Blowup Estimates for a Mutualistic Model in Ecology *

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Abstract. The cooperating two-species Lotka-Volterra model is discussed. We study the blowup properties of the solution to a parabolic system with homogeneous Dirichlet boundary conditions. The upper and lower bounds of blowup rate are obtained.

Key words: reaction diffusion system, blowup estimates, upper and lower bounds.

AMS subject classifications: 35K15, 35K65.

1 Introduction and main results

The well-known Lotka-Volterra ecological model, which involves a coupled system of two ordinary differential equations, has been given an enormous attention in the past decades. When the effect of dispersion is taken into consideration the densities u, v of the species are governed by

$$\begin{cases} u_t - d_1\Delta u = u(a_1 - b_1 u - c_1 v), & x \in \Omega,\ t > 0, \\ v_t - d_2\Delta v = v(a_2 - b_2 u - c_2 v), & x \in \Omega,\ t > 0, \\ u(x) = v(x) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where Δ is the Laplacian operator, Ω is a bounded domain in $\mathcal{R}^N$ with $\partial\Omega$ uniformly $C^{2+\alpha}$-smooth, $u_0(x)$ and $v_0(x)$ are nonnegative smooth functions with

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$u_0(x) = v_0(x) = 0$ on $\partial\Omega$. d_i, a_i, b_i and c_i $(i = 1, 2)$ are positive constants. d_i represents its respective diffusion rate and the real number a_i, its net birth rate. b_1 and c_2 are the coefficients of intra-specific competitions and b_2, c_1 are that of inter-specific competitions. Here we consider the case with homogeneous Dirichlet boundary conditions, which implies that the habitat is surrounded by a totally hostile environment.

If the presence of one species encourages the growth of the other species then the system (1.1) becomes so-called mutualistic model:

$$\begin{cases} u_t - d_1\Delta u = u(a_1 - b_1 u + c_1 v), & x \in \Omega,\ t > 0, \\ v_t - d_2\Delta v = v(a_2 + b_2 u - c_2 v), & x \in \Omega,\ t > 0, \\ u(x) = v(x) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{\Omega}. \end{cases} \tag{1.2}$$

Because of the quasimonotone nondecreasing of reaction functions in (1.2), there is a quite different behavior of solutions compared with the solutions of (1.1). The solution of (1.1) with any nonnegative initial data is unique and global, while the blowup solutions are possible when the two species are strongly mutualistic ($b_2c_1 > b_1c_2$), which means that the geometric mean of the interaction coefficients exceeds that of population regulation coefficients. Here we give only the related result of Pao [20].

Theorem 1.1 *(i) If $b_2c_1 < b_1c_2$, the problem (1.2) has a unique global solution (u, v), which is uniformly bounded in $[0,\infty) \times \overline{\Omega}$;*

(ii) If $b_2c_1 > b_1c_2$ and $a_1 \geq \lambda_1, a_2 \geq \lambda_2$, there exists a finite time T such that the unique solution to (1.2) exists in $[0, T) \times \overline{\Omega}$ and blows up in the meaning that $\lim_{t\to T} \max(|u(x,t)| + |v(x,t)|) = \infty$;

(iii) If $b_2c_1 > b_1c_2$, the solution will blow up for any $a_1 \geq 0$ and $a_2 \geq 0$ under suitable initial data.

Based on the above result, we are chiefly interested in studying the blowup properties of the solution. We derive the upper and lower bounds of blowup rate, that is, there are positive constants c and C such that

$$c(T-t)^{-1} \leq \max_{\overline{\Omega}} u(x,t) \leq C(T-t)^{-1}, \quad c(T-t)^{-1} \leq \max_{\overline{\Omega}} v(x,t) \leq C(T-t)^{-1}$$

for $t \in (0, T)$ if $N = 1$.

There are some related results on the blowup of solutions to nonlinear parabolic systems, see for example [19] and [24]. In a recent paper, Lou etc. in [18] considered (1.2) with homogeneous Neumann boundary conditions and gave a sufficient condition on the initial data for the solution to blow up in finite time. For the blowup estimates, as we know, no result has been given owing to the cross-coupled reactions.

For the related elliptic systems, there is an extensive literature regarding the existence and uniqueness of positive solutions, the reader can see [1, 10, 12, 14, 15, 16, 17, 20, 23] and the references therein.

The paper is arranged as follows. In §2 the comparison principles for bounded and unbounded domains are given. In §3, we derive the lower bound of blowup rate and §4 deals with its upper bound.

2 Comparison principles

In this section, we show the comparison principle for unbounded domains, which will be used in the sequel. For completeness, we also give the comparison principle for bounded domains.

Lemma 2.1 *Let Ω be a bounded domain with smooth boundary $\partial\Omega$. $u_i, v_i \in C(\overline{\Omega}\times[0,T))\bigcap C^{2,1}(\Omega\times(0,T))$ (i=1,2) and satisfy*

$$\begin{cases} u_{1t}-d_1\Delta u_1 \geq u_1(a_1-b_1u_1+c_1v_1), & x\in\Omega,\ t>0,\\ v_{1t}-d_2\Delta v_1 \geq v_1(a_2+b_2u_1-c_2v_1), & x\in\Omega,\ t>0,\\ u_{2t}-d_1\Delta u_2 \leq u_2(a_1-b_1u_2+c_1v_2), & x\in\Omega,\ t>0,\\ v_{2t}-d_2\Delta v_2 \leq v_2(a_2+b_2u_2-c_2v_2), & x\in\Omega,\ t>0,\\ u_1(x,t)\geq u_2(x,t),\quad v_1(x,t)\geq v_2(x,t), & x\in\partial\Omega,\ t>0,\\ u_1(x,0)\geq u_2(x,0),\quad v_1(x,0)\geq v_2(x,0), & x\in\overline{\Omega}. \end{cases} \tag{2.1}$$

Then $u_1(x,t)\geq u_2(x,t)$ and $v_1(x,t)\geq v_2(x,t)$ in $\overline{\Omega}\times[0,T)$. Moreover, if $u_2(x,0)\not\equiv u_1(x,0)\geq u_2(x,0)$ and $v_2(x,0)\not\equiv v_1(x,0)\geq v_2(x,0)$, then $u_1(x,t)>u_2(x,t)$ and $v_1(x,t)>v_2(x,t)$ in $\Omega\times(0,T)$.

Lemma 2.2 *Let Ω_u be a unbounded domain with boundary $\partial\Omega\in C^{2+\alpha}$. $u_i, v_i\in C(\overline{\Omega}_u\times[0,T))\bigcap C^{2,1}(\Omega_u\times(0,T))$ (i=1,2) and satisfy*

$$\begin{cases} u_{1t}-d_1\Delta u_1 \geq u_1(a_1-b_1u_1+c_1v_1), & x\in\Omega_u,\ t>0,\\ v_{1t}-d_2\Delta v_1 \geq v_1(a_2+b_2u_1-c_2v_1), & x\in\Omega_u,\ t>0,\\ u_{2t}-d_1\Delta u_2 \leq u_2(a_1-b_1u_2+c_1v_2), & x\in\Omega_u,\ t>0,\\ v_{2t}-d_2\Delta v_2 \leq v_2(a_2+b_2u_2-c_2v_2), & x\in\Omega_u,\ t>0,\\ u_1(x,t)\geq u_2(x,t),\quad v_1(x,t)\geq v_2(x,t), & x\in\partial\Omega_u,\ t>0,\\ u_1(x,0)\geq u_2(x,0),\quad v_1(x,0)\geq v_2(x,0), & x\in\overline{\Omega}_u \end{cases} \tag{2.2}$$

and there exist positive constants A and γ such that

$$\begin{array}{l} |u_i(x,t)| \leq A\exp(\gamma|x|^2) \\ |v_i(x,t)| \leq A\exp(\gamma|x|^2) \end{array} \quad as \quad |x| \to \infty \ (0 < t < T). \tag{2.3}$$

Then $u_1(x,t) \geq u_2(x,t)$ and $v_1(x,t) \geq v_2(x,t)$ in $\overline{\Omega} \times [0,T)$.

Lemma 2.1 is followed by the strong Maximum principle and Lemma 2.2 is followed by the Phragman-Lindelöf principle (see [21], [22]).

Remark 2.1 *When $\Omega = \mathcal{R}^N$, the boundary inequality in 2.2 is redundant. The condition in 2.3 is called the growth condition.*

Remark 2.2 *Since $(0,0)$ is unique solution of (1.2) with $u(x,0) \equiv 0$ and $v(x,0) \equiv 0$. Lemma 2.1 implies that if (u,v) be the nonnegative solution of (1.2), then $u, v \equiv 0$ or $u, v > 0$ in $\Omega \times (0,T)$.*

Remark 2.3 *Lemmas 2.1 and 2.2 hold for the more general case. For example, for the system*

$$\begin{cases} u_t - d_1\Delta u = f(x,t,u,v), & x \in \Omega,\ t > 0, \\ v_t - d_2\Delta v = g(x,t,u,v), & x \in \Omega,\ t > 0, \end{cases} \tag{2.4}$$

Lemmas 2.1 and 2.2 hold if f, g are quasi-monotone nondecreasing, i.e. f is nondecreasing with respect to the component of v and g is nondecreasing with respect to the component of u, see [21] in detail.

3 Lower blowup estimate

We first establish the relationship between u and v as the solution (u,v) of (1.2) near the blow-up time.

Lemma 3.1 *Let (u,v) be the nonnegative solution of (1.2), which blows up at $t = T$. Then there exists δ such that*

$$\delta \max_{\overline{\Omega}\times[0,t]} v(x,\tau) \leq \max_{\overline{\Omega}\times[0,t]} u(x,\tau) \leq \frac{1}{\delta} \max_{\overline{\Omega}\times[0,t]} v(x,\tau), \quad t \in (T/2, T). \tag{3.1}$$

Proof: Let

$$U(t) = \max_{\overline{\Omega}\times[0,t]} u(x,\tau), \quad V(t) = \max_{\overline{\Omega}\times[0,t]} v(x,\tau).$$

As in [2] or [3], we argue by contradiction. Without loss of generality we may assume that there exists a sequence $\{t_n\}$ with $t_n \to T$ as $n \to \infty$ such that

$$V(t_n)U^{-1}(t_n) \to 0. \tag{3.2}$$

For each t_n, there exists

$$(\hat{x}_n, \hat{t}_n) \in \Omega \times (0, t_n] \text{ such that } u(\hat{x}_n, \hat{t}_n) = U(t_n). \tag{3.3}$$

Since (u, v) blows up, we have that $U(t_n) \to \infty$ as $t_n \to T$ and $\hat{t}_n \to T$ as $n \to \infty$. Let d_n denote the distant of $\hat{x}_n$ to $\partial\Omega$. Similarly as in [4], we distinguish two cases:

$$(i) \quad \limsup_{n\to\infty} \frac{d_n}{\lambda_n} = \infty \quad \text{and} \quad (ii) \quad \limsup_{n\to\infty} \frac{d_n}{\lambda_n} < \infty.$$

Case (i) Choose a subsequence (denoted again by $\{t_n\}$) such that

$$\lim_{n\to\infty} \frac{d_n}{\lambda_n} = \infty.$$

We now introduce the scaling argument inspired by [9]. Let

$$\begin{aligned} \lambda_n &:= \lambda(t_n) := U^{-1/2}(t_n), & (3.4)\\ \phi^{\lambda_n}(y,s) &:= \lambda_n^2 u(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n),\ (y,s) \in \overline{\Omega_n} \times I_n(T), & (3.5)\\ \psi^{\lambda_n}(y,s) &:= \lambda_n^2 v(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n),\ (y,s) \in \overline{\Omega_n} \times I_n(T), & (3.6) \end{aligned}$$

where

$$I_n(t) := (-\lambda_n^{-2}\hat{t}_n, \lambda_n^{-2}(t - \hat{t}_n)), \quad \Omega_n := \{y : \lambda_n y + \hat{x}_n \in \Omega\}.$$

Clearly, $\lambda_n \to 0$ as $n \to \infty$ and $(\phi^{\lambda_n}, \psi^{\lambda_n})$ solves

$$\begin{aligned} \phi_s - d_1\Delta\phi &= \phi(a_1\lambda_n^2 - b_1\phi + c_1\psi), \qquad && y \in \Omega_n,\ s \in I_n(T),\\ \psi_s - d_2\Delta\psi &= \psi(a_2\lambda_n^2 + b_2\phi - c_2\psi), && y \in \Omega_n,\ s \in I_n(T) \end{aligned}$$

and satisfies

$$\begin{aligned} \phi^{\lambda_n}(0,0) &= 1, \\ 0 \le \phi^{\lambda_n} &\le 1, && y \in \Omega_n,\ s \in (-\lambda_n^{-2}\hat{t}_n, 0],\\ 0 \le \psi^{\lambda_n} &\le V(t_n)U^{-1}(t_n), && y \in \Omega_n,\ s \in (-\lambda_n^{-2}\hat{t}_n, 0]. \end{aligned}$$

It follows from the parabolic estimates [11] that there is a $\mu \in (0,1)$ such that for any $K > 0$,

$$||\phi^{\lambda_n}||_{C^{2+\mu,1+\mu/2}(\overline{\Omega}_n \cap |y| \le K \times [-K,0])} \le C_K,$$

$$||\psi^{\lambda_n}||_{C^{2+\mu,1+\mu/2}(\overline{\Omega}_n\cap|y|\le K\times[-K,0])} \le C_K,$$

where the constant C_K is independent of n. Hence, we obtain a sequence converging to a solution (ϕ,ψ) of

$$\begin{aligned}
\phi_s - d_1\Delta\phi &= \phi(-b_1\phi + c_1\psi), \qquad y\in\mathcal{R}^N,\ s\in(-\infty,0], &(3.7)\\
\psi_s - d_2\Delta\psi &= \psi(b_2\phi - c_2\psi), \qquad y\in\mathcal{R}^N,\ s\in(-\infty,0] &(3.8)
\end{aligned}$$

such that $\phi(0,0)=1$ and $\phi\le 1, \psi\equiv 0$, which leads to a contradiction. In fact, ϕ achieves its maximum at $(0,0)$; therefore $[\phi_s - d_1\Delta\phi](0,0)\ge 0$, but $[\phi(-b_1\phi+c_1\psi)](0,0) = -b_1 < 0$. This proves (3.1) in Case (i).

Case (ii) Choose a subsequence (denoted again by $\{t_n\}$) such that

$$\lim_{n\to\infty}\frac{d_n}{\lambda_n} = c \ge 0.$$

Let $\tilde{x}_n\in\partial\Omega$ such that $d_n = |\hat{x}_n - \tilde{x}_n|$ and let R_n be an orthonormal transformation in $\mathcal{R}^N$ that maps $-e_1 := (-1,0,\cdots,0)$ onto the outer normal vector to $\partial\Omega$ at $\tilde{x}_n$. We now introduce the new scaling. Let

$$\begin{aligned}
\phi^{\lambda_n}(y,s) &:= \lambda_n^2 u(\lambda_n R_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n),\quad (y,s)\in\overline{\Omega_n}\times I_n(T), &(3.9)\\
\psi^{\lambda_n}(y,s) &:= \lambda_n^2 v(\lambda_n R_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n),\quad (y,s)\in\overline{\Omega_n}\times I_n(T), &(3.10)
\end{aligned}$$

where

$$I_n(t) := (-\lambda_n^{-2}\hat{t}_n, \lambda_n^{-2}(t-\hat{t}_n)),\quad \Omega_n := \{y : \lambda_n R_n y + \hat{x}_n\in\Omega\}.$$

Clearly, $\lambda_n\to 0$ as $n\to\infty$, Ω_n approaches the halfspace $H_c = \{y_1 > -c\}$ as $n\to\infty$ and $(\phi^{\lambda_n},\psi^{\lambda_n})$ solves

$$\begin{aligned}
\phi_s - d_1\Delta\phi &= \phi(a_1\lambda_n^2 - b_1\phi + c_1\psi), \qquad y\in\Omega_n,\ s\in I_n(T),\\
\psi_s - d_2\Delta\psi &= \psi(a_2\lambda_n^2 + b_2\phi - c_2\psi), \qquad y\in\Omega_n,\ s\in I_n(T),\\
\phi = \psi &= 0, \qquad y\in\partial\Omega_n,\ s\in I_n(T)
\end{aligned}$$

and satisfies

$$\begin{aligned}
\phi^{\lambda_n}(0,0) &= 1,\\
0\le\phi^{\lambda_n} &\le 1,\quad y\in\Omega_n,\ s\in(-\lambda_n^{-2}\hat{t}_n,0],\\
0\le\psi^{\lambda_n} &\le V(t_n)U^{-1}(t_n),\quad y\in\Omega_n,\ s\in(-\lambda_n^{-2}\hat{t}_n,0].
\end{aligned}$$

Noticing that $\partial\Omega$ is of $C^{2+\alpha}$, then uniform Schauder's estimates for $\phi^{\lambda_n},\psi^{\lambda_n}$ yield a subsequence converging to a solution (ϕ,ψ) of

$$\begin{aligned}
\phi_s - d_1\Delta\phi &= \phi(-b_1\phi + c_1\psi), \qquad y\in H_c,\ s\in(-\infty,0], &(3.11)\\
\psi_s - d_2\Delta\psi &= \psi(b_2\phi - c_2\psi), \qquad y\in H_c,\ s\in(-\infty,0], &(3.12)\\
\phi = \psi &= 0, \qquad y_1 = -c,\ s\in(-\infty,0] &(3.13)
\end{aligned}$$

such that $\phi(0,0) = 1$ and $\phi \le 1, \psi \equiv 0$, which leads to a contradiction as in Case (i). This prove (3.1) in Case (ii).

Remark 3.1 *We claim from (3.1) that u and v blow up at the same finite time T if (u,v) solves (1.2), that is*

$$\lim_{t\to T} \sup u(x,t) = \lim_{t\to T} \sup v(x,t) = \infty.$$

Now we first give the lower bound of the blowup rate using the integral equation.

Theorem 3.1 *Let (u,v) be the nonnegative solution of (1.2), which blows up at $t = T$. Then there exists a constant c such that*

$$\max_{\overline{\Omega}\times[0,t]} u(x,\tau) \ge c(T-t)^{-1}, \quad 0 < t < T,$$

$$\max_{\overline{\Omega}\times[0,t]} v(x,\tau) \ge c(T-t)^{-1}, \quad 0 < t < T.$$

Proof: Let $G_i(x,t;y,\tau)(i = 1,2)$ be the Green's function of the parabolic operator $(\partial/\partial t - d_i\Delta)$ in the bounded domain $\Omega\times(0,T]$ under the homogeneous Dirichlet boundary condition on $\partial\Omega\times(0,T]$. Then we have the representation formula of (1.2):

$$\begin{aligned}
u(x,t) &= \int_\Omega G_1(x,t;y,z)u(y,z)dy \\
&\quad + \int_z^t \int_\Omega u(a_1 - b_1u + c_1v)G_1(x,t;y,\tau)dyd\tau, \\
v(x,t) &= \int_\Omega G_2(x,t;y,z)v(y,z)dy \\
&\quad + \int_z^t \int_\Omega v(a_2 + b_2u - c_2v)G_2(x,t;y,\tau)dyd\tau
\end{aligned}$$

for $0 < z < t < T$ and $x \in \overline{\Omega}$.

Noticing that $\int_\Omega G_i(x,t;y,\tau)dy \le 1$ and the relationship (3.1), we have

$$\begin{aligned}
U(t) &\le U(z) + \int_z^t U(a_1 + b_1U + c_1V)(\tau)d\tau \\
&\le U(z) + (t-z)U(a_1 + b_1U + c_1V)(t) \\
&\le U(z) + (T-z)U(a_1 + b_1U + c_1V)(t) \\
&\le U(z) + (T-t)U(a_1 + b_1U + \frac{c_1}{\delta}U)(t), \\
V(t) &\le V(z) + (T-t)V(a_2 + \frac{b_2}{\delta}V + c_2V)(t).
\end{aligned}$$

Next we use the argument as in [9]. By assumption, T is the blowup time, so $U(t) \to +\infty$ as $t \to T^-$. Then we can choose $z < t < T$ such that $U(t) = 2U(z)$, and hence the above inequality for U becomes

$$2U(z) \leq U(z) + (T - z)2U(a_1 + 2b_1 U + 2\frac{c_1}{\delta}U)(z),$$

which implies that

$$(T - z) \geq (4a_1)^{-1} \quad \text{or} \quad U(z) \geq (2b_1 + 2\frac{c_1}{\delta})^{-1}(T - z)^{-1}, \quad 0 < z < T.$$

Take c such that $c \leq (2b_1 + 2\frac{c_1}{\delta})^{-1}$ and $c \leq \frac{1}{4a_1} \max_\Omega u_0$; then

$$U(t) \geq c(T - t)^{-1}, \quad 0 < t < T.$$

The proof for $V(t)$ is similar.

4 Upper blowup estimate

For the upper bound of the blowup rate, we assume that $b_2c_1 > b_1c_2$ and $N = 1$. The former assumption $b_2c_1 > b_1c_2$ is the sufficient condition for the solution of (1.2) to have a finite time blowup, see Theorem 1.1 and the latter $N = 1$ is restriction for the solution of the related scalar problem to blow up in a finite time, see Lemma 4.3.

Theorem 4.1 *Let (u, v) be the nonnegative solution of (1.2), which blows up at $t = T$. If $b_2c_1 > b_1c_2$ and $N = 1$, then there exists a constant C such that*

$$\max_{\overline{\Omega}\times[0,t]} u(x, \tau) \leq C(T - t)^{-1}, \quad 0 < t < T,$$

$$\max_{\overline{\Omega}\times[0,t]} v(x, \tau) \leq C(T - t)^{-1}, \quad 0 < t < T.$$

Proof: From Lemma 3.1 we only need to prove that $U(t) \leq C(T - t)^{-1}$. We use a scaling argument inspired by [8]. Noticing that $U(t) \to \infty$ as $t \to T$, for any given $t_0 \in (\frac{T}{2}, T)$ we can define

$$t_0^+ := t^+(t_0) := \max\{t \in (t_0, T) : U(t) = 2U(t_0)\}.$$

Choose $\lambda_0 = \lambda(t_0) = U^{-1/2}(t_0)$ as before. We claim that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \leq D, \quad t_0 \in (\frac{T}{2}, T), \tag{4.1}$$

where the constant D depends only N (it is independent of t_0).

Suppose that (4.1) is not true, then there exists $t_n \to T$ such that

$$\lambda_n^{-2}(t_n)(t_n^+ - t_n) \to \infty.$$

For each t_n, choose $(\hat{x}_n, \hat{t}_n)$ as in (3.3) and let d_n denote the distant of $\hat{x}_n$ to $\partial\Omega$. Similarly as in [4], we distinguish two cases:

$$(i) \quad \limsup_{n\to\infty} \frac{d_n}{\lambda_n} = \infty \quad \text{and} \quad (ii) \quad \limsup_{n\to\infty} \frac{d_n}{\lambda_n} < \infty.$$

Case (i) Choose a subsequence (denoted again by $\{t_n\}$) such that

$$\lim_{n\to\infty} \frac{d_n}{\lambda_n} = \infty.$$

We introduce the scaling functions as before. Let

$$\begin{aligned}
\lambda_n &:= \lambda(t_n) := U^{-1/2}(t_n), & (4.2)\\
\phi^{\lambda_n}(y,s) &:= \lambda_n^2 u(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n), \quad (y,s) \in \overline{\Omega_n} \times I_n(T), & (4.3)\\
\psi^{\lambda_n}(y,s) &:= \lambda_n^2 v(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n), \quad (y,s) \in \overline{\Omega_n} \times I_n(T), & (4.4)
\end{aligned}$$

where

$$I_n(t) := (-\lambda_n^{-2}\hat{t}_n, \lambda_n^{-2}(t - \hat{t}_n)), \quad \Omega_n := \{y : \lambda_n y + \hat{x}_n \in \Omega\}.$$

Clearly, $(\phi^{\lambda_n}, \psi^{\lambda_n})$ has a sequence converging to a solution (ϕ, ψ) of

$$\begin{aligned}
\phi_s - d_1\Delta\phi &= \phi(-b_1\phi + c_1\psi), \qquad y \in \mathcal{R}^N,\ s \in (-\infty, \infty), & (4.5)\\
\psi_s - d_2\Delta\psi &= \psi(b_2\phi - c_2\psi), \qquad y \in \mathcal{R}^N,\ s \in (-\infty, \infty) & (4.6)
\end{aligned}$$

such that $\phi(0,0) = 1$ and $\phi \le 1, \psi \le \frac{1}{\delta}$. Moreover, since that ϕ achieves its maximum at $(0,0)$, ψ must be nontrivial as in Lemma 3.1. Therefore ϕ and ψ are nontrivial nonnegative bounded functions, which leads to a contradiction to the following Theorem 4.2 if $N \le 2$. This prove (4.1) in Case (i).

For the Case (ii), it is easy to show as in Case (i) that there is nontrivial nonnegative solution (ϕ, ψ) of

$$\begin{aligned}
\phi_s - d_1\Delta\phi &= \phi(-b_1\phi + c_1\psi), \qquad y \in H_c,\ s \in (-\infty, \infty), & (4.7)\\
\psi_s - d_2\Delta\psi &= \psi(b_2\phi - c_2\psi), \qquad y \in H_c,\ s \in (-\infty, \infty), & (4.8)\\
\phi = \psi &= 0, \qquad y_1 = -c,\ s \in (-\infty, \infty) & (4.9)
\end{aligned}$$

such that $\phi(0,0) = 1$ and $\phi \le 1, \psi \le \frac{1}{\delta}$, which leads to a contradiction to Theorem 4.3 if $N = 1$. This prove (4.1) in Case (ii). Thus (4.1) is established. Step 3 of proof of Theorem 2.1 in [8] shows that (4.1) implies that $U(t) \le C(T-t)^{-1}$ for $0 \le t < T$.

Theorem 4.2 *If $b_2c_1 > b_1c_2$ and $N \leq 2$, then any nontrivial nonnegative solution of*

$$\begin{cases} u_t - d_1\Delta u = u(-b_1u + c_1v), & x \in \mathcal{R}^N,\ t > 0, \\ v_t - d_2\Delta v = v(b_2u - c_2v), & x \in \mathcal{R}^N,\ t > 0, \\ u(x,0) \geq 0, \quad v(x,0) \geq 0, & x \in \mathcal{R}^N, \\ u(x,0),\ v(x,0) \in L^\infty(\mathcal{R}^N) \end{cases} \tag{4.10}$$

is nonglobal.

To prove Theorem 4.2, it suffices to find a lower solution of (4.10) that blows up at a finite time T_0. First we show the following three useful Lemmas:

Lemma 4.1 *Any nontrivial nonnegative solution of (4.10) is positive for $t > 0$.*

Proof: If there exist $x_0 \in R^N$ and $t_0 > 0$ such that $u(x_0, t_0) = 0$, then there exist $R > 0$ and T_1 with $t_0 < T_1 < T$ such that $(x_0, t_0) \in B_R \times (0, T_1)$ and $u(x,t) \not\equiv 0$ in $\overline{B}_R \times [0, T_1]$. Now let $B = b_1 \max_{\overline{B}_R \times [0,T_1]} u(x,t)$ and define the function

$$w(x,t) = u(x,t)e^{Bt}.$$

We find from a straightforward computation that

$$\begin{cases} w_t - d_1\Delta w = w[-b_1u + c_1v + B] \geq 0, & x \in B_R,\ 0 < t \leq T_1, \\ w(x,0) \geq 0, & x \in B_R. \end{cases}$$

It follows form the strong maximum principle that $w \equiv 0$ in $B_R \times [0, T_1]$ or $w > 0$ in $B_R \times (0, T]$. It leads to a contradiction. So $u(x,t) > 0$ for $t > 0$ and also $v(x,t) > 0$ for $t > 0$ similarly.

Lemma 4.2 *Let $w(x,t)$ be a nontrivial nonnegative solution of*

$$\begin{cases} dw_t - \Delta w = bw^2, & x \in \mathcal{R}^N,\ t > 0, \\ w(x,0) \geq 0, & x \in \mathcal{R}^N, \\ w(x,0) \in L^\infty(\mathcal{R}^N). \end{cases} \tag{4.11}$$

(i) If $\Delta w(x,0) + bw^2(x,0) \geq 0$, then $w_t(x,t) \geq 0$ in $R^n \times (0,T)$;

(ii) If $w(x,0)$ is radially symmetric, then $w(x,t)$ is radial. Moreover, if $\frac{\partial w(r,0)}{\partial r} \leq 0$ for $r \geq 0$, then $\frac{\partial w(r,t)}{\partial r} \leq 0$ for $r \geq 0, t > 0$, where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$.

Proof: Since w satisfy the growth condition, using the comparison principle (see Lemma 2.2 for the system) and the assumptions on $w(x,0)$ yield $w(x,t) \geq w(x,0)$ in $R^N \times (0,T)$. Using again the comparison principle gives that $w(x, t+\varepsilon) \geq w(x,t)$ in $R^N \times (0, T-\varepsilon)$ for $\varepsilon > 0$ arbitrarily small. Hence $w_t(x,t) \geq 0$ in $R^N \times (0,T)$.

The result that the solution is radial follows by the uniqueness and the rotation invariance of problem (4.11) in the case that $w(x,0)$ is radial. Furthermore, if the initial data $w(x,0)$ is radially nonincreasing, then the solution $w(x,t)$ is also radially nonincreasing.

Lemma 4.3 *All nontrivial nonnegative solutions of*

$$\begin{cases} dw_t - \Delta w = bw^2, & x \in \mathcal{R}^N,\ t > 0, \\ w(x,0) \geq 0, & x \in \mathcal{R}^N \end{cases} \tag{4.12}$$

are nonglobal if $N \leq 2$; all nontrivial nonnegative solutions of

$$\begin{cases} dw_t - \Delta w = bw^2, & x \in H_c,\ t > 0, \\ w(x,t) = 0, & x_1 = 0,\ t > 0, \\ w(x,0) \geq 0, & x \in H_c \end{cases} \tag{4.13}$$

are nonglobal if $N = 1$, where $H_c := \{x_1 > -c\}$.

The former blowup result is followed from the well-known result of the general case $w_t - \Delta w = w^p$ shown in [6] for $1 < p < 1 + \frac{2}{N}$ and [7] for $p = 1 + \frac{2}{N}$, the latter is followed from the result of the general case $w_t - \Delta w = w^p$ shown in [13] for $1 < p \leq 1 + \frac{2}{N+1}$.

Proof of Theorem 4.2 We look for a lower solution $(\underline{u}, \underline{v})$ of (4.10) such that $(\underline{u}, \underline{v})$ blow up in finite time. Let $(\underline{u}, \underline{v}) = (\delta_1 w, \delta_2 w)$, where δ_1 and δ_2 are some positive constants to be chosen later and w is a nonnegative function in $\overline{\Omega} \times (0, T_0)$ and unbounded in Ω at some $T_0 < +\infty$. From Lemma 2.2, $(\underline{u}, \underline{v})$ is a lower solution of (4.10) in $\overline{\Omega} \times [0, T_0)$ if

$$w_t - d_1 \Delta w \leq w(-b_1\delta_1 w + c_1 \delta_2 w), \qquad \mathcal{R}^N \times (0, T_0), \tag{4.14}$$

$$w_t - d_2 \Delta w \leq w(b_2\delta_2 w - c_2 \delta_2 w), \qquad \mathcal{R}^N \times (0, T_0), \tag{4.15}$$

$$\delta_1 w(x,0) \leq u(x,0), \quad \delta_2 w(x,0) \leq v(x,0), \quad x \in \mathcal{R}^N. \tag{4.16}$$

Since $b_2 c_1 > b_1 c_2$, choose δ_1, δ_2 as in [20] such that $c_2/b_2 < \delta_1/\delta_2 < c_1/b_1$ and set

$$d = \max\{d_1^{-1},\ d_2^{-1}\},$$
$$b = \min\{(c_1\delta_2 - b_1\delta_1)/d_1,\ (b_2\delta_1 - c_2\delta_2)/d_2\}.$$

Then $d, b > 0$ and (4.14), (4.15) hold if

$$d_1^{-1} w_t - \Delta w \leq bw^2,$$
$$d_2^{-1} w_t - \Delta w \leq bw^2.$$

By choosing w as the solution of the scalar problem

$$dw_t - \Delta w = bw^2, \tag{4.17}$$

(4.14), (4.15) hold provided that $w_t \geq 0$.

Now for arbitrary nontrivial nonnegative solution (u, v) of (4.10), by Lemma 4.1, the solution is positive for $t > 0$. Without loss of generality, we may assume that $u(x, 0) > 0$ and $v(x, 0) > 0$ for $x \in R^N$, otherwise replace the initial function $(u(x, 0), v(x, 0))$ by $(u(x, t_1), v(x, t_1))$ for $t_1 > 0$. Since the initial data is positive, there exists a radially symmetric, radially nondecreasing function $\psi(x)$ such that

$$\begin{array}{ll} \delta_1\psi(x) \leq u(x, 0), \quad \delta_2\psi(x) \leq v(x, 0), & x \in \mathcal{R}^N, \\ \Delta\psi(x) + b\psi^2(x) \geq 0, & x \in \mathcal{R}^N \end{array}$$

and define w^* be the solution of (4.17) when $w(x, 0) = \psi(x)$. By Lemma 4.2, w^* is monotone nondecreasing in t. Moreover, w^* is radially symmetric, radially nondecreasing and therefore satisfies the growth condition. It follows from comparison principle Lemma 2.2 that $u(x, t) \geq \delta_1 w^*(x, t)$ and $v(x, t) \geq \delta_2 w^*(x, t)$ in $R^N \times [0, T_0)$. Hence $(\underline{u}, \underline{v}) = (\delta_1 w^*, \delta_2 w^*)$ is a lower solution of (4.10).

On the other hand, Lemma 4.3 ensures the existence of a finite T_0 such that the solution w^* exists in $R^N \times [0, T_0)$ and is unbounded in R^N as $t \to T_0$ if $N \leq 2$. Thus the solution of (4.10) cannot exist beyond T_0 and is nonglobal.

Theorem 4.3 *If $b_2c_1 > b_1c_2$ and $N = 1$, then any nontrivial nonnegative solution of*

$$\left\{ \begin{array}{ll} u_t - d_1\Delta u = u(-b_1u + c_1v), & x \in H_c, \ t > 0, \\ v_t - d_2\Delta v = v(b_2u - c_2v), & x \in H_c, \ t > 0, \\ u(x, t) = 0, \quad v(x, t) = 0, & x_1 = -c, \ t > 0, \\ u(x, 0) \geq 0, \quad v(x, 0) \geq 0, & x \in H_c, \\ u(x, 0), \ v(x, 0) \in L^\infty(H_c) & \end{array} \right. \tag{4.18}$$

is nonglobal.

Proof: The proof of Theorem 4.3 is similar to that of Theorem 4.2. The only difference is that in the proof of Theorem 4.2 the related scalar problem (4.12) is nonglobal if $N \leq 2$ and in the proof of Theorem 4.3, the related scalar problem (4.13) is nonglobal if $N = 1$, see Lemma 4.3.

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