METHOD OF THE QUASILINEARIZATION FOR NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH LINEAR BOUNDARY CONDITIONS

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Abstract

The method of quasilinearization for nonlinear impulsive differential equations with linear boundary conditions is studied. The boundary conditions include periodic boundary conditions. It is proved that the convergence is quadratic.

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1 Introduction

In this paper a boundary value problem (BVP) for impulsive differential equations with a family of linear two point boundary conditions is studied. An existence theorem is proved. An algorithm, based on methods of quasilinearization, for constructing successive approximations of the solution of the considered problem is given. The quadratic convergence of the iterates is proved. The obtained results are generalizations of the known results for initial value problems as well as boundary value problems for ordinary differential equations and impulsive differential equations.

The method of quasilinearization has recently been studied and extended extensively. It is generating a rich history beginning with the works by Bellman and Kalaba [1]. Lakshmikantham and Vatsala, and many co-authors have extensively developed the method and have applied the method to a wide range of problems. We refer the reader to the recent work by Lakshmikantham and Vatsala [9] and the extensive bibliography found there. The method has been applied to two-point boundary value problems for ordinary differential equations and we refer the reader to the papers, [2, 3, 4, 8, 10, 11, 12], for example.

Likewise impulsive equations have been generating a rich history. We refer the reader to the monograph by Lakshmikantham, Bainov, and Simeonov [6] for a thorough introduction to the material and an introduction to the literature. Methods of quasilinearization have been applied to impulsive differential equations with various initial or boundary conditions. We refer the reader to [9] for references and we refer the reader to [2, 3, 13] in our bibliography. In this paper, we consider a family of

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boundary value conditions that contain periodic boundary conditions. A quasilinearization method has been applied to problems with periodic boundary conditions, [8]; to our knowledge, this is the first application to impulsive problems with periodic boundary conditions.

2 Preliminary notes and definitions

Let the points $\tau_k \in (0,T), k = 1,2,...,p$ be such that

$$\tau_{k+1} > \tau_k, \quad k = 1,2,\ldots,p-1.$$

Consider the nonlinear impulsive differential equation (BVP)

$$x' = f(t,x(t)) \quad \text{for} \quad t \in [0,T], \quad t \neq \tau_k, \tag{1}$$

$$x(\tau_k + 0) = I_k(x(\tau_k)), \quad k = 1,2,\ldots,p \tag{2}$$

with the linear boundary value condition

$$Mx(0) - Nx(T) = c, \tag{3}$$

where $x \in \mathbf{R}$, $f : [0,T] \times \mathbf{R} \to \mathbf{R}$, $I_k : \mathbf{R} \to \mathbf{R}$, $(k = 1,2,\ldots,p)$, c, M, N are constants.

We consider the set $PC(X,Y)$ of all functions $u : X \to Y, (X \subset \mathbf{R}, Y \subset \mathbf{R})$ which are piecewise continuous in X with points of discontinuity of first kind at the points $\tau_k \in X$, i.e. there exist the limits $\lim_{t \downarrow \tau_k} u(t) = u(\tau_k + 0) < \infty$ and $\lim_{t \uparrow \tau_k} u(t) = u(\tau_k - 0) = u(\tau_k)$.

We consider the set $PC^1(X,Y)$ of all functions $u \in PC(X,Y)$ that are continuously differentiable for $t \in X, t \neq \tau_k$.

Definition 1 . *The function $\alpha(t) \in PC^1([0,T],\mathbf{R})$ is called a lower solution of the BVP (1)-(3), if the following inequalities are satisfied:*

$$\alpha'(t) \leq f(t,\alpha(t)) \quad \textit{for} \quad t \in [0,T], t \neq \tau_k, \tag{4}$$

$$\alpha(\tau_k + 0) \leq I_k(\alpha(\tau_k)), \quad k = 1,2,\ldots,p \tag{5}$$

$$M\alpha(0) - N\alpha(T) \leq c. \tag{6}$$

Definition 2 . *The function $\beta(t) \in PC^1([0,T],\mathbf{R})$ is called an upper solution of the BVP (1)-(3), if the inequalities (4), (5), (6) are satisfied in the opposite direction.*

Let the functions $\alpha, \beta \in PC([0,T],\mathbf{R})$ be such that $\alpha(t) \leq \beta(t)$.

Consider the sets:

$$S(\alpha,\beta) = \{u \in PC([0,T],\mathbf{R}) : \quad \alpha(t) \leq u(t) \leq \beta(t) \ \text{ for } \ t \in [0,T]\},$$

$$\Omega(\alpha,\beta) = \{(t,x) \in [0,T] \times \mathbf{R} : \quad \alpha(t) \leq x \leq \beta(t)\},$$

$$D_k(\alpha,\beta) = \{x \in \mathbf{R} : \alpha(\tau_k) \leq x \leq \beta(\tau_k)\}, k = 1,2,\ldots,p.$$

Consider the linear boundary value problem for the linear impulsive differential equation (LBVP)

$$u'(t) = g(t)u(t) + \sigma(t), \quad t \in [0,T],\ t \neq \tau_k \tag{7}$$
$$u(\tau_k + 0) = b_k u(\tau_k) + \gamma_k,\ k = 1,2,\dots,p \tag{8}$$
$$Mu(0) - Nu(T) = 0. \tag{9}$$

Using the results for the initial value problem for the linear impulsive differential equation (7),(8) (Corollary 1.6.1 [6]) we can easily prove the following existence result for the LBVP (7), (8), (9) and obtain the formula for the solution.

Lemma 2.1 . *Let the functions $g, \sigma \in PC([0,T],\mathbf{R})$ and $M, N, b_k, \gamma_k, (k = 1,2,\dots p)$ be constants such that $N(\prod_{k=1}^{p} b_k) exp\Big(\int_0^T g(s)ds\Big) \neq M$.*

Then the LBVP (7), (8), (9) has a unique solution $u(t)$ on the interval $[0,T]$, where

$$\begin{aligned} u(t) &= u(0)\Big(\prod_{0<\tau_k<t} b_j\Big) exp(\int_0^t g(\tau)d\tau) \\ &\quad + \sum_{0<\tau_k<t} \gamma_k \Big(\prod_{\tau_k<\tau_j<t} b_j\Big) exp(\int_{\tau_k}^t g(\tau)d\tau)ds \\ &\quad + \int_0^t \sigma(s)\Big(\prod_{s<\tau_k<t} b_k\Big) exp(\int_s^t g(\tau)d\tau)ds, \end{aligned}$$

$$\tau_0 = 0, \quad b_0 = 1, \quad \prod_{j=k}^{n} f(j) = 1 \ for\ k > n,$$

$$\begin{aligned} u(0) &= \Big[1 - \frac{N}{M}\Big(\prod_{k=1}^{p} b_k\Big) exp\Big(\int_0^T g(s)ds\Big)\Big]^{-1} \\ &\quad \times\Big\{\sum_{i=1}^{p} \gamma_i \Big(\prod_{j=i+1}^{p} b_j\Big) exp(\int_{\tau_i}^T g(\tau)d\tau)ds \\ &\quad + \int_0^T \sigma(s)\Big(\prod_{s<\tau_j<T} b_j\Big) exp(\int_s^T g(\tau)d\tau)\Big\}. \end{aligned}$$

We will need the following results for differential inequalities.

Lemma 2.2 *(Theorem 1.4.1 [6]). Let the following conditions be satisfied:*
1. $u, g, \sigma \in PC([0,T],\mathbf{R})$.
2. The function $m \in PC^1[\mathbf{R}_+,\mathbf{R}]$ and

$$m'(t) \leq p(t)m(t) + q(t), \quad t \in [0,T],\ t \neq \tau_k$$
$$m(\tau_k + 0) \leq d_k u(\tau_k) + b_k,\ k = 1,2,\dots,p,$$

where $d_k,\ b_k\ \ (k=1,2,\dots,p)$ *are constants,* $d_k \ge 0,\ p,q \in C[\mathbf{R}_+,\mathbf{R}]$.

Then for $t \ge 0$ *the inequality*

$$\begin{aligned} m(t) \ \le\ & m(0)\Big(\prod_{0<\tau_k<t} d_j\Big) exp\Big(\int_0^t p(\tau)d\tau\Big) \\ & + \sum_{0<\tau_k<t} b_k\Big(\prod_{\tau_k<\tau_j<t} d_j\Big) exp\Big(\int_{\tau_k}^t p(\tau)d\tau\Big)ds \\ & + \int_0^t \Big(\prod_{s<\tau_k<t} d_k\Big) exp\Big(\int_s^t p(\tau)d\tau\Big)q(s)ds \end{aligned}$$

is valid.

In the proof of the main results we will use the following comparison result.

Lemma 2.3 . *Assume that the function* $m \in PC^1([0,T],\mathbf{R})$ *satisfies the inequalities*

$$m'(t) \le \phi(t)m(t), \qquad t \in [0,T], t \ne t_k, \tag{10}$$

$$m(t_k+0) \le \alpha_k m(t_k), \quad k=1,2,\dots,p \tag{11}$$

$$Mm(0) - Nm(T) \le 0, \tag{12}$$

where $\alpha_k \ge 0, M > 0, N \ge 0$ *are constants such that*

$$M - N\Big(\prod_{k=1}^{p} \alpha_k\Big) exp\Big(\int_0^T \phi(s)ds\Big) > 0. \tag{13}$$

Then $m(t) \le 0$ *for* $t \in [0,T]$.

Proof: According to Lemma 2.2 the function $m(t)$ satisfies the inequality

$$m(t) \le m(0)\Big(\prod_{k:0<t_k<t} \alpha_k\Big) exp\Big(\int_0^t \phi(s)ds\Big) \ \text{ for } \ t \in [0,T]. \tag{14}$$

From inequality (12) we have

$$m(0) \le \frac{N}{M} m(T)$$

and therefore

$$m(0) \le \frac{N}{M} m(0)\Big(\prod_{k=1}^{p} \alpha_k\Big) exp\Big(\int_0^T \phi(s)ds\Big). \tag{15}$$

From the inequalities (13) and (15) it follows that $m(0) \le 0$. Therefore according to (14) the inequality $m(t) \le 0$ holds for $t \in [0,T]$.

As a partial case of Lemma 2.3 we obtain the following result:

Corollary 1 . *Let the function $m \in PC^1([0,T],\mathbf{R})$ satisfies the inequalities (10) - (12), where $\int_0^T \phi(s)ds \le 0$, $0 \le \alpha_k < 1$ and $M > 0,\ N \ge 0,\ M \ge N$.*

Then the function $m(t)$ is nonpositive on the interval $[0,T]$.

3 Main Results

We will obtain sufficient conditions for existence of a solution of the BVP (1)-(3). The obtained result will be useful not only for the proof of the method of quasilinearization but for different qualitative investigation of nonlinear boundary value problem for impulsive differential equations.

Theorem 3.1 . *Let the following conditions be fulfilled:*
1. The functions $\alpha, \beta \in PC^1([0,T],\mathbf{R})$ are lower and upper solutions of the BVP (1)-(3) and $\alpha(t) \le \beta(t)$ for $t \in [0,T]$.
2. The function $f \in C(\Omega(\alpha,\beta),\mathbf{R})$.
3. The functions $I_k : D_k(\alpha,\beta) \to \mathbf{R}, (k = 1,2,\dots)$ are nondecreasing in $D_k(\alpha,\beta)$.
4. The constants $M > 0, N \ge 0$.

Then the BVP (1)-(3) has a solution $u \in S(\alpha,\beta)$.

Proof: Without loss of generality we will consider the case when $p = 1$, i.e. $0 < t_1 < T$. Let x_0 be an arbitrary point such that $\alpha(0) \le x_0 \le \beta(0)$. Define a function $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ by the equality

$$F(t,x) = \begin{cases} f(t,\beta(t)) + \frac{\beta(t)-x}{1+|x|} & \text{for } x > \beta(t) \\ f(t,x) & \text{for } \alpha(t) \le x \le \beta(t) \\ f(t,\alpha(t)) + \frac{\alpha(t)-x}{1+|x|} & \text{for } x < \alpha(t). \end{cases}$$

From the condition 2 of the Theorem 3.1 it follows that the function $f(t,x)$ is bounded on $S(\alpha,\beta)$ and therefore there exists a function $\mu \in C([0,T],[0,\infty))$ such that $sup\{|F(t,x)| : x \in \mathbf{R}\} \le \mu(t)$ for $t \in [0,T]$.

Therefore, the initial value problem for the ordinary differential equation $x' = F(t,x),\ \ x(0) = x_0$ has a solution $X(t;x_0)$ for $t \in [0,t_1]$.

Consider the function $m(t) = X(t;x_0) - \beta(t)$. We will prove that the function $m(t)$ is non-positive on $[0,t_1]$. Assume the opposite, i.e. $sup\{m(t) : t \in [0,t_1]\} > 0$. Therefore, there exists a point $t^* \in (0,t_1)$ such that $m(t^*) > 0$ and $m'(t^*) \ge 0$. From the definition of the function $X(t;x_0)$ it also follows that

$$m'(t^*) \le f(t^*,\beta(t^*)) + \frac{\beta(t^*) - X(t^*;x_0)}{1 + |X(t^*;x_0)|} - f(t^*,\beta(t^*)) = \frac{-m(t^*)}{1 + |X(t^*;x_0)|} < 0.$$

According to the obtained contradiction, the assumption is not true. Therefore,

$$X(t;x_0) \leq \beta(t), \quad t \in [0,t_1].$$

Analogously, we can prove that $X(t;x_0) \geq \alpha(t)$, $t \in [0,t_1]$.

Let $y_0 = I_1(X(t_1;x_0))$. We note that y_0 depends on x_0. From the monotonicity of the function $I_1(x)$ we obtain

$$\alpha(t_1+0) \quad \leq \quad I_1(\alpha(t_1)) \leq I_1(X(t_1;x_0)) \leq I_1(\beta(t_1)) \leq \beta(t_1+0),$$

i.e.

$$\alpha(t_1+0) \leq y_0 \leq \beta(t_1+0).$$

Consider the initial value problem for the ordinary differential equation $x' = F(t,x)$, $x(t_1) = y_0$ for $t \in [t_1,T]$. This initial value problem has a solution $Y(t;y_0)$ for $t \in [t_1,T]$. Using the same ideas as above we can prove that the inequalities $\alpha(t) \leq Y(t;y_0) \leq \beta(t)$ for $t \in [t_1,T]$ hold. At the same time $Y(t_1;y_0) = I_1(X(t_1;x_0))$.

Define the function

$$x(t;x_0) = \begin{cases} X(t;x_0) & \text{for} \quad t \in [0,t_1] \\ Y(t;y_0) & \text{for} \quad t \in (t_1,T]. \end{cases}$$

The function $x(t;x_0) \in S(\alpha,\beta)$ is a solution of the impulsive differential equation (1), (2) with the initial condition $x(0) = x_0$.

From the inequality $\alpha(t) \leq \beta(t)$ for $t \in [0,T]$ it follows that the following two cases are possible:

Case 1. Let $\alpha(0) = \beta(0)$. Then $x_0 = \alpha(0) = \beta(0)$. Therefore

$$Mx(0;x_0) - Nx(T;x_0) = Mx_0 - Nx(T;x_0) \leq M\alpha(0) - N\alpha(T) \leq c$$

and

$$Mx(0;x_0) - Nx(T;x_0) \geq Mx_0 - N\beta(T) \geq c.$$

Therefore $Mx(0;x_0) - Nx(T;x_0)) = c$, i.e. the function $x(t;x_0)$ is a solution of the BVP (1)-(3).

Case 2. Let $\alpha(0) < \beta(0)$. We will prove that there exists a point $x_0 \in [\alpha(0),\beta(0)]$ such that the solution $x(t;x_0)$ of the impulsive differential equation (1), (2) with initial condition $x(0) = x_0$ satisfies the boundary condition (3). Assume the opposite, i.e. for every point $x_0 \in [\alpha(0),\beta(0)]$ the inequality $Mx(0;x_0) - Nx(T;x_0)) \neq c$ holds, where $x(t;x_0)$ is the solution of the impulsive equation (1),(2).

If $x_0 = \beta(0)$ then from the relation $x(t;x_0) \in S(\alpha,\beta)$ we obtain that

$$Mx(0;x_0) - Nx(T;x_0)) = M\beta(0) - Nx(T;x_0) \geq M\beta(0) - N\beta(T) \geq c.$$

According to the assumption and the above inequality we obtain

$$Mx(0;x_0) - Nx(T;x_0) > c. \tag{16}$$

Then there exists a number $\delta : 0 < \delta < \beta(0) - \alpha(0)$, such that for $x_0 : 0 \leq \beta(0) - x_0 < \delta$ the corresponding solution $x(t;x_0)$ of the impulsive differential equation (1), (2) satisfies the inequality

$$Mx(0;x_0) - Nx(T;x_0) > c. \tag{17}$$

Indeed, assume that for every natural number n there exists a point $z_n : 0 \leq \beta(0) - z_n < \frac{1}{n}$ such that the corresponding solution $x^{(n)}(t;z_n)$ of the impulsive equation (1),(2) with the initial condition $x(0) = z_n$ satisfies the inequality

$$Mx^{(n)}(0;z_n) - Nx^{(n)}(T;z_n) < c.$$

Let $\{z_{n_j}\}$ is a subsequence such that $lim_{j\to\infty} z_{n_j} = \beta(0)$ and $lim_{j\to\infty} x^{(n_j)}(t;z_{n_j}) = x(t)$ uniformly on the intervals $[0,t_1]$ and $(t_1,T]$. The function $x(t)$ is a solution of the impulsive differential equation (1), (2) such that $x(0) = \beta(0)$, $x(t) \in S(\alpha,\beta)$ and

$$Mx(0) - Nx(T) \leq c. \tag{18}$$

The inequality (18) contradicts the inequality (16) and therefore the assumption is not true.

Let

$$\delta^* = \quad sup\{\delta \in (0, \beta(0) - \alpha(0)] : \textit{ for which there exists a point } x_0 \in (\beta(0) - \delta, \beta(0)] \\ \textit{such that the solution } \; x(t;x_0) \textit{ satisfies the inequality (17)}\}.$$

Choose a sequence of points $x_n \in (\alpha(0), \beta(0) - \delta^*)$ such that $\lim_{n\to\infty} x_n = \beta(0) - \delta^*$. From the choice of δ^* and the assumption it follows that the corresponding solutions $x^{(n)}(t;x_n)$ satisfy the inequality

$$Mx^{(n)}(0;x_n) - Nx^{(n)}(T;x_n) < c.$$

There exists a subsequence $\{x_{n_j}\}_0^\infty$ of the sequence $\{x_n\}_0^\infty$ such that

$lim_{j\to\infty}\ x^{(n_j)}(t;x_{n_j}) = x^*(t)$ uniformly on the intervals $[0,t_1]$ and $(t_1,T]$. The function $x^*(t) \in S(\alpha,\beta)$ is a solution of the impulsive equation (1), (2) with the initial condition $x(0) = \beta(0) - \delta^*$ and satisfies the inequality $Mx^*(0) - Nx^*(T) \leq c$. The last inequality contradicts the choice of δ^*.

Therefore, there exists a point $x_0 \in [\alpha(0), \beta(0)]$ such that the solution $x(t;x_0)$ of the impulsive differential equation (1), (2) satisfies the condition (3), i.e. the function $x(t;x_0)$ is a solution of the BVP (1), (2), (3). This completes the proof of Theorem 3.1.

We will construct the method of quasilinearization to approximate the solution of the BVP (1), (2), (3). We will prove that the convergence of the successive approximations is quadratic.

Theorem 3.2 *Let the following conditions hold:*
1. The functions $\alpha_0(t), \beta_0(t)$ *are lower and upper solutions of the BVP (1), (2), (3) and* $\alpha_0(t) \le \beta_0(t)$ *for* $t \in [0, T]$.
2. The function $f \in C^{0,2}(\Omega(\alpha_0, \beta_0), \mathbf{R})$ *and there exist two functions* $F, g \in C^{0,2}(\Omega(\alpha_0, \beta_0), \mathbf{R})$ *such that* $F(t, x) = f(t, x) + g(t, x)$, $F''_{xx}(t, x) \ge 0$, $g''_{xx}(t, x) \ge 0$,

$$\int_0^T [F'_x(s, \beta_0(s)) - g'_x(s, \alpha_0(s))]ds < 0.$$

3. The functions $I_k \in C^2(D_k(\alpha_0, \beta_0), \mathbf{R}), k = 1, 2, \ldots, p$, *and there exist functions* $G_k, J_k \in C^2(D_k(\alpha_0, \beta_0), \mathbf{R})$ *such that* $G_k(x) = I_k(x) + J_k(x)$, $G''_k(x) \ge 0$, $J''_k(x) \ge 0$,

$$G'_k(\beta_0(\tau_k)) - J'_k(\alpha_0(\tau_k)) < 1,$$

$$G'_k(\alpha_0(\tau_k)) - J'_k(\beta_0(\tau_k)) \ge 0.$$

4. The constants $M > 0, N \ge 0, M \ge N$.
Then there exist two sequences of functions $\{\alpha_n(t)\}_0^\infty$ *and* $\{\beta_n(t)\}_0^\infty$ *such that:*
a. The sequences are increasing and decreasing respectively.
b. The functions $\alpha_n(t)$ *are lower solutions and the functions* $\beta_n(t)$ *are upper solutions of the BVP (1), (2), (3).*
c. Both sequences are uniformly convergent on the intervals $(\tau_k, \tau_{k+1}]$, *for* $k = 0, 1, 2, \ldots, p$, *to the unique solution of the BVP (1), (2), (3) in* $S(\alpha_0, \beta_0)$.
d. The convergence is quadratic.

Proof: From the condition 2 of Theorem 3.2 it follows that if $(t, x_1), (t, x_2) \in \Omega(\alpha_0, \beta_0)$ and $x_1 \ge x_2$ then

$$f(t, x_1) \ \ge \ f(t, x_2) + F'_x(t, x_2)(x_1 - x_2) + g(t, x_2) - g(t, x_1), \tag{19}$$

$$g(t, x_1) \ \ge \ g(t, x_2) + g'_x(t, x_2)(x_1 - x_2). \tag{20}$$

From the condition 3 of Theorem 3.2 it follows that if $x_1 \ge x_2$, $x_1, x_2 \in D_k(\alpha_0, \beta_0)$, then

$$I_k(x_1) \ge I_k(x_2) + G'_k(x_2)(x_1 - x_2) + J_k(x_2) - J_k(x_1), \tag{21}$$

and

$$G_k(x_1) \ge G_k(x_2) + G'_k(x_2)(x_1 - x_2). \tag{22}$$

From the condition 3 it follows that the functions $G'_k(x)$ and $J'_k(x)$ are nondecreasing in $D_k(\alpha_o, \beta_0)$. Therefore for $x \in D_k(\alpha_0, \beta_0)$ the inequality $I'_k(x) = G'_k(x) - J'_k(x) \ge$

$G_k'(\alpha_0(\tau_k)) - J_k'(\beta_0(\tau_k)) \geq 0$ holds, which proves that the functions $I_k(x)$ are nondecreasing, $k = 1, 2, \ldots, p$.

According to Theorem 3.1 the BVP (1), (2), (3) has a solution in $S(\alpha_0, \beta_0)$.

We consider the linear boundary value problem for the impulsive linear differential equation (LBVP)

$$
\begin{aligned}
& x'(t) = f(t, \alpha_0(t)) + Q_0(t)(x - \alpha_0(t)) \quad \text{for} \quad t \in [0, T], t \neq \tau_k, && (23)\\
& x(\tau_k + 0) = I_k(\alpha_0(\tau_k)) + B_k^0[x(\tau_k) - \alpha_0(\tau_0)], && (24)\\
& Mx(0) - Nx(T) = c, && (25)
\end{aligned}
$$

where

$$
\begin{aligned}
& Q_0(t) = F_x'(t, \alpha_0(t)) - g_x'(t, \beta_0(t)),\\
& B_k^0 = G_k'(\alpha_0(\tau_k)) - J_k'(\beta_0(\tau_k)), \quad k = 1, 2, \ldots, p.
\end{aligned}
$$

It is easy to verify that the function $\alpha_0(t)$ is a lower solution of the LBVP (23), (24), (25).

According to the condition 1 of Theorem 3.2, inequalities (19) and (21) we obtain the inequalities

$$
\begin{aligned}
\beta_0'(t) \quad \geq \quad & f(t, \alpha_0(t)) + Q_0(t)(\beta_0(t) - \alpha_0(t))\\
& -[F(t, \alpha_0(t)) - F(t, \beta_0(t)) + F_x'(t, \alpha_0(t))(\beta_0(t) - \alpha_0(t))]\\
& +g(t, \alpha_0(t)) - g(t, \beta_0(t)) + g_x^{'}(t, \beta_0(t))(\alpha_0(t) - \beta_0(t))\\
& \geq f(t, \alpha_0(t)) + Q_0(t)(\beta_0(t) - \alpha_0(t)) \quad \text{for} \quad t \in [0, T], \ t \neq \tau_k, \quad (26)\\
\beta_0(\tau_k + 0) \quad \geq \quad & I_k(\alpha_0(\tau_k)) + [I_k(\beta_0(\tau_k)) - I_k(\alpha_0(\tau_k))]\\
& \geq I_k(\alpha_0(\tau_k)) + [G_k'(\alpha_0(\tau_k)) - J_k'(\beta_0(\tau_k))](\beta_0(\tau_k) - \alpha_0(\tau_k))\\
& \geq I_k(\alpha_0(\tau_k)) + B_k^0(\beta_0(\tau_k) - \alpha_0(\tau_k)). \quad (27)
\end{aligned}
$$

From the inequalities (26), (27) it follows that the function $\beta_0(t)$ is an upper solution of the LBVP (23), (24), (25).

According to the Lemma 2.1 the LBVP (23), (24), (25) has a unique solution $\alpha_1(t) \in S(\alpha_0, \beta_0)$.

We consider the linear boundary value problem for the impulsive linear differential equation (LBVP)

$$
\begin{aligned}
& x'(t) = f(t, \beta_0(t)) + Q_0(t)(x(t) - \beta_0(t)) \quad \text{for} \quad t \in [0, T], t \neq \tau_k, && (28)\\
& x(\tau_k + 0) = I_k(\beta_0(\tau_k)) + B_k^0(x(\tau_k) - \beta_0(\tau_k)), && (29)\\
& Mx(0) - Mx(T) = c. && (30)
\end{aligned}
$$

The functions $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of the LBVP (28), (29), (30) and according to Lemma 2.1 there exists a unique solution $\beta_1(t) \in S(\alpha_0, \beta_0)$.

We will prove that $\alpha_1(t) \le \beta_1(t)$ for $t \in [0,T]$.

Define the function $u(t) = \alpha_1(t) - \beta_1(t)$ for $t \in [0,T]$. From the choice of the functions $\alpha_1(t)$ and $\beta_1(t)$ and the inequality (20) we obtain that the function $u(t)$ satisfies the inequalities

$$\begin{aligned} u' &= f(t,\alpha_0(t)) - f(t,\beta_0(t)) + Q_0(t)u(t) + Q_0(t)(\beta_0(t) - \alpha_0(t)) \\ &\le Q_0(t)u(t) \quad \text{for} \quad t \in [0,T], t \neq \tau_k. \end{aligned} \tag{31}$$

According to the inequality (21) for $x_2 = \beta_0(t_k)$ and $x_1 = \alpha_0(t_k)$ and the definition of the functions α_1, β_1 we obtain

$$\begin{aligned} u(\tau_k + 0) \quad &\le I_k(\alpha_0(\tau_k)) - I_k(\beta_0(\tau_k)) + B_k^0 u(\tau_k) + B_k^0[\beta_0(\tau_k) - \alpha_0(\tau_k)] \\ &\le B_k^0 u(\tau_k). \end{aligned} \tag{32}$$

From the boundary value condition for the functions α_1, β_1 and the condition 4 we obtain the inequality

$$Mu(0) - Nu(T) = M\alpha_1(0) - N\alpha_1(T) - (M\beta_1(0) - N\beta_1(T)) = c - c = 0. \tag{33}$$

From the inequalities (31), (32) and boundary condition (33), according to Lemma 2.3, the function $u(t)$ is non-positive, i.e. $\alpha_1(t) \le \beta_1(t)$.

The function $\alpha_1(t)$ is a lower solution of the BVP (1), (2), (3). Indeed, for $t \in [0,T], t \neq \tau_k$,

$$\begin{aligned} \alpha_1' \;\le\; & f(t,\alpha_1(t)) + F_x'(t,\alpha_0(t))(\alpha_0(t) - \alpha_1(t)) \\ & -g(t,\alpha_0(t)) + g(t,\alpha_1(t)) + Q_0(t)(\alpha_1(t) - \alpha_0(t)) \\ & \le f(t,\alpha_1(t)). \end{aligned} \tag{34}$$

From the inequality (21) and the choice of the function $\alpha_1(t)$ we obtain the inequalities

$$\begin{aligned} \alpha_1(\tau_k + 0) \;\le\; & I_k(\alpha_1(\tau_k)) + [G_k'(\alpha_0(\tau_k)) - J_k'(\alpha_1(\tau_k)) \\ & -B_k^0](\alpha_0(\tau_k) - \alpha_1(\tau_k)) \\ \le\; & I_k(\alpha_1(\tau_k)) - [J_k'(\alpha_0(\tau_k)) - J_k'(\beta_0(\tau_k))](\alpha_0(\tau_k) - \alpha_1(\tau_k)) \\ \le\; & I_k(\alpha_1(\tau_k)), \quad k = 1, 2, \ldots, p. \end{aligned} \tag{35}$$

From the inequalities (34), (35) and the boundary condition for the function $\alpha_1(t)$ it follows that the function $\alpha_1(t)$ is a lower solution of the BVP (1), (2), (3).

Analogously, it can be proved that the function $\beta_1(t)$ is an upper solution of the BVP (1), (2), (3).

By this way we can construct two sequences of functions $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$, $\alpha_n, \beta_n \in S(\alpha_{n-1}, \beta_{n-1})$. The function $\alpha_{n+1}(t)$ is the unique solution of the linear boundary value problem for the impulsive linear differential equation (LBVP)

$$x'(t) = f(t, \alpha_n(t)) + Q_n(t)(x - \alpha_n(t)) \text{ for } \ t \in [0, T], t \neq \tau_k, \tag{36}$$
$$x(\tau_k + 0) = I_k(\alpha_n(\tau_k)) + B_k^n(x(\tau_k) - \alpha_n(\tau_k)), \tag{37}$$
$$Mx(0) - Nx(T) = c \tag{38}$$

and the function $\beta_{n+1}(t)$ is the unique solution of the linear boundary value problem for the impulsive linear differential equation (LBVP)

$$x'(t) = f(t, \beta_n(t)) + Q_n(t)(x - \beta_n(t)) \text{ for } \ t \in [0, T], t \neq \tau_k, \tag{39}$$
$$x(\tau_k + 0) = I_k(\beta_n(\tau_k)) + B_k^n(x(\tau_k) - \beta_n(\tau_k)), \tag{40}$$
$$Mx(0) - Nx(T) = c, \tag{41}$$

where

$$Q_n(t) = F'_x(t, \alpha_n(t)) - g'_x(t, \beta_n(t)),$$
$$B_k^n = G'_k(\alpha_n(\tau_k)) - J'_k(\beta_n(\tau_k)).$$

As in the case $n = 0$ it can be proved that the functions $\alpha_{n+1}(t)$ and $\beta_{n+1}(t)$ are lower and upper solutions of the BVP (1), (2), (3) and the inequalities

$$\alpha_0(t) \le \alpha_1(t) \le \ldots \le \alpha_n(t) \le \beta_n(t) \le \ldots \le \beta_0(t) \tag{42}$$

hold.

Therefore, the sequences $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ are uniformly bounded and equi-continuous on the intervals $(\tau_k, \tau_{k+1}], k = 0, 1, 2, \ldots, p$ and they are uniformly convergent.

Denote

$$\lim_{n\to\infty} \alpha_n(t) = u(t), \qquad \lim_{n\to\infty} \beta_n(t) = v(t).$$

From the uniform convergence and the definition of the functions $\alpha_n(t)$ and $\beta_n(t)$ it follows that

$$\alpha_0(t) \le u(t) \le v(t) \le \beta_0(t). \tag{43}$$

From the LBPVPs (36)-(38) and (39)-(41) we obtain that the functions $u(t)$ and $v(t)$ are solutions of the BVP (1), (2), (3) in $S(\alpha_0, \beta_0)$ and therefore $u(t) = v(t)$.

We will prove the convergence is quadratic.

Define the functions $a_{n+1}(t) = u(t) - \alpha_{n+1}(t)$ and $b_{n+1}(t) = \beta_{n+1}(t) - u(t), \ \ t \in [0, T]$. For $t \in [0, T], t \neq \tau_k$ we obtain the inequalities

$$\begin{aligned} a'_{n+1} \ &\le \ Q_n(t)a_{n+1}(t) + [F'_x(t, u(t)) - g'_x(t, \alpha_n(t)) - Q_n(t)]a_n(t) \\ &= \ Q_n(t)a_{n+1}(t) + F''_{xx}(t, \xi_1)a_n^2(t) \\ &\quad +g''_{xx}(t, \eta_1)a_n(t)(\beta_n(t) - \alpha_n(t)) \end{aligned} \tag{44}$$

where $u(t) \le \xi_1 \le \alpha_n(t)$, $\alpha_n(t) \le \eta_1 \le \beta_n(t)$.

It is easy to verify that the inequality

$$a_n(t)(\beta_n(t)-\alpha_n(t)) = a_n(t)(b_n(t)+a_n(t)) \leq \frac{1}{2}b_n^2(t)+\frac{3}{2}a_n^2(t). \tag{45}$$

From the inequalities (44) and (45) it follows that for $t \in [0,T], t \neq \tau_k$ the inequality

$$a'_{n+1}(t) \leq Q_n(t)a_{n+1}(t)+\sigma_n(t), \tag{46}$$

holds, where

$$\sigma_n(t) \;=\; [F''_{xx}(t,\xi_1)+\frac{3}{2}g''_{xx}(t,\eta_1)]a_n^2+\frac{1}{2}g''_{xx}(t,\eta_1)b_n^2.$$

Analogously, it can be proved that

$$a_{n+1}(\tau_k+0) \leq B_k^n a_{n+1}(\tau_k)+\gamma_k, \tag{47}$$

where

$$\gamma_k = [G''_k(\omega_k)+\frac{3}{2}J''_k(\nu_k)]a_n^2(\tau_k)+\frac{1}{2}J''_k(\nu_k)b_n^2(\tau_k),$$
$$\alpha_n(\tau_k) \leq \omega_k \leq u(\tau_k), \alpha_n(\tau_k) \leq \kappa_k \leq \beta_n(\tau_k), k=1,2,\ldots,p.$$

From the boundary conditions for the functions $u(t)$ and $\alpha_n(t)$ we obtain the equality

$$Ma_{n+1}(0)-Na_{n+1}(T)=0. \tag{48}$$

From the inequalities (46), (47) according to Lemma 2.2 it follows that the function $a_{n+1}(t)$ satisfies the estimate

$$\begin{aligned} a_{n+1}(t) \;\leq\; & a_{n+1}(0)\Big(\prod_{0<\tau_k<t} B_k^n\Big)exp\Big(\int_0^t Q_n(\tau)d\tau\Big) \\ & +\sum_{0<\tau_k<t}\gamma_k\Big(\prod_{\tau_k<\tau_j<t} B_j^n\Big)exp\Big(\int_{\tau_k}^t Q_n(\tau)d\tau\Big)ds \\ & +\int_0^t \sigma_n(s)\Big(\prod_{s<\tau_k<t} B_k^n\Big)exp\Big(\int_s^t Q_n(\tau)d\tau\Big)ds. \end{aligned} \tag{49}$$

From the boundary condition (48) we have $a_{n+1}(0)=\frac{N}{M}a_{n+1}(T)$ and therefore

$$\begin{aligned} a_{n+1}(0) \;\leq\; & \Big[1-\frac{N}{M}\Big(\prod_{k=1}^{p} B_k^n\Big)exp\Big(\int_0^T Q_n(s)ds\Big)\Big]^{-1} \\ & \times\Big\{\sum_{i=0}^{p}\gamma_i\Big(\prod_{j=i+1}^{p} B_j^n exp\Big(\int_{\tau_i}^T Q_n(\tau)d\tau\Big)ds \\ & +\int_0^T \sigma_n(s)\Big(\prod_{s<\tau_j<T} B_j^n\Big)exp\Big(\int_s^T Q_n(\tau)d\tau\Big). \end{aligned} \tag{50}$$

From the properties of the functions $F(t,x)$ and $g(t,x)$, the definition of $\sigma_n(t)$ and the inequalities (49), (50) it follows that there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$||a_{n+1}|| \leq \lambda_1 ||a_n||^2 + \lambda_2 ||b_n||^2. \tag{51}$$

Analogously, it can be proved that there exists constants $\mu_1 > 0$ and $\mu_2 > 0$ such that

$$||b_{n+1}|| \leq \mu_1 ||b_n||^2 + \mu_2 ||a_n||^2. \tag{52}$$

The inequalities (51) and (52) prove that the convergence is quadratic.

Remark 1 *In the case when $N = 0$ the BVP (1), (2), (3) is reduced to an initial value problem for impulsive differential equations for which the quasilinearization is applied in [9].*

In the case when $M = 1, N = 1, c = 0$ the BVP (1), (2), (3) is reduced to the periodic boundary value problem for an impulsive differential equations.

We also note that some of the results for ordinary differential equations, obtained in [5, 7, 8, 9] are partial cases of the obtained results when $I_k(x) = x$.

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