_{ii}(t,s)| + \sum_{\substack{j=1\\ j\neq i}}^{n}|b_{ij}(t,s)|\right]ds\,. \tag{18}$$

Suppose $\beta_i(t,\mathbf{x}) \leq 0$ *for* $i = 1,\ldots,n$, $t \geq 0$ *and* $\mathbf{x} \in \mathbf{R}^n$. *Then the zero solution of system (2) is stable.*

Proof. Consider the functional

$$V_5(t,\mathbf{x}(\cdot)) = \frac{1}{2}\sum_{i=1}^{n}x_i^2(t)$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_0^t\int_t^{\infty}\left[|b_{ii}(u,s)|e_{ik}^2(\mathbf{x}(s)) + \sum_{\substack{j=1\\ j\neq i}}^{n}|b_{ij}(u,s)|e_{jk}^2(\mathbf{x}(s))\right]du x_k^2(s)ds\,.$$

Now,

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\frac{d}{dt}\left[x_i^2/2\right]_{(2)} = \sum_{i=1}^{n} x_i x_i' \\
= \;& \sum_{i=1}^{n} x_i \Bigg\{ a_{ii}(t)\Bigg[d_{ii}(\mathbf{x})x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}(\mathbf{x})x_j \Bigg] \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)\Bigg[d_{ji}(\mathbf{x})x_i + \sum_{\substack{k=1\\ k\neq i}}^{n} d_{jk}(\mathbf{x})x_k \Bigg] \\
&+ \sum_{k=1}^{n}\int_0^t \Bigg[b_{ii}(t,s)e_{ik}(\mathbf{x}(s)) + \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}(t,s)e_{jk}(\mathbf{x}(s)) \Bigg] x_k(s)ds \Bigg\} \\
\leq \;& \sum_{i=1}^{n}\Bigg\{ a_{ii}(t)d_{ii}(\mathbf{x}) + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)d_{ji}(\mathbf{x}) \Bigg\} x_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ii}(t)d_{ij}(\mathbf{x})x_j x_i \\
&+ \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{k=1\\ k\neq i}}^{n} a_{ij}(t)d_{jk}(\mathbf{x})x_k x_i \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_0^t |b_{ii}(t,s)|[e_{ik}^2(\mathbf{x}(s))x_k^2(s) + x_i^2]ds \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\int_0^t |b_{ij}(t,s)|[e_{jk}^2(\mathbf{x}(s))x_k^2(s) + x_i^2]ds \\
= \;& \sum_{i=1}^{n}\Bigg\{ a_{ii}(t)d_{ii}(\mathbf{x}) + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)d_{ji}(\mathbf{x}) + \frac{n}{2}\int_0^t \Bigg[|b_{ii}(t,s)| + \sum_{\substack{j=1\\ j\neq i}}^{n} |b_{ij}(t,s)| \Bigg] ds \Bigg\} x_i^2 \\
&+ \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ii}(t)d_{ij}(\mathbf{x})x_j x_i + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{k=1\\ k\neq i}}^{n} a_{ij}(t)d_{jk}(\mathbf{x})x_k x_i \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_0^t \Bigg[|b_{ii}(t,s)|e_{ik}^2(\mathbf{x}(s)) + \sum_{\substack{j=1\\ j\neq i}}^{n} |b_{ij}(t,s)|e_{jk}^2(\mathbf{x}(s)) \Bigg] x_k^2(s)ds\,,
\end{aligned}
$$

where

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} a_{ii}(t)d_{ij}(\mathbf{x})x_jx_i \quad\leq\quad \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\left[|a_{ii}(t)d_{ij}(\mathbf{x})|+|a_{jj}(t)d_{ji}(\mathbf{x})|\right]x_i^2\,,$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\k\neq i}}^{n} a_{ij}(t)d_{jk}(\mathbf{x})x_kx_i\\
=\;& \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} a_{ji}(t)d_{ii}(\mathbf{x})x_jx_i+\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\k\neq i\\k\neq j}}^{n} a_{ik}(t)d_{kj}(\mathbf{x})x_jx_i\\
\leq\;& \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\left[|a_{ji}(t)d_{ii}(\mathbf{x})|+|a_{ij}(t)d_{jj}(\mathbf{x})|\right]x_i^2\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\left[|a_{ik}(t)d_{kj}(\mathbf{x})|+|a_{jk}(t)d_{ki}(\mathbf{x})|\right]x_i^2\,.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{d}{dt}\left[\int_0^t\int_t^\infty\left(|b_{ii}(u,s)|e_{ik}^2(\mathbf{x}(s))+\sum_{\substack{j=1\\j\neq i}}^{n}|b_{ij}(u,s)|e_{jk}^2(\mathbf{x}(s))\right)du\,x_k^2(s)ds\right]\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\left\{\int_t^\infty\left[|b_{ii}(u,t)|e_{ik}^2(\mathbf{x}(t))+\sum_{\substack{j=1\\j\neq i}}^{n}|b_{ij}(u,t)|e_{jk}^2(\mathbf{x}(t))\right]du\,x_k^2(t)\right.\\
&\left.-\int_0^t\left[|b_{ii}(t,s)|e_{ik}^2(\mathbf{x}(s))+\sum_{\substack{j=1\\j\neq i}}^{n}|b_{ij}(t,s)|e_{jk}^2(\mathbf{x}(s))\right]x_k^2(s)ds\right\}\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\left\{\int_t^\infty\left[|b_{kk}(u,t)|e_{ki}^2(\mathbf{x}(t))+\sum_{\substack{j=1\\j\neq i}}^{n}|b_{kj}(u,t)|e_{ji}^2(\mathbf{x}(t))\right]du\,x_i^2(t)\right.\\
&\left.-\int_0^t\left[|b_{ii}(t,s)|e_{ik}^2(\mathbf{x}(s))+\sum_{\substack{j=1\\j\neq i}}^{n}|b_{ij}(t,s)|e_{jk}^2(\mathbf{x}(s))\right]x_k^2(s)ds\right\}\,.
\end{aligned}$$

Thus,

$$\frac{d}{dt}[V_5]_{(2)} \le \sum_{i=1}^{n} \beta_i(t, \mathbf{x}) x_i^2 \le 0 \,.$$

Moreover, V_5 is clearly positive definite, given that

$$V_5(t, \mathbf{x}(\cdot)) \ge \frac{1}{2} \sum_{i=1}^{n} x_i^2(t) \,.$$

Hence, we obtain the conclusion of Theorem 5. $\square$

Putting $n = 1$ in Theorem 5 yields a new stability criterion for the scalar case (3),

$$x'(t) = A(t) f(x(t)) + \int_0^t B(t, s) g(x(s)) ds \,,$$

rewritten as

$$x'(t) = A(t) D(x(t)) x(t) + \int_0^t B(t, s) E(x(s)) x(s) ds,$$

on the assumption that $f, g \in C^1[\mathbf{R}, \mathbf{R}]$ and $f(0) = g(0) = 0$, and on letting

$$D(x) = \begin{cases} \dfrac{f(x)}{x}, & x \ne 0 \,, \\ f'(0), & x = 0 \,, \end{cases} \qquad \text{and} \qquad E(x) = \begin{cases} \dfrac{g(x)}{x}, & x \ne 0 \,, \\ g'(0), & x = 0 \,. \end{cases}$$

Now, if $n = 1$, then, from (18),

$$\beta_1(t, x_1) = a_{11}(t) d_{11}(x_1) + \frac{1}{2} \int_t^{\infty} |b_{11}(u, t)| du e_{11}^2(x_1) + \frac{1}{2} \int_0^t |b_{11}(t, s)| ds \,.$$

Putting $x_1 = x$, $a_{11}(t) = A(t)$, $d_{11}(x_1) = D(x)$, $e_{11}(x_1) = E(x)$ and $b_{11}(t, s) = B(t, s)$, we have the following result:

Corollary 1. *Let $f, g \in C^1[\mathbf{R}, \mathbf{R}]$, $f(0) = g(0) = 0$ and*

$$\beta(t, x) = A(t) D(x) + \frac{1}{2} E^2(x) \int_t^{\infty} |B(u, t)| \, du \ + \frac{1}{2} \int_0^t |B(t, s)| \, ds \,.$$

If $\beta(t, x) \le 0$ for $t \ge 0$ and $x \in \mathbf{R}$, then the zero solution of (3) is stable.

Proof. Using V_5, with $n = 1$, so that

$$V_5(t, x(\cdot)) = \frac{1}{2}x^2(t) + \frac{1}{2}\int_0^t \int_t^\infty |B(u, s)| du E^2(x(s))x^2(s) ds \,,$$

we get

$$\frac{d}{dt}[V_5]_{(3)} \leq \beta(t, x)x^2 \leq 0 \,.$$

Corollary 1 is thus proved. $\square$

Example 3. For the equation

$$x' = -e^t \left(x + \frac{\sin x}{2} \right) + k \int_0^t e^{-(t-s)} [1 - \cos x(s)] \, ds \,, \quad k > 0 \,, \tag{19}$$

Corollary 1 is easier than either Theorem 1 or Theorem 2 to apply. Thus, we have, for all $t \geq 0$ and $x \neq 0$,

$$\begin{aligned} \beta(t, x) &= -e^t \left(1 + \frac{1}{2} \frac{\sin x}{x} \right) + \frac{1}{2} \left[k \left(\frac{1 - \cos x}{x} \right)^2 + k \left(1 - e^{-t} \right) \right] \\ &\leq -e^t + \frac{1}{2} e^t \left| \frac{\sin x}{x} \right| + \frac{k}{2} \left(\frac{1 - \cos x}{x} \right)^2 + \frac{k}{2} \left| 1 - e^{-t} \right| \\ &\leq -e^t + \frac{1}{2} e^t + k \leq -\frac{1}{2} + k \,, \end{aligned}$$

so that $\beta(t, x) \leq 0$ if $0 < k \leq 1/2$. Moreover, for these values of k,

$$\beta(t, 0) = -\frac{3}{2}e^t + \frac{k}{2}|1 - e^{-t}| \leq -\frac{3}{2} + \frac{k}{2} < 0 \,.$$

Hence, by Corollary 1, the zero solution of (19) is stable if $k \in (0, 1/2]$.

Example 4. The system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} f_1(x_1(t),x_2(t)) \\ f_2(x_1(t),x_2(t)) \end{bmatrix} + \int_0^t \begin{bmatrix} b_{11}(t,s) & b_{12}(t,s) \\ b_{21}(t,s) & b_{22}(t,s) \end{bmatrix} \begin{bmatrix} g_1(x_1(s),x_2(s)) \\ g_2(x_1(s),x_2(s)) \end{bmatrix} ds\,,$$

is stable if $\mathbf{f},\mathbf{g} \in C^1[\mathbf{R}^2,\mathbf{R}^2]$, with $\mathbf{f}(\mathbf{0}) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$, and if for $c_1, c_2 > 0$, $t \geq 0$ and $\mathbf{x} \in \mathbf{R}^2$, we have, using (18) and condition (b) of Theorem 5,

$$\begin{aligned}
&\beta_1(t,\mathbf{x})x_1^2 = \{a_{11}(t)d_{11}(\mathbf{x}) + a_{12}(t)d_{21}(\mathbf{x}) \\
&+ \frac{1}{2}\left[|a_{11}(t)d_{12}(\mathbf{x})| + |a_{22}(t)d_{21}(\mathbf{x})| + |a_{21}(t)d_{11}(\mathbf{x})| + |a_{12}(t)d_{22}(\mathbf{x})|\right] \\
&+ \frac{1}{2}\int_t^\infty \left[|b_{11}(u,t)e_{11}^2(\mathbf{x})| + |b_{12}(u,t)e_{21}^2(\mathbf{x})| + 2|b_{22}(u,t)e_{21}^2(\mathbf{x})|\right] du \\
&+ \int_0^t \left[|b_{11}(t,s)| + |b_{12}(t,s)|\right] ds\} x_1^2 \leq -c_1 x_1^2\,,
\end{aligned}$$

and

$$\begin{aligned}
&\beta_2(t,\mathbf{x})x_2^2 = \{a_{22}(t)d_{22}(\mathbf{x}) + a_{21}(t)d_{12}(\mathbf{x}) \\
&+ \frac{1}{2}\left[|a_{22}(t)d_{21}(\mathbf{x})| + |a_{11}(t)d_{12}(\mathbf{x})| + |a_{12}(t)d_{22}(\mathbf{x})| + |a_{21}(t)d_{11}(\mathbf{x})|\right] \\
&+ \frac{1}{2}\int_t^\infty \left[2|b_{11}(u,t)e_{12}^2(\mathbf{x})| + |b_{22}(u,t)e_{22}^2(\mathbf{x})| + |b_{21}(u,t)e_{12}^2(\mathbf{x})|\right] du \\
&+ \int_0^t \left[|b_{22}(t,s)| + |b_{21}(t,s)|\right] ds\} x_2^2 \leq -c_2 x_2^2\,.
\end{aligned}$$

The following simple, but illustrative, case is one such stable system:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \dfrac{10}{t+1} & -20 \\ -20 & \dfrac{10}{t+1} \end{bmatrix} \begin{bmatrix} x_2(t) - \dfrac{1}{20}x_1 \tanh(x_1(t)) \\ x_1(t) + \dfrac{1}{20}x_2 \tanh(x_2(t)) \end{bmatrix} + \int_0^t \begin{bmatrix} \dfrac{1}{(1+t-s)^2} & 0 \\ 0 & \dfrac{1}{4[(t-s)^2+1]} \end{bmatrix} \begin{bmatrix} x_1(s)+x_2(s) \\ x_1(s)+x_2(s) \end{bmatrix} ds\,, \tag{20}$$

or, in the form of (17),

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \dfrac{10}{t+1} & -20 \\ -20 & \dfrac{10}{t+1} \end{bmatrix} \begin{bmatrix} -\dfrac{1}{20}\tanh(x_1(t)) & 1 \\ 1 & \dfrac{1}{20}\tanh(x_2(t)) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

$$
+ \int_0^t \begin{bmatrix} \dfrac{1}{(1+t-s)^2} & 0 \\ 0 & \dfrac{1}{4[(t-s)^2+1]} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} ds\,.
$$

Now, for all $t \geq 0$ and for all $\mathbf{x} \in \mathbf{R}^2$, we have,

$$
\begin{aligned}
\beta_1(t,\mathbf{x})x_1^2 &= \left\{ \left(\frac{10}{t+1}\right)\left(-\frac{\tanh x_1}{20}\right) + (-20)\cdot 1 \right. \\
&+ \frac{1}{2}\left[\left|\frac{10}{t+1}\cdot 1\right| + \left|\frac{10}{t+1}\cdot 1\right| + \left|-20\left(-\frac{\tanh x_1}{20}\right)\right| + \left|-20\cdot\frac{\tanh x_2}{20}\right|\right] \\
&+ \frac{1}{2}\left[\int_t^\infty \frac{1}{(1+u-t)^2}\cdot 1\, du + 2\int_t^\infty \frac{1}{4[(u-t)^2+1]}\cdot 1\, du\right] \\
&\left. + \int_0^t \frac{1}{(1+t-s)^2}\, ds \right\} x_1^2 \\
&= \left\{ -\frac{\tanh x_1}{2(t+1)} - 20 + \frac{10}{t+1} + \frac{1}{2}\left[|\tanh x_1| + |\tanh x_2|\right] + \frac{1}{2}\left[1 + \frac{2\pi}{8}\right] \right. \\
&\left. + \left[1 - \frac{1}{1+t}\right] \right\} x_1^2\,,
\end{aligned}
$$

which clearly shows that $\beta_1(t,\mathbf{x})x_1^2$ is continuous on $[0,\infty)\times\mathbf{R}^2$. Moreover

$$
\beta_1(t,\mathbf{x})x_1^2 \leq -\left(6 - \frac{\pi}{8}\right)x_1^2 \leq 0\,.
$$

Next, we have,

$$
\begin{aligned}
&\beta_2(t,\mathbf{x})x_2^2 = \left\{\left(\frac{10}{t+1}\right)\left(\frac{\tanh x_2}{20}\right) + (-20)\cdot 1\right. \\
&+\frac{1}{2}\left[\left|\frac{10}{t+1}\cdot 1\right| + \left|\frac{10}{t+1}\cdot 1\right| + \left|-20\left(\frac{\tanh x_2}{20}\right)\right| + \left|-20\left(-\frac{\tanh x_1}{20}\right)\right|\right] \\
&+\frac{1}{2}\left[2\int_t^\infty \frac{1}{(1+u-t)^2}\cdot 1\,du + \int_t^\infty \frac{1}{4[(u-t)^2+1]}\cdot 1\,du\right] \\
&\left.+\int_0^t \frac{1}{4[(t-s)^2+1]}\,ds\right\} x_2^2 \\
&= \left\{\frac{\tanh x_2}{2(t+1)} - 20 + \frac{10}{t+1} + \frac{1}{2}\left[|\tanh x_1| + |\tanh x_2|\right] + \frac{1}{2}\left[2+\frac{\pi}{8}\right]\right. \\
&\left.+\frac{\tan^{-1} t}{4}\right\} x_2^2\,,
\end{aligned}
$$

which shows that $\beta_2(t,\mathbf{x})x_2^2$ is continuous on $[0,\infty)\times\mathbf{R}^2$. Moreover

$$
\beta_2(t,\mathbf{x})x_2^2 \le -\left(7-\frac{\pi}{4}\right)x_2^2 \le 0\,.
$$

Hence, we have shown that $\beta_i(t,\mathbf{x})x_i^2 \le 0$ for $i=1,2$, $t\ge 0$ and $\mathbf{x}\in\mathbf{R}^2$. The zero solution of system (20) is therefore stable by Theorem 5.

3.2 Perturbed Case

We finally consider system (1), where $\mathbf{h}(t) = (h_1(t),\ldots,h_n(t))^T$.

Theorem 6. *Let the conditions of Theorem 5 hold, with the last condition replaced by the assumption that there are constants $c_i > 0$ such that $\beta_i(t,\mathbf{x}) \le -c_i$ for $i=1,\ldots,n$, $t\ge 0$ and $\mathbf{x}\in\mathbf{R}^n$. If $h_i(\cdot) \in L^2[0,\infty)$ for $i=1,\ldots,n$, then all solutions of (1) are bounded.*

Proof. Let $\epsilon > 0$ and consider the functional

$$
V_6(t,\mathbf{x}(\cdot)) = V_5(t,\mathbf{x}(\cdot)) + \frac{1}{4\epsilon}\sum_{i=1}^n \int_t^\infty h_i^2(u)du\,,
$$

which is clearly radially unbounded. Now, we have, for $c = -\min\{c_1,\ldots,c_n\}$,

$t \geq 0$ and $\mathbf{x} \in \mathbf{R}^n$,

$$\begin{aligned}\frac{d}{dt}[V_6]_{(1)} &\leq \sum_{i=1}^{n} \beta_i(t, \mathbf{x}) x_i^2 + \sum_{i=1}^{n} x_i h_i(t) - \frac{1}{4\epsilon} \sum_{i=1}^{n} h_i^2(t) \\ &\leq -(c - \epsilon) \sum_{i=1}^{n} x_i^2 \,.\end{aligned}$$

We have thus proved the boundedness of solutions of (1), since we can always find $\epsilon > 0$ such that $(c - \epsilon) \geq 0$.

□

The following corollary follows directly from Theorem 6 by putting $n = 1$.

Corollary 2. *Let the conditions of Corollary 1 hold, with the last condition replaced by the assumption that there is a constant $c > 0$ such that $\beta(t, x) \leq -c$ for $t \geq 0$ and $x \in \mathbf{R}$. If $h(\cdot) \in L^2[0, \infty)$, then all solutions of the scalar equation (15) are bounded.*

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