Elastic Membrane Equation in Bounded and Unbounded Domains *

by

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Abstract

The small-amplitude motion of a thin elastic membrane is investigated in n-dimensional bounded and unbounded domains, with $n \in \mathbb{N}$. Existence and uniqueness of the solutions are established. Asymptotic behavior of the solutions is proved too.

1 Introduction

The one-dimensional equation of motion of a thin membrane fixed at both ends and undergoing cylindrical bending can be written as

$$\begin{aligned} u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx + \sigma \int_\Omega u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) \\ + u_{xxxx}(x,t) + \nu u_{xxxxt}(x,t) = 0 \quad \text{in} \quad Q, \end{aligned} \tag{1.1}$$

where u is the plate transverse displacement, x is the spatial coordinate in the direction of the fluid flow, and t is the time. The viscoelastic structural damping terms are denote by σ and ν, ζ_1 is the nonlinear stiffness of the membrane, ζ_0 is an in-plane tensile load, and (x,t) belongs to

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$Q = \Omega \times [0, \infty[$ with $\Omega = (0,1)$. All quantities are physically non-dimensionalized, ν, σ, ζ_1 are fixed positive and ζ_0 is fixed non-negative.

Equation (1.1) is related to the flutter panel equation, i.e.,

$$\begin{aligned} &u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx + \sigma \int_\Omega u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) \\ &+\sqrt{\rho}\delta u_t(x,t) + \rho u_x(x,t) + u_{xxxx}(x,t) + \nu u_{xxxxt}(x,t) = 0 \quad \text{in} \quad Q, \end{aligned} \tag{1.2}$$

when the internal aerodynamic pressure of plate motion ρ is assumed negligible. In equation (1.2) it means that the sum $\sqrt{\rho}\delta u_t + \rho u_x$ is negligible. From the mathematical viewpoint, this hypothesis does not have a significant influence in the formulation (1.2) when the interest is to obtain existence and asymptotic behavior of the solutions, because the hight-order sum $u_{xxxx} + \nu u_{xxxxt}$ has a dominant performance about $\sqrt{\rho}\delta u_t + \rho u_x$.

Equation (1.2) arises in a wind tunnel experiment for a panel at supersonic speeds. For a derivation of this model see, for instance, Dowell [12] and Holmes [15].

Existence of global solutions for the mixed problem associated with equation (1.2) was investigated by Hughes & Marsden [18], and with respect to asymptotic stability of solutions, Holmes & Marsden [16] supposing some restrictive hypotheses about the aerodynamic pressure ρ the authors concluded that the derivative of the solution is negative.

Equation (1.1) includes some special situations in elasticity. Namely, omitting the dissipative terms we have the beam equation

$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx\Big) u_{xx}(x,t) + u_{xxxx}(x,t) = 0 \quad \text{in} \quad Q. \tag{1.3}$$

The beam equation (1.3) has been studied by several authors, among them, Ball [2, 3], Biler [4], Brito [5, 6], Pereira [27] and Medeiros [24] . The precedent works investigate the equation in several contexts.

By omission of the term u_{xxxx} in (1.3) we have the well known Kirchhoff equation

$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx\Big) u_{xx}(x,t) = 0 \quad \text{in} \quad Q. \tag{1.4}$$

Equation (1.4) has also been extensively studied by several authors in both $\{1, 2, \cdots, n\}$-dimensional cases and general mathematical models in a Hilbert space H. Both local and global solutions have been shown to exist in several physical-mathematical contexts. Among them, Arosio-Spagnolo [1], Carrier [7], Clark [8, 9], Dickey [10], Kirchhoff [19], Matos-Pereira [23], Narashinham [26], Pohozaev [28] and a number of other interesting references cited in the previously mentioned papers, mainly in Medeiros at al [25].

The investigation of existence of a solution for the Cauchy problem associated with equation (1.1) in n-dimensional bounded and unbounded domain will be made by the application of diagonalization theorem of Dixmier & Von Neumann. The use of the diagonalization theorem allows us to study the Cauchy problem associated with equation (1.1) independently of compactness, and in this way, the result obtained here leads to the conclusion that the inherent properties of such problem are valid in bounded, unbounded, and exteriors domains.

The use of the diagonalization theorem in the study of Cauchy problem associated with the Kirchhoff equation was initially utilized by Matos [22] to prove existence of a local solution. Subsequently, Clark [8] also utilizing the diagonalization theorem proved existence and uniqueness of a global classical solutions supposing that the initial datum are A-analytics such as Arosio-Spagnolo [1] on bounded domain.

This paper is divided in four sections, where the emphasis is to describe the properties in a mathematically rigorous fashion. In §2, the basic notations are laid out. Section §3 is devoted to investigate the existence and uniqueness of global solutions of the Cauchy problem associated with the equation (1.1). In §4, the asymptotic behavior to the energy of the solutions of the section 3 is established. Finally, in §4 is concerned with applications.

2 Notation and terminology

We shall use, throughout this paper, the following terminology. Let X be a Banach or Hilbert space, T is a positive real number or $T = +\infty$ and $1 \leq p \leq \infty$. $L^p(0,T;X)$ denotes the Banach space of all measurable functions $u :]0,T[\to X$ such that $t \mapsto \|u(t)\|_X$ belongs to $L^p(0,T)$ and the norm in $L^p(0,T;X)$ is defined by

$$\|u\|_p = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty,$$

and if $p = \infty$ then

$$\|u\|_\infty = \operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|_X.$$

For $m \in \mathbb{N}$, $C^m([0,T];X)$ represents the space of m-times continuously differentiable functions $v : [0,T] \to X$, where X can be either $\mathbb{R}$, a Banach space or a Hilbert space.

In the context of hilbertian integral a field of the Hilbert space is, by definition, a mapping $\lambda \to \mathcal{H}(\lambda)$ where $\mathcal{H}(\lambda)$ is a Hilbert space for each $\lambda \in \mathbb{R}$. A vector field on $\mathbb{R}$ is a mapping $\lambda \to u(\lambda)$ defined on $\mathbb{R}$ such that $u(\lambda) \in \mathcal{H}(\lambda)$.

We represent by $\mathcal{F}$ the real vector space of all vector fields on $\mathbb{R}$ and by μ a positive real measure on $\mathbb{R}$.

A field of the Hilbert spaces $\lambda \to \mathcal{H}(\lambda)$ is said to be μ-measurable when there exists a subspace $\mathcal{M}$ of $\mathcal{F}$ satisfying the following conditions

- *The mapping* $\lambda \to \|u(\lambda)\|_{\mathcal{H}(\lambda)}$ *is μ-measurable for all* $u \in \mathcal{M}$.
- *If* $u \in \mathcal{F}$ *and* $\lambda \to (u(\lambda), v(\lambda))_{\mathcal{H}(\lambda)}$ *is μ-measurable for all* $v \in \mathcal{M}$ *then* $u \in \mathcal{M}$.
- *There exists in* $\mathcal{M}$ *a sequence* $(u_n)_{n\in\mathbb{N}}$ *such that* $(u_n(\lambda))_{n\in\mathbb{N}}$ *is total in* $\mathcal{H}(\lambda)$ *for each* $\lambda \in \mathbb{R}$.

The objects of $\mathcal{M}$ are called μ-measurable vector fields. In the following, $\lambda \to \mathcal{H}(\lambda)$ represents a μ-measurable field of the Hilbert spaces and all the vector fields considered are μ-measurable.

The space $\mathcal{H}_0 = \displaystyle\int^{\oplus} \mathcal{H}(\lambda) d\mu(\lambda)$ will be defined by

A vector field $\lambda \to u(\lambda)$ *is in* $\mathcal{H}_0$ *if, and only if* $\displaystyle\int_{\mathbb{R}} \|u(\lambda)\|^2_{\mathcal{H}(\lambda)} d\mu(\lambda) < \infty$.

The scalar product in $\mathcal{H}_0$ is defined by

$$(u,v)_0 = \int_{\mathbb{R}} ((u(\lambda), v(\lambda)))_{\mathcal{H}(\lambda)} \, d\mu(\lambda) \quad \text{for all} \quad u,\, v \in \mathcal{H}_0. \tag{2.1}$$

With (2.1) the vector space $\mathcal{H}_0$ becomes a Hilbert space which is called the hilbertian integral, or measurable hilbertian sum, of the field $\lambda \to \mathcal{H}(\lambda)$.

Given a real number ℓ, we denote by $\mathcal{H}_\ell$ the Hilbert space of the vector fields u such that the field $\lambda \to \lambda^\ell u(\lambda)$ belongs to $\mathcal{H}_0$. In $\mathcal{H}_\ell$ we define the norms by

$$|u|^2_\ell = |\lambda^\ell u|^2_0 = \int_{\mathbb{R}} \lambda^{2\ell} \|u(\lambda)\|^2_{\mathcal{H}(\lambda)} \, d\mu(\lambda) \quad \text{for all} \quad u \in \mathcal{H}_\ell. \tag{2.2}$$

Let us fix a separable Hilbert space H with scalar product $(\cdot,\cdot)$ and norm $|\cdot,\cdot|$. We consider a self-adjoint operator A in H such that

$$(Au, u) \geq \vartheta |u|^2 \quad \text{for all} \quad u \in D(A) \quad \text{and} \quad \vartheta > 0. \tag{2.3}$$

With the hypothesis (2.3) the operator A satisfies all the hypotheses of the diagonalization theorem, cf. Dixmier [11], Gelfand & Vilenkin [13], Huet [17] and Lions & Magenes [21]. Thus, it follows that there exists a Hilbertian integral

$$\mathcal{H}_0 = \int^{\oplus} \mathcal{H}(\lambda) d\mu(\lambda),$$

and an unitary operator $\mathcal{U}$ from H onto $\mathcal{H}_0$ such that

$$\mathcal{U}(A^\ell u) = \lambda^\ell \mathcal{U}(u) \quad \text{for all} \quad u \in D(A^\ell) \quad \text{and} \quad \ell \geq 0, \tag{2.4}$$

$$\mathcal{U} : D(A^\ell) \to \mathcal{H}_\ell \quad \text{is an isomorphism.} \tag{2.5}$$

The domain $D(A^\ell)$ is equipped with the graph norm and μ is a positive Radon measure with support in $]\lambda_0, \infty[$ for $0 < \lambda_0 < \vartheta$ where ϑ is the constant defined in (2.3).

Observe that as $\operatorname{supp}(\mu) \subset]\lambda_0, \infty[$ then for $\ell \geq \tau$, where ℓ and τ are real numbers, we have

$$\mathcal{H}_\ell \hookrightarrow \mathcal{H}_\tau, \tag{2.6}$$

$$|u|_\tau^2 \leq \lambda_0^{(\ell-\tau)} |u|_\ell^2 \quad \text{for all} \quad u \in \mathcal{H}_\tau. \tag{2.7}$$

3 Existence and uniqueness of solutions

Our goal in this section is to inquire the existence and uniqueness of solutions of the Cauchy problem associated with equation (1.1) in a bounded and unbounded domain. Thus, changing the Laplace operator $-\frac{\partial^2}{\partial x^2}$ by a self-adjoint, positive and unbounded operator $\mathcal{A}$ in a real Hilbert space H, satisfying the hypothesis (2.3) we have the following Cauchy problem

$$\begin{gathered} u''(t) + \left\{ M\left(|\mathcal{A}^{1/2} u(t)|^2 \right) + \sigma\left(\mathcal{A}u(t), u'(t) \right) \right\} \mathcal{A}u(t) + \\ \mathcal{A}^2 u(t) + \nu \mathcal{A}^2 u'(t) = 0 \quad \text{for all} \quad t \geq 0, \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1, \end{gathered} \tag{3.1}$$

where $\mathcal{A}^{1/2}$ represents the square root of $\mathcal{A}$, $\mathcal{A}^2$ is defined by $\langle \mathcal{A}^2 v, w \rangle = (\mathcal{A}v, \mathcal{A}w)$, $D(\mathcal{A})'$ is the dual of $D(\mathcal{A})$, $\langle \cdot, \cdot \rangle$ denotes the duality pairing

$D(\mathcal{A})' \times D(\mathcal{A})$, $M(\xi) = \zeta_0 + \zeta_1 \xi$ for all $\xi \in [0, \infty[$, and finally, the symbol $'$ represents the derivative in the sense of distributions.

We assume the following natural hypothesis on the real function M

$$M \quad \text{is locally Lipschitz function on} \quad [0, \infty[. \tag{3.2}$$

Definition 3.1 *A function u is solution of (3.1) if, and only if $u : [0, \infty[\to H$ satisfies*

$$\begin{gathered} (u''(t), w)_0 + M\left(|\mathcal{A}^{1/2} u(t)|_0^2\right)\left(\mathcal{A}^{1/2} u(t), \mathcal{A}^{1/2} w\right)_0 + \\ \sigma\left(\mathcal{A} u(t), u'(t)\right)_0 \left(\mathcal{A}^{1/2} u(t), \mathcal{A}^{1/2} w\right)_0 + \\ \left(\mathcal{A} u(t), \mathcal{A} w\right)_0 + \nu\left(\mathcal{A} u'(t), \mathcal{A} w\right)_0 = 0, \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1, \end{gathered} \tag{3.3}$$

for all $w \in D(\mathcal{A})$.

The existence and uniqueness of solutions for the Cauchy problem (3.1) is guaranteed by

Theorem 3.1 *Let u_0 and u_1 belong to $D(\mathcal{A})$. If the operator $\mathcal{A}$ is a self-adjoint, unbounded and positive on a real Hilbert space H satisfying the hypothesis (2.3) and the real function M satisfies the hypothesis (3.2), then there exists a unique function $u : [0, \infty[\to H$ solution of the Cauchy problem (3.1) in the sense of the definition 3.1 in the class*

$$u \in C^0\left([0, \infty[; D\left(\mathcal{A}\right)\right), \quad u' \in C^0\left([0, \infty[; D\left(\mathcal{A}^{1/2}\right)\right), \tag{3.4}$$

$$u'' \in L^2([0, \infty[; H). \tag{3.5}$$

Existence. Assuming the hypotheses on the operator $\mathcal{A}$ we have by diagonalization theorem that there exists an unitary operator $\mathcal{U}$ defined in (2.4) and (2.5) such that $\mathcal{U} : H \to \mathcal{H}_0$ is an isomorphism. Thus, *u is a solution of the Cauchy problem (3.1) if, and only if $v = \mathcal{U}(u)$ is a solution of the following system of ordinary differential equations*

$$\begin{gathered} v''(t) + \left\{ M\left(|v(t)|_{1/2}^2\right) + \sigma\left(\lambda v(t), v'(t)\right)_0 \right\} \lambda v(t) \\ + \lambda^2 v(t) + \nu \lambda^2 v'(t) = 0 \quad \text{for all} \quad t \geq 0, \\ v(0) = v_0 = \mathcal{U}(u_0) \in \mathcal{H}_1 \quad \text{and} \quad v'(0) = v_1 = \mathcal{U}(u_1) \in \mathcal{H}_1, \end{gathered} \tag{3.6}$$

where equation $(3.6)_1$ is verified in the sense of $L^2\left([0,\infty[;\mathcal{H}_0\right)$.

Our next task is to prove the existence and uniqueness of local solution for the system (3.6) in the class $C^2\left([0,T_p]\,;\mathcal{H}_{0,p}\right)$ for $p\in\mathbb{N}$ fixed.

Truncated problem - Local solution. Let $p\in\mathbb{N}$ be. We denote by $\mathcal{H}_{0,p}$ the subspace of $\mathcal{H}_0$ of the fields $v(\lambda)$ such that $v(\lambda)=0$, μ - a. e. on the interval $[p,+\infty[$. Under these conditions $\mathcal{H}_{0,p}$ equipped with the norm of $\mathcal{H}_0$ is a Hilbert space.

For each vector field $v\in\mathcal{H}_\ell$ with $\ell\in\mathbb{R}$ we denote by v_p the truncated field associated with v, which is defined on the following way

$$v_p=\begin{cases} v & \mu - \text{a. e. on} \quad]\lambda_0,p[,\\ 0 & \mu - \text{a. e. on} \quad [p,+\infty[,\end{cases}$$

where $0<\lambda_0<\vartheta$ and $\vartheta>0$ is the constant defined in (2.3). Hence, it is easy to prove that $v_p\in\mathcal{H}_{0,p}$ and $v_p\longrightarrow v$ strongly in $\mathcal{H}_\ell$ for all $\ell\in\mathbb{R}$.

The truncated problem associated with (3.6) consists of finding a function $v_p:[0,T_p]\to\mathcal{H}_{0,p}$ such that $v_p\in C^2([0,T_p];\mathcal{H}_{0,p})$ for all $T_p>0$ satisfying

$$\begin{aligned} &v_p''(t)+\left\{M\left(|v_p(t)|_{1/2}^2\right)+\sigma\left(\lambda v_p(t),v_p'(t)\right)_0\right\}\lambda v_p(t)\\ &\quad+\lambda^2 v_p(t)+\nu\lambda^2 v_p'(t)=0 \quad \text{for all} \quad t\geq 0,\\ &v_p(0)=v_{0p}\longrightarrow v_0 \quad \text{strongly in} \quad \mathcal{H}_1,\\ &v_p'(0)=v_{1p}\longrightarrow v_1 \quad \text{strongly in} \quad \mathcal{H}_1.\end{aligned}\tag{3.7}$$

Let $\mathcal{V}_p=\begin{pmatrix} v_p\\ v_p'\end{pmatrix}$ be. Hence the system (3.7) is equivalent to

$$\begin{aligned} &\frac{d}{dt}\mathcal{V}_p=\mathcal{F}\left(\mathcal{V}_p\right),\\ &\mathcal{V}_p(0)=\mathcal{V}_{0p}\longrightarrow\mathcal{V}_0,\end{aligned}\tag{3.8}$$

where

$$\mathcal{F}(\mathcal{V}_p)=\begin{pmatrix} v_p'\\ -\left\{M\left(|v_p|_{1/2}^2\right)+\sigma\left(\lambda v_p,v_p'\right)_0\right\}\lambda v_p-\lambda^2 v_p-\nu\lambda^2 v_p'\end{pmatrix},$$

and

$$\mathcal{V}_{0p} = \begin{pmatrix} v_{0p} \\ v'_{0p} \end{pmatrix}.$$

As M is a locally Lipschitz function then $\mathcal{F}$ is also locally Lipschitz function and by Cauchy theorem it follows that there exists a unique local solution $\mathcal{V}_p$ of the Cauchy problem (3.8) in the class $C^1\left([0,T_p]\,;\mathcal{H}_{0,p}\times\mathcal{H}_{0,p}\right)$.

The interval $[0,T_p]$ will be extended to the whole interval $[0,\infty[$ as a consequence of the first estimate below.

Estimate I. Taking the scalar product on $\mathcal{H}_0$ of $2v'_p$ with both sides of $(3.7)_1$ and integrating from 0 to $t\le T_p$ yields

$$\left|v'_p(t)\right|_0^2 + \widehat{M}\left(\left|v_p(t)\right|_{1/2}^2\right) + \sigma\int_0^t \left(\lambda v_p(s), v'_p(s)\right)_0^2 ds +$$
$$\left|v_p(t)\right|_1^2 + \nu\int_0^t \left|v'_p(s)\right|_1^2 ds = \left|v_{1p}\right|_0^2 + \hat{M}\left(\left|v_{0p}\right|_{1/2}^2\right) + \left|v_{0p}\right|_1^2,$$

where $\widehat{M}(\xi) = \displaystyle\int_0^\xi M(\tau)d\tau$. Hence, from initial conditions $(3.7)_{2,3}$ and as $\sigma\displaystyle\int_0^t \left(\lambda v_p(s), v'_p(s)\right)_0^2 ds \ge 0$ we have the estimates

$$\begin{aligned} &\left(v_p\right)_{p\in\mathbb{N}} \quad \text{belongs to} \quad L^\infty\left(0,\infty;\mathcal{H}_1\right), \\ &\left(v'_p\right)_{p\in\mathbb{N}} \quad \text{belongs to} \quad L^\infty\left(0,\infty;\mathcal{H}_0\right)\cap L^2\left(0,\infty;\mathcal{H}_1\right). \end{aligned} \tag{3.9}$$

Estimate II. Taking the scalar product on $\mathcal{H}_0$ of $2v''_p$ with both sides of $(3.7)_1$, using Cauchy-Schwartz inequality in some terms and the identity

$$2\left(\lambda^2 v_p(t), v''_p(t)\right)_0 = 2\frac{d}{dt}\left(\lambda v_p(t), \lambda v'_p(t)\right)_0 - 2\left|v'_p(t)\right|_1^2,$$

yields

$$\left|v''_p(t)\right|_0^2 + 2\frac{d}{dt}\left(\lambda v_p(t), \lambda v'_p(t)\right)_0 + \nu\frac{d}{dt}\left|v'_p(t)\right|_1^2 \le 2\left|v'_p(t)\right|_1^2 +$$
$$2\left\{M\left(\left|v_p(t)\right|_{1/2}^2\right) + \frac{\sigma}{2}\left(\left|v_p(t)\right|_1^2 + \left|v'_p(t)\right|_0^2\right)\right\}\left\{\frac{1}{4\delta}\left|v_p(t)\right|_1^2 + \delta\left|v''_p(t)\right|_1\right\},$$

where δ is a suitable positive constant to be chosen later. Hence, using the estimate (3.9), the continuous injection (2.6) and hypothesis (3.2) we obtain

$$(2-c_1\delta)\left|v_p''(t)\right|_0^2+2\frac{d}{dt}\left(\lambda v_p(t),\lambda v_p'(t)\right)_0+\nu\frac{d}{dt}\left|v_p'(t)\right|_1^2\leq c_0+2\left|v_p'(t)\right|_1^2,$$

where c_0 and c_1 represent positive constants dependent only on initial data. Hence, integrating from 0 to $t\leq T$ we have

$$(2-c_1\delta)\int_0^t\left|v_p''(s)\right|_0^2ds+2\left(v_p(t),v_p'(t)\right)_1+\nu\left|v_p'(t)\right|_1^2\leq$$
$$c_0T+2\left(v_0,v_1\right)_1+\nu\left|v_1\right|_1^2+2\int_0^t\left|v_p'(s)\right|_1^2ds.$$

Choosing $0<c_1\delta<\frac{1}{2}$, using the initial conditions $(3.7)_{2,3}$ and the estimate (3.9) we have the estimates

$$\begin{array}{lll}\left(v_p'\right)_{p\in\mathbb{N}} & \text{belongs to} & L^\infty\left(0,T;\mathcal{H}_1\right),\\ \left(v_p''\right)_{p\in\mathbb{N}} & \text{belongs to} & L^2\left(0,T;\mathcal{H}_0\right).\end{array}\tag{3.10}$$

The estimates (3.9) and (3.10) are sufficient to take the limit in $(3.7)_1$.

Limit of the truncated solutions. From estimates (3.9) and (3.10) it follows

$$\begin{array}{lll}\left(v_p\right)_{p\in\mathbb{N}} & \text{belongs to} & C^0\left([0,T];\mathcal{H}_1\right),\\ \left(v_p'\right)_{p\in\mathbb{N}} & \text{belongs to} & C^0\left([0,T];\mathcal{H}_{1/2}\right).\end{array}$$

Hence, the sequences of real functions $\left(f_p\right)_{p\in\mathbb{N}}$ and $\left(g_p\right)_{p\in\mathbb{N}}$ defined by

$$f_p(t)=\left|v_p(t)\right|_{1/2}^2\quad\text{for all}\quad t\in[0,T],$$

$$g_p(t)=\left(v_p(t),v_p'(t)\right)_{1/2}\quad\text{for all}\quad t\in[0,T],$$

are continuous on $[0,T]$.

On the other hand, given t and s in $[0,T]$ we get

$$\begin{aligned}\left|f_p(t)-f_p(s)\right|_{\mathbb{R}}^2 &\leq 2\left(\left|v_p(t)\right|_{1/2}^2+\left|v_p(s)\right|_{1/2}^2\right)\left|v_p(t)-v_p(s)\right|_{1/2}^2\\ &\leq c_0\left|v_p(t)-v_p(s)\right|_{1/2}^2\leq\int_s^t\left|v_p(\tau)\right|_{1/2}^2d\tau\\ &\leq c|t-s|_{\mathbb{R}}^2\quad\text{for all}\quad p\in\mathbb{N}.\end{aligned}$$

That is,

$$|f_p(t) - f_p(s)|_{\mathbb{R}} \leq \sqrt{c}|t-s|_{\mathbb{R}} \ \text{ for all } \ p \in \mathbb{N} \ \text{ and } \ s, \ t \in [0,T]. \tag{3.11}$$

Analogously,

$$|g_p(t) - g_p(s)|_{\mathbb{R}} \leq \sqrt{c}|t-s|_{\mathbb{R}} \ \text{ for all } \ p \in \mathbb{R} \ \text{ and } \ s, \ t \in [0,T]. \tag{3.12}$$

As a consequence of (3.11), (3.12) and the Arzelá-Ascoli theorem there are subsequences of $(f_p)_{p\in\mathbb{N}}$ and of $(g_p)_{p\in\mathbb{N}}$, which we will still continue to denote by $(f_p)_{p\in\mathbb{N}}$ and $(g_p)_{p\in\mathbb{N}}$ respectively, and functions $f, \ g \in C^0([0,T];\mathbb{R})$ such that

$$f_p \longrightarrow f \quad \text{and} \quad g_p \longrightarrow g \quad \text{uniformly in} \quad C^0([0,T];\mathbb{R}). \tag{3.13}$$

Hence, from estimates (3.9), (3.10) and hypothesis (3.2) we obtain that there exists a function v such that

$$\begin{aligned} v_p &\longrightarrow v \qquad \text{weak star in} \quad L^\infty\left(0,T;\mathcal{H}_1\right), \\ v'_p &\longrightarrow v' \qquad \text{weak star in} \quad L^\infty\left(0,T;\mathcal{H}_1\right), \\ v''_p &\longrightarrow v'' \qquad \text{weakly in} \quad L^2\left(0,T;\mathcal{H}_0\right), \\ M\left(|v_p|^2_{1/2}\right) &\longrightarrow M\left(f\right) \quad \text{in} \quad C^0\left([0,T];\mathbb{R}\right), \\ g_p = \left(v_p, v'_p\right)_{1/2} &\longrightarrow g \quad \text{in} \quad C^0\left([0,T];\mathbb{R}\right). \end{aligned} \tag{3.14}$$

From the preceding convergence (3.14) and taking the limit in $(3.7)_1$ yields

$$\begin{gathered} v''(t) + \left\{M\left(f(t)\right) + \sigma g(t)\right\}\lambda v(t) \\ +\lambda^2 v(t) + \nu\lambda^2 v'(t) = 0 \quad \text{in} \quad L^2\left(0,T;\mathcal{H}_0\right). \end{gathered} \tag{3.15}$$

Our next task is to prove that $f(t) = |v(t)|^2_{1/2}$ and $g(t) = (v(t), v'(t))_{1/2}$ for all $t \in [0,T]$. To do this we will prove the following result

Lemma 3.1 *Let $p \in \mathbb{N}$ be. If $w_p = v_p - v$ where v_p and v satisfy $(3.7)_1$ and (3.15) respectively, then*

$$\left|w'_p(t)\right|^2_0 + |w_p(t)|^2_1 \longrightarrow 0 \quad \textit{as} \quad p \longrightarrow \infty \quad \textit{for all} \quad t \in [0,T]. \tag{3.16}$$

Proof. The function w_p previously defined satisfies

$$
\begin{aligned}
& w_p''(t) + M\left(f(t)\right)\lambda w_p(t) + \sigma g(t)\lambda w_p(t) + \lambda^2 w_p(t) + \nu\lambda^2 w_p'(t) = \\
& \qquad \left\{ M\left(f(t)\right) - M\left(\left|v_p(t)\right|_{1/2}^2\right)\right\}\lambda v_p(t) + \\
& \qquad \sigma\left\{\left(v_p(t), v_p'(t)\right)_{1/2} - g(t)\right\}\lambda v_p(t), \\
& w_p(0) \longrightarrow 0 \quad \text{strongly} \quad \text{in} \quad \mathcal{H}_1 \quad \text{as} \quad p \longrightarrow \infty, \\
& w_p'(0) \longrightarrow 0 \quad \text{strongly} \quad \text{in} \quad \mathcal{H}_1 \quad \text{as} \quad p \longrightarrow \infty.
\end{aligned}
\tag{3.17}
$$

Taking the scalar product on $\mathcal{H}_0$ of w_p' with both sides of $(3.17)_1$ yields

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left\{\left|w_p'(t)\right|_0^2 + \left|w_p(t)\right|_1^2\right\} + \nu\left|w_p'(t)\right|_1^2 \\
\leq \;& M\left(f(t)\right)\left|w_p(t)\right|_1\left|w_p'(t)\right|_0 + \sigma|g(t)|\left|w_p(t)\right|_1\left|w_p'(t)\right|_0 \\
+ \;& \left|M\left(f(t)\right) - M\left(\left|v_p(t)\right|_{1/2}^2\right)\right|_{\mathbb{R}}\left|v_p(t)\right|_1\left|w_p'(t)\right|_0 \\
+ \;& \sigma\left|\left(v_p(t), v_p'(t)\right)_{1/2} - g(t)\right|_{\mathbb{R}}\left|v_p(t)\right|_1\left|w_p'(t)\right|_0 .
\end{aligned}
\tag{3.18}
$$

Using the estimate (3.9), the convergence (3.13) and the continuity of the function M we have

$$
\begin{aligned}
& \left|M\left(f(t)\right) - M\left(\left|v_p(t)\right|_{1/2}^2\right)\right|_{\mathbb{R}}\left|v_p(t)\right|_1\left|w_p'(t)\right|_0 \leq c\epsilon + \tfrac{1}{2}\left|w_p'(t)\right|_0^2, \\
& \sigma\left|\left(v_p(t), v_p'(t)\right)_{1/2} - g(t)\right|_{\mathbb{R}}\left|v_p(t)\right|_1\left|w_p'(t)\right|_0 \leq c\epsilon + \tfrac{1}{2}\left|w_p'(t)\right|_0^2,
\end{aligned}
\tag{3.19}
$$

where $c > 0$ is a constant independent of p and $0 < \epsilon \leq 1$ is a suitable constant. Observing that

$$
\nu\left|w_p'(t)\right|_1^2 \geq 0 \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad g \in C^0([0, T]; \mathbb{R}),
$$

we get by substitution of (3.19) into (3.19) that

$$
\frac{1}{2}\frac{d}{dt}\left\{\left|w_p(t)\right|_1^2 + \left|w_p'(t)\right|_0^2\right\} \leq c\epsilon + c\left\{\left|w_p(t)\right|_1^2 + \left|w_p'(t)\right|_0^2\right\},
$$

where c is a general positive real constant independent of p. Hence, integrating form 0 to $t \leq T$, using $(3.17)_{2,3}$ and applying the Gronwall's inequality we have

$$
\left|w_p(t)\right|_1^2 + \left|w_p'(t)\right|_0^2 \leq c\epsilon T \exp\left(cT\right).
$$

Hence, for $0 < \epsilon \leq 1$ arbitrarily small we conclude that (3.16) holds.

Now, we can justify that $|f(t)| = |v(t)|_{1/2}^2$ and $g(t) = (v(t), v'(t))_{1/2}$. In fact, given $t \in [0,T]$ we have

$$\begin{aligned} &\left| f(t) - |v(t)|_{1/2}^2 \right|_{\mathbb{R}}^2 \\ &\leq \; 2\,|f(t) - f_p(t)|_{\mathbb{R}}^2 + 2\left| f_p(t) - |v(t)|_{1/2}^2 \right|_{\mathbb{R}}^2 \\ &\leq \; 2\,|f(t) - f_p(t)|_{\mathbb{R}}^2 + 4\left\{ |v_p(t)|_{1/2}^2 + |v(t)|_{1/2}^2 \right\} |v_p(t) - v(t)|_{\mathbb{R}}^2 . \end{aligned}$$

Hence, from estimate (3.9) and inequality (2.7) we get

$$\left| f(t) - |v(t)|_{1/2}^2 \right|_{\mathbb{R}}^2 \leq 2\,|f(t) - f_p(t)|_{\mathbb{R}}^2 + c\lambda_0^{-1}\,|w_p(t)|_1^2 .$$

Thus, from convergence (3.13) and (3.16) we obtain

$$f(t) = |v(t)|_{1/2}^2 \quad \text{for all} \quad t \in [0,T]. \tag{3.20}$$

Analogously, we obtain

$$g(t) = (v(t), v'(t))_{1/2} \quad \text{for all} \quad t \in [0,T]. \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.15) we can conclude the existence of the function $v : [0,T] \to \mathcal{H}_0$ satisfying $(3.6)_1$ in $L^2([0,T]; \mathcal{H}_0)$.

As a consequence of the convergence

$$\left(v_p(t), v_p'(t)\right)_{1/2} \longrightarrow (v(t), v'(t))_{1/2} \quad \text{for all} \quad t \in [0,T],$$

the initial conditions $(3.6)_2$ hold. Therefore, the function $v : [0,T] \to \mathcal{H}_0$ is a solution of the Cauchy problem (3.6) in the sense of definition 3.1.

Uniqueness. If v_1 and v_2 are two solutions of the Cauchy problem (3.6) then the function $w = v_1 - v_2$ satisfies the problem

$$\begin{gathered} w''(t) + \lambda^2 w(t) + M\left(|v_1(t)|_{1/2}^2\right)\lambda w(t) + \\ \sigma\,(v_1(t), v_1'(t))_{1/2}\,\lambda w(t) + \nu\lambda^2 w'(t) = \\ \left\{ M\left(|v_2(t)|_{1/2}^2\right) - M\left(|v_1(t)|_{1/2}^2\right) \right\} \lambda v_2(t) + \\ \sigma\left\{ (v_1(t), v_1'(t))_{1/2} - (v_2(t), v_2'(t))_{1/2} \right\} \lambda v_2(t), \\ w(0) = w'(0) = 0. \end{gathered} \tag{3.22}$$

Taking the scalar product on $\mathcal{H}_0$ of w' with both sides of the equation $(3.22)_1$, observing that $\nu\,|w'(t)|_1^2 \geq 0$ for all $t \in [0,T]$, v and v' belong to $L^\infty(0,T;\mathcal{H}_1)$ and also using (2.7) and (3.2) yields

$$\begin{aligned}&\left| M\left(|v_2(t)|_{1/2}^2\right) - M\left(|v_1(t)|_{1/2}^2\right)\right|_{\mathbb{R}}\\ &\quad\leq\; c|w(t)|_{1/2} \leq c\lambda_0^{-1/2}|w(t)|_1 \quad \text{for all} \quad t \in [0,T].\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{|w'(t)|_0^2 + |w(t)|_1^2\right\}\\ &\leq\; \frac{1}{2}\Big(c_0 + \sigma c_1\Big)\left\{|w'(t)|_0^2 + |w(t)|_1^2\right\}\\ &+\; \frac{1}{2}c_2\left\{|w'(t)|_0^2 + \lambda_0^{-1}|w(t)|_1^2\right\} + \frac{1}{2}\left(\sigma c_3 + 1\right)|w'(t)|_0^2.\end{aligned}$$

Hence, integrating from 0 to $t \leq T$ yields

$$|w'(t)|_0^2 + |w(t)|_1^2 \leq c_4 \int_0^t \left\{|w'(s)|_0^2 + |w(s)|_1^2\right\} ds,$$

where $c_4 = \max\left\{c_0 + \sigma\left(c_1 + c_3\right) + c_2,\ \ c_0 + \sigma c_1 + c_2\lambda_0^{-1}\right\}$. From this and Gronwall's inequality we have $w = 0$ for all $t \in [0,T]$.

Global solutions. Let us justify that the function limit v is a solution of the Cauchy problem (3.6) for all t in $[0,\infty[$. In fact, we proved that the function limit v is a unique solution of the Cauchy problem (3.6) in the sense of $L^2(0,T;\mathcal{H}_0)$. It means that for all $\theta \in \mathcal{D}(0,T)$ and ϕ eigenvector of $\mathcal{A}$ the following equation holds

$$\begin{gathered}\int_0^T \left(v''(t),\phi\right)\theta(t)dt + \int_0^T \left(\lambda v(t),\lambda\phi\right)\theta(t)dt +\\ \nu\int_0^T \left(\lambda v(t),\lambda\phi'\right)\theta(t)dt + \int_0^T M\left(|v(t)|_{1/2}^2\right)\left(\lambda v(t),\phi\right)\theta(t)dt +\\ \int_0^T \sigma\left(\lambda v(t),v'(t)\right)\left(\lambda v(t),\phi\right)\theta(t)dt = 0.\end{gathered} \tag{3.23}$$

From estimate (3.9) we have that v belongs to $L^\infty(0,\infty;\mathcal{H}_1)$ and v' belongs to $L^\infty(0,\infty;\mathcal{H}_0)\cap L^2(0,\infty;\mathcal{H}_1)$. Thus, we obtain that

$$M\left(|v|^2_{1/2}\right)+\sigma(\lambda v,v') \quad \text{belongs to} \quad L^\infty(0,\infty),$$

and consequently

$$\left\{M\left(|v|^2_{1/2}\right)+\sigma(\lambda v,v')\right\}\lambda v \quad \text{belongs to} \quad L^\infty(0,\infty;\mathcal{H}_0).$$

Therefore, from (3.10), (3.23) and uniqueness of solutions we have that the equation $(3.6)_1$ is verified in $L^2(0,\infty;\mathcal{H}_0)$. This way we conclude that

$$v'' \quad \text{belongs to} \quad L^2(0,\infty;\mathcal{H}_0), \tag{3.24}$$

and from convergence $(3.14)_1$-$(3.14)_3$ we have that

$$\begin{aligned} v &\quad \text{belongs to} \quad C^0([0,\infty[;\mathcal{H}_1),\\ v' &\quad \text{belongs to} \quad C^0\left([0,\infty[;\mathcal{H}_{1/2}\right). \end{aligned} \tag{3.25}$$

Finally, as the operator $\mathcal{U}:\mathcal{D}\left(\mathcal{A}^\ell\right)\to\mathcal{H}_\ell$, for all $\ell\in\mathbb{R}$, is an isomorphism we conclude from (3.6), (3.24), (3.25) and uniqueness of solutions that the vector function $u:[0,\infty[\,\to H$ defined by $u=\mathcal{U}^{-1}(v)$ is the unique global solution of the Cauchy problem (3.1) in the sense of definition 3.1 in the class (3.4) and (3.5) ■

4 Asymptotic behavior

The aim of this section is to prove that the total energy associated with the solutions of the Cauchy problem (3.1) has exponential decay when the time t goes to $+\infty$.

For the sake of simplicity we will utilize for $(3.1)_1$ the representation

$$\begin{aligned} u''(t)+\left\{\zeta_0+\zeta_1|\mathcal{A}^{1/2}u(t)|^2+\sigma(\mathcal{A}u(t),u'(t))\right\}\mathcal{A}u(t)&\\ +\mathcal{A}^2u(t)+\nu\mathcal{A}^2u'(t)=0 \quad \text{for all} \quad t\geq 0.& \end{aligned} \tag{4.1}$$

The total energy of the system (4.1) is given, for all $t\geq 0$, by

$$E(t)=\frac{1}{2}\left\{\left|u'(t)\right|^2+\zeta_0\left|\mathcal{A}^{1/2}u(t)\right|^2+\frac{\zeta_1}{2}\left|\mathcal{A}^{1/2}u(t)\right|^4+\left|\mathcal{A}u(t)\right|^2\right\}. \tag{4.2}$$

Taking the scalar product on H of u' with both sides of (4.1) and observing that $\sigma\left(\mathcal{A}u(t), u'(t)\right)^2 \geq 0$ for all $t \geq 0$ we obtain

$$\frac{d}{dt}E(t) \leq -\nu \left|\mathcal{A}u'(t)\right|^2 \quad \text{for all} \quad t \geq 0. \tag{4.3}$$

Therefore, the energy $E(t)$ is not increasing. To obtain the asymptotic behavior of $E(t)$ we will use the method idealized by Haraux-Zuazua [14], see also, Komornik-Zuazua [20]. Thus, we can prove the following result

Theorem 4.1 *If u is the solution of the Cauchy problem (3.1), guaranteed by Theorem 3.1, then the energy $E(t)$ defined in (4.2) satisfies*

$$E(t) \leq \Lambda \exp(-\omega t) \quad \textit{for all} \quad t \geq 0, \tag{4.4}$$

where $\omega = \omega(\epsilon) > 0$, $\Lambda = 2E(0) + \epsilon F(0)$. The function $F(t)$ and the constants ϵ are defined by

$$F(t) = \frac{1}{2}\left\{\nu \left|\mathcal{A}u(t)\right|^2 + \frac{\sigma}{2}\left|\mathcal{A}^{1/2}u(t)\right|^4\right\}, \quad \epsilon = \min\left\{\frac{1}{\sqrt{c}},\ c\nu\right\}, \tag{4.5}$$

and $c > 0$ is the constant of the immersion of $D(\mathcal{A})$ into H.

Proof. For each $\epsilon > 0$ we consider the auxiliary function:

$$E_\epsilon(t) = E(t) + \frac{\epsilon}{2}\left(u'(t), u(t)\right) \quad \text{for all} \quad t \geq 0. \tag{4.6}$$

As $D(\mathcal{A})$ is continuously embedded in H we obtain after application of the usual inequalities that

$$\frac{\epsilon}{2}\left(u'(t), u(t)\right) \leq \frac{1}{4}\left|u'(t)\right| + \frac{\epsilon^2 c}{4}\left|\mathcal{A}u(t)\right|^2. \tag{4.7}$$

Combining this with the hypothesis $(4.5)_2$ we get

$$E_\epsilon(t) \leq \frac{3}{2}E(t) \quad \text{for all} \quad t \geq 0.$$

On the other hand, we get from (4.6) and (4.7) the following inequality

$$\begin{aligned} E_\epsilon(t) \quad &\geq \quad E(t) - \frac{\epsilon}{2}\left|\left(u'(t), u(t)\right)\right| \\ &\geq \quad \frac{1}{2}E(t) \quad \text{for all} \quad t \geq 0. \end{aligned}$$

Thus, the function E_ϵ satisfies both inequalities, namely

$$\frac{1}{2}E_\epsilon(t) \le E(t) \le \frac{3}{2}E_\epsilon(t) \quad \text{for all} \quad t \ge 0. \tag{4.8}$$

Now, differentiating the function E_ϵ with respect to t yields

$$E'_\epsilon(t) = E'(t) + \frac{\epsilon}{2}\left(u''(t), u(t)\right) + \frac{\epsilon}{2}\left|u'(t)\right|^2 \quad \text{for all} \quad t \ge 0.$$

Replacing u'' by $-\left\{\zeta_0 + \zeta_1|\mathcal{A}^{1/2}u|^2 + \sigma\left(\mathcal{A}u, u'\right)\right\}\mathcal{A}u - \mathcal{A}^2u - \nu\mathcal{A}^2u'$ in the second term of the right-hand side of the identity above and using the definition of the function F yields

$$E'_\epsilon(t) + \frac{\epsilon}{2}F'(t) =$$
$$E'(t) + \frac{\epsilon}{2}\left|u'(t)\right|^2 - \frac{\epsilon}{2}\left\{\zeta_0\left|\mathcal{A}^{1/2}u(t)\right|^2 + \frac{\zeta_1}{2}\left|\mathcal{A}^{1/2}u(t)\right|^4 + \left|\mathcal{A}u(t)\right|^2\right\},$$

for all $t \ge 0$. Hence, (4.3) and $(4.5)_2$ we obtain

$$E'_\epsilon(t) + \frac{\epsilon}{2}F'(t) \le -\frac{\epsilon}{2}E(t) \quad \text{for all} \quad t \ge 0. \tag{4.9}$$

Using the definitions of the functions E, E_ϵ and F it is easy to see that there exists a real positive constant $c_0 = c_0(\epsilon) > 0$ such that

$$E_\epsilon(t) + \frac{\epsilon}{2}F(t) \le c_0E(t) \quad \text{for all} \quad t \ge 0. \tag{4.10}$$

Thus, from inequalities (4.9) and (4.10) there exists a suitable real positive constant $\omega = \omega(\epsilon) > 0$ such that

$$E'_\epsilon(t) + \frac{\epsilon}{2}F'(t) - \omega\left\{E_\epsilon(t) + \frac{\epsilon}{2}F(t)\right\} \le 0 \quad \text{for all} \quad t \ge 0.$$

Therefore,

$$E_\epsilon(t) + \frac{\epsilon}{2}F(t) \le \left\{E_\epsilon(0) + \frac{\epsilon}{2}F(0)\right\}\exp\left(-\omega t\right) \quad \text{for all} \quad t \ge 0.$$

From this and inequality (4.8) we conclude that the inequality (4.4) holds. Thus, the demonstration of the Theorem 4.1 is completed ■

5 Comments & Applications

(I) The theorem 3.1 is still valid if we suppose the operator $\mathcal{A}$ satisfying the property $(\mathcal{A}u, u) \geq 0$ for all $u \in D(\mathcal{A})$ instead of ellipticity's propriety (2.3). In fact, in this case, we consider the operator $\mathcal{A}_\epsilon = \mathcal{A} + \epsilon I$, where I is the identity operator on the Hilbert space H, and ϵ is a suitable constant such that $0 < \epsilon \leq 1$. Under these conditions the operator $\mathcal{A}_\epsilon$ satisfies the hypotheses of the diagonalization theorem and the solution for the Cauchy problem (3.1) will be obtained as a limit of the family u_ϵ of the solutions to the Cauchy problem

$$
\begin{aligned}
&u_\epsilon''(t) + \left\{ M\left(|\mathcal{A}_\epsilon^{1/2} u_\epsilon(t)|^2 \right) + \sigma\left(\mathcal{A}_\epsilon u_\epsilon(t), u_\epsilon'(t) \right) \right\} \mathcal{A}_\epsilon u_\epsilon(t) \\
&\qquad + \mathcal{A}_\epsilon^2 u_\epsilon(t) + \nu \mathcal{A}_\epsilon^2 u_\epsilon'(t) = 0 \quad \text{for all} \quad t \geq 0, \\
&\qquad\qquad u_\epsilon(0) = u_0 \quad \text{and} \quad u_\epsilon'(0) = u_1.
\end{aligned}
\tag{5.1}
$$

The convergence of the family u_ϵ of the solutions of (5.1) to the function solution u of the Cauchy problem (3.1) is guaranteed by convergence (3.14) and identities (3.20) and (3.21). That is, from (3.14) and the Uniform boundedness theorem the following estimates hold

$$
\begin{aligned}
|u_\epsilon(t)|^2 &= |v_\epsilon(t)|^2 \leq \liminf_{p\to\infty} |v_{\epsilon p}(t)|^2 \leq c, \\
|u_\epsilon'(t)|^2 &= |v_\epsilon'(t)|^2 \leq \liminf_{p\to\infty} \left|v_{\epsilon p}'(t)\right|^2 \leq c, \\
|u_\epsilon''(t)|^2 &= |v_\epsilon''(t)|^2 \leq \liminf_{p\to\infty} \left|v_{\epsilon p}''(t)\right|^2 \leq c,
\end{aligned}
\tag{5.2}
$$

where $v_\epsilon(t) = \mathcal{U}\left(u_\epsilon(t)\right)$ and c is a positive constant independent of ϵ and t. Consequently, we have the same convergence of (3.14) for u_ϵ.

As an application of Theorem 3.1, Theorem 4.1 and the preceding commentary **(I)** we have the particular cases.

(II) Let Ω a smooth-bounded-open set of $\mathbb{R}^n$, $n \in \mathbb{N}$, with C^2 boundary Γ and $Q = \Omega \times [0, \infty[$. If $\mathcal{A}$ is the Laplace operator $-\Delta$ defined by the triplet $\left\{ H_0^1(\Omega),\, L^2(\Omega),\, ((,))_{H_0^1(\Omega)} \right\}$ we have as a consequence of Theorem 3.1 and

Theorem 4.2 that the mixed problem

$$u''(t) - \left\{ M\left(\int_\Omega |\nabla u(t)|^2 \, dx \right) + \sigma \int_\Omega \nabla u(t) \nabla u'(t) dx \right\} \Delta u(t)$$

$$+\Delta^2 u(t) + \nu \Delta^2 u'(t) = 0 \quad \text{in} \quad Q,$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega,$$

$$u(x,t) = u_x(x,t) = 0 \quad \text{on} \quad \Gamma \times]0,\infty[,$$

has a unique solution $u : [0,\infty[\to L^2(\Omega)$ and the energy associated with the solutions is asymptotically stable.

(III) Let $n \in \mathbb{N}$ and $\Omega = \mathbb{R}^n$ be. We consider in $L^2(\mathbb{R}^n)$ the Laplace operator $\mathcal{A} = -\Delta$ with domain $D(\mathcal{A}) = H^2(\mathbb{R}^n)$. Under these conditions there exists a unique global solution $u : [0,\infty[\to L^2(\mathbb{R}^n)$ of the Cauchy problem

$$u''(t) - \left\{ M\left(\int_{\mathbb{R}^n} |\nabla u(t)|^2 \, dx \right) + \sigma \int_{\mathbb{R}^n} \nabla u(t) \nabla u'(t) dx \right\} \Delta u(t) +$$

$$\Delta^2 u(t) + \nu \Delta^2 u'(t) = 0 \quad \text{in} \quad \mathbb{R}^n \times [0,\infty[,$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \quad \text{in} \quad \mathbb{R}^n,$$

and the total energy of the system decay exponentially.

(IV) Finally, let Ω^{C} the complement of Ω in $\mathbb{R}^n$, where Ω is a smooth-bounded set. In $L^2(\Omega^{\mathrm{C}})$ we consider the Laplace operator $\mathcal{A} = -\Delta$ with domain $D(\mathcal{A}) = H^2(\Omega^{\mathrm{C}})$. Thus, we obtain the same results obtained in the preceding applications for the Cauchy problem

$$u''(t) - \left\{ M\left(\int_{\Omega^{\mathrm{C}}} |\nabla u(t)|^2 \, dx \right) + \sigma \int_{\Omega^{\mathrm{C}}} \nabla u(t) \nabla u'(t) dx \right\} \Delta u(t) +$$

$$\Delta^2 u(t) + \nu \Delta^2 u'(t) = 0 \quad \text{in} \quad \Omega^{\mathrm{C}} \times [0,\infty[,$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega^{\mathrm{C}} \quad \blacksquare$$

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