Decay rates for solutions of semilinear wave equations with a memory condition at the boundary *

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Abstract

In this paper, we study the stability of solutions for semilinear wave equations whose boundary condition includes an integral that represents the memory effect. We show that the dissipation is strong enough to produce exponential decay of the solution, provided the relaxation function also decays exponentially. When the relaxation function decays polynomially, we show that the solution decays polynomially and with the same rate.

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1. Introduction

The main purpose of this work is to study the asymptotic behavior of solution of the semilinear wave equation with a boundary condition of memory type. For this, we consider the following initial boundary-value problem

$$u_{tt} - (\mu(x,t)u_x)_x + h(u) = 0 \quad \text{in} \quad (0,1)\times(0,\infty), \tag{1.1}$$

$$u(0,t) = 0, \quad u(1,t) + \int_0^t g(t-s)\mu(1,s)u_x(1,s)ds = 0, \quad \forall t > 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad (0,1). \tag{1.3}$$

The integral equation (1.2) is a boundary condition which includes the memory effect. Here, by u we are denoting the displacement and by g the relaxation function. By $\mu = \mu(x,t)$ we represent a function of $W^{1,\infty}_{loc}(0,\infty : H^1(0,1))$, such that $\mu(x,t) \geq \mu_0 > 0$, $\mu_t(x,t) \leq 0$ and $\mu_x(x,t) \leq 0$ for all $(x,t) \in (0,1)\times(0,\infty)$. We will assume in the sequel that the function $h \in C^1(\mathbb{R})$ satisfies the following conditions:

$$h(s)s \geq 0 \quad \forall s \in \mathbb{R}, \tag{1.4}$$

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$$\exists \delta > 0 : h(s)s \geq (2+\delta)H(s), \quad \forall s \in \mathbb{R}, \tag{1.5}$$

$$|h(s) - h(t)| \leq b(1 + |s|^{\rho-1} + |t|^{\rho-1})|s-t|, \quad \forall s, t \in \mathbb{R}, \tag{1.6}$$

$$b > 0, \quad \rho \geq 1$$

where

$$H(z) = \int_0^z h(s)ds.$$

We refer to [4] for the physical motivation of this model.
Frictional dissipative boundary condition for the wave equation was studied by several authors, see for example [4, 5, 8, 9, 10, 11, 12, 16, 17] among others. In these works existence of solutions and exponential stabilization were proved for linear or nonlinear equations. In contrast with the large literature for frictional dissipative, for boundary condition with memory, we have only a few works as for example [2, 3, 7, 13, 14]. Let us explain briefly each of the above works. In [2] Ciarletta established theorems of existence, uniqueness and asymptotic stability for a linear model of heat conduction. In this case the memory condition describes a boundary that can absorb heat and due to the hereditary term, can retain part of it. In [3] Fabrizio & Morro consider a linear electromagnetic model with boundary condition of memory type and proved the existence, uniqueness and asymptotic stability of the solutions. While in [13] was shown the existence of global smooth solution for the one dimensional nonlinear wave equation, provided the initial data (u_0, u_1) is small in the $H^3 \times H^2$-norm and also that the solution tends to zero as time goes to infinity. In all the above works was left open the rate of decay. In [7] Rivera & Doerty consider a nonlinear one dimensional wave equation with a viscoelastic boundary condition and proved the existence, uniqueness of global smooth solution, provided the initial data (u_0, u_1) is small in the $H^2 \times H^1$-norm and also that the solution decays uniformly in time(exponentially or algebraically). Finally, in [14] Qin proved a blow up result for the nonlinear one dimensional wave equation with boundary condition and memory. Our main result is to show that the solution of system (1.1)- (1.3) decays uniformly in time, with rates depending on the rate of decay of the relaxation function. More precisely, denoting by k the resolvent kernel of g' (the derivative of the relaxation function) we show that the solution decays exponentially to zero provided k decays exponentially to zero. When k decays polynomially, we show that the corresponding solution also decays polynomially to zero with the same rate of decay.

The method used here is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^{\alpha+1}}$$

for some positive constants c_1, c_2, α and γ. To study the existence of solution of (1.1)- (1.3) we introduce the space

$$V := \{v \in H^1(0,1); v(0) = 0\}.$$

The notation used in this paper is standard and can be found in Lions's book [6]. In the sequel by c (sometime $c_1, c_2, \ldots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follows. In section 2 we establish a existence and regularity result. In section 3 we prove the uniform rate of exponential decay. Finally in section 4 we prove the uniform rate of polynomial decay.

2. Existence and Regularity

In this section we shall study an existence and regularity of solutions to equation (1.1)-(1.3). To this end we will assume that the relaxation function g is positive and non decreasing and we shall use (1.2) to estimate the value of $\mu(1,t)u_x(1,t)$. Denoting by

$$(f * \varphi)(t) \quad = \quad \int_0^t f(t-s)\varphi(s)ds,$$

the convolution product operator and differentiating (1.2) we arrive at the following Volterra equation

$$\mu(1,t)u_x(1,t) + \frac{1}{g(0)} g' * \mu(1,t)u_x(1,t) = -\frac{1}{g(0)} u_t(1,t).$$

Using the Volterra inverse operator, we obtain

$$\mu(1,t)u_x(1,t) = -\frac{1}{g(0)}\{u_t(1,t) + k * u_t(1,t)\},$$

where the resolvent kernel satisfies

$$k + \frac{1}{g(0)} g' * k = \frac{1}{g(0)} g'.$$

With $\tau = \frac{1}{g(0)}$ and using the above identity, we write

$$\mu(1,t)u_x(1,t) = -\tau\{u_t(1,t) + k(0)u(1,t) - k(t)u_0(1) + k' * u(1,t)\}. \tag{2.1}$$

Reciprocally, taking initial data such that $u_0(1) = 0$, the identity (2.1) implies (1.2). Since we are interested in relaxation function of exponential or polynomial type and the identity (2.1) involve the resolvent kernel k, we want to know if k has the same properties. The following Lemma answers this question. Let g be a relaxation function and k its resolvent kernel, that is

$$k(t) - k * g(t) = g(t). \tag{2.2}$$

Lemma 2.1 *If g is a positive continuous function, then k also is a positive continuous function. Moreover,*

1. *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

$$g(t) \leq c_0 e^{-\gamma t},$$

then, the function k satisfies

$$k(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$

for all $0 < \epsilon < \gamma - c_0$.

2. *Given $p > 1$, let us denote by $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p}\, ds$. If there exists a positive constant c_0 with $c_0 c_p < 1$ such that*

$$g(t) \leq c_0 (1+t)^{-p},$$

then, the function k satisfies

$$k(t) \leq \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}.$$

Proof. Note that $k(0) = g(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}$, so $k(t) > 0$ for all $t \in [0, t_0[$. If $t_0 \in \mathbb{R}^+$, from equation (2.2) we get that $-k * g(t_0) = g(t_0)$ but this is a contradiction. Therefore $k(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, let us fix ϵ, such that $0 < \epsilon < \gamma - c_0$ and denote by

$$k_\epsilon(t) := e^{\epsilon t} k(t), \quad g_\epsilon(t) := e^{\epsilon t} g(t).$$

Multiplying equation (2.2) by $e^{\epsilon t}$ we get $k_\epsilon(t) = g_\epsilon(t) + k_\epsilon * g_\epsilon(t)$, hence

$$\sup_{s \in [0,t]} k_\epsilon(s) \;\leq\; \sup_{s \in [0,t]} g_\epsilon(s) + \int_0^\infty c_0 e^{(\epsilon - \gamma)s} ds \sup_{s \in [0,t]} k_\epsilon(s) \;\leq\; c_0 + \frac{c_0}{(\gamma - \epsilon)} \sup_{s \in [0,t]} k_\epsilon(s).$$

Therefore

$$k_\epsilon(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0},$$

which implies our first assertion. To show the second part let us consider the following notation

$$k_p(t) := (1+t)^p k(t), \quad g_p(t) := (1+t)^p g(t).$$

Multiplying equation (2.2) by $(1+t)^p$ we get $k_p(t) = g_p(t) + \int_0^t k_p(t-s)(1+t-s)^{-p}(1+t)^p g(s)\, ds$, hence

$$\sup_{s \in [0,t]} k_p(s) \;\leq\; \sup_{s \in [0,t]} g_p(s) + c_0 c_p \sup_{s \in [0,t]} k_p(s) \;\leq\; c_0 + c_0 c_p \sup_{s \in [0,t]} k_p(s).$$

Therefore

$$k_p(t) \leq \frac{c_0}{1 - c_0 c_p},$$

which proves our second assertion. ■

Remark: The finiteness of the constant c_p can be found in [15, Lemma 7.4].

Due to this Lemma, in the remainder of this paper, we shall use (2.1) instead of (1.2). Let us denote by

$$(f \Box \varphi)(t) \quad = \quad \int_0^t f(t-s)|\varphi(t) - \varphi(s)|^2 ds.$$

The following lemma states an important property of the convolution operator.

Lemma 2.2 *For $f, \varphi \in C^1([0, \infty[: \mathbb{R})$ we have*

$$\begin{aligned} \int_0^t f(t-s)\varphi(s)ds\varphi_t \quad &= \quad -\frac{1}{2}f(t)|\varphi(t)|^2 + \frac{1}{2}f' \Box \varphi \\ &\quad -\frac{1}{2}\frac{d}{dt}\left[f \Box \varphi - (\int_0^t f(s)ds)|\varphi|^2\right]. \end{aligned}$$

The proof of this lemma follows by differentiating the term $f \Box \varphi$.

The first order energy of system (1.1)-(1.3) is given by

$$E(t) \quad = \quad \frac{1}{2}\int_0^1 [|u_t|^2 + \mu(x,t)|u_x|^2 + 2H(u)]dx + \frac{\tau}{2}(k(t)|u(1,t)|^2 - k'(t)\Box u(1,t)).$$

We summarize the well-posedness of (1.1)-(1.3) in the following theorem.

Theorem 2.1 *Let $k \in C^2(\mathbb{R}^+)$ be such that*

$$k, -k', k'' \geq 0. \tag{2.3}$$

If $(u_0, u_1) \in H^2(0,1) \cap V \times V$ satisfies the compatibility condition

$$\mu(1,0)\frac{\partial u_0}{\partial x}(1) = -\tau u_1(1), \tag{2.4}$$

then there exists only one solution u of the system (1.1)-(1.3) satisfying

$$u \in L^\infty(0,T; H^2(0,1) \cap V) \cap W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T; L^2(0,1)).$$

Proof. To prove this Theorem we shall use the Galerkin method. Let $\{w_j\}$ be a complete orthogonal system of V such that

$$\{u_0, u_1\} \in Span\{w_0, w_1\}.$$

Let us consider the following Galerkin approximation

$$u^m(t) := \sum_{j=0}^m h_{j,m}(t) w_j.$$

Standard results about ordinary differential equations guarantee that there exists only one solution of the approximate system,

$$\begin{aligned}
&\int_0^1 u_{tt}^m w_j dx + \int_0^1 \mu(x,t)\frac{\partial u^m}{\partial x}\frac{\partial w_j}{\partial x}dx + \int_0^1 h(u^m)w_j dx \\
&= -\tau\{u_t^m(1,t) + k(0)u(1,t) - k(t)u_0(1) + k' * u(1,t)\}w_j(1), \qquad (2.5)
\end{aligned}$$

for $0 \le j \le m$, satisfying the following initial conditions

$$u^m(0) = u_0, \quad u_t^m(0) = u_1.$$

Our first step is to show that the approximate solutions remain bounded for any $m \in \mathbb{N}$. To do this, let us multiply equation (2.5) by $h'_{j,m}(t)$ and summing up the product results in j we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \{|u_t^m|^2 + \mu(x,t)|u_x^m|^2 + 2H(u^m)\}dx = \int_0^1 \mu_t(x,t)|u_x^m|^2 dx \\
&-\tau\{|u_t^m(1,t)|^2 + k(0)u(1,t)u_t(1,t) - k(t)u_0(1)u_t(1,t) + k' * u(1,t)u_t(1,t)\}. \qquad (2.6)
\end{aligned}$$

Using Lemma 2.2 for the term

$$k' * u(1,t)u_t(1,t)$$

and the properties of k, k' and k'' from assumption (2.3) we conclude by (2.6)

$$\frac{d}{dt}E(t,u^m) \le cE(0,u^m).$$

Taking into account the definition of the initial data of u^m we conclude that

$$E(t,u^m) \le c, \quad \forall t \in [0,T], \quad \forall m \in \mathbb{N}. \qquad (2.7)$$

Next, we shall find an estimate for the second order energy. First, let us estimate the initial data $u_{tt}^m(0)$ in the $L^2 - norm$. Letting $t \to 0^+$ in the equation (2.5), multiplying the result by $h''_{j,m}(0)$ and using the compatibility condition (2.4) we get

$$||u_{tt}^m(0)||_2^2 = \int_0^1 \mu(x,0)u_{xx}^m(0)u_{tt}^m(0)dx + \int_0^1 \mu_x(x,0)u_x^m(0)u_{tt}^m(0)dx - \int_0^1 h(u_0)u_{tt}^m(0)dx.$$

Since $u_0 \in H^2(0,1)$, the hypothesis (1.6) for the function h together with the Sobolev's imbedding imply that $h(u_0) \in L^2(0,1)$. Hence

$$||u_{tt}^m(0)||_2^2 \le c_1, \quad \forall m \in \mathbb{N}.$$

Differentiating the equation (2.5) with respect to the time, multiplying by $h''_{j,m}(t)$ and summing up the product results in j, we arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \{|u_{tt}^m|^2 + \mu(x,t)|u_{xt}^m|^2\}dx \;=\; & \frac{1}{2}\int_0^1 \mu_t(x,t)|u_{xt}^m|^2 dx - \int_0^1 \mu_t(x,t)u_x^m u_{xtt}^m dx \\
& - \int_0^1 h'(u^m)u_t^m u_{tt}^m dx - \tau|u_{tt}^m(1,t)|^2 \\
& -\tau k(0)u_t^m(1,t)u_{tt}^m(1,t) + \tau k'(t)u_0(1)u_{tt}^m(1,t) \\
& -\tau(k' * u(1,t))_t u_{tt}^m(1,t). \qquad (2.8)
\end{aligned}$$

Note that

$$\begin{aligned}\int_0^1 \mu_t u_x^m u_{xtt}^m dx &= \mu_t(1,t)u_x^m(1,t)u_{tt}^m(1,t) - \int_0^1 \mu_{xt} u_x^m u_{tt}^m dx \\ &\quad - \int_0^1 \mu_t u_{xx}^m u_{tt}^m dx.\end{aligned}$$

Using the elementary inequality $2ab \le a^2 + b^2$, the hypothesis on μ and the inequality

$$\begin{aligned}|u_x^m(1,t)|^2 &\le c\{|u_t^m(1,t)|^2 + k^2(t)|u^m(1,t)|^2 \\ &\quad k(0)|k'|\Box u^m(1,t) + k(0)k(t)|u^m(1,t)|^2\}\end{aligned}$$

we obtain

$$\begin{aligned}\int_0^1 \mu_t u_x^m u_{xtt}^m dx &\le c\{|u_t^m(1,t)|^2 + k^2(t)|u_0(1)|^2 + k(0)|k'|\Box u^m(1,t) \\ &\quad + k(0)k(t)|u^m(1,t)|^2\} + \frac{\tau}{2}|u_{tt}^m(1,t)|^2 + c\int_0^1 |u_x^m|^2 dx \\ &\quad + c\int_0^1 |u_{tt}^m|^2 dx + \frac{1}{2}\int_0^1 |u_{xx}^m|^2 dx. \qquad (2.9)\end{aligned}$$

Note that

$$\begin{aligned}(k' * u^m(1,t))_t &= k'(0)u^m(1,t) + \int_0^t k''(t-s)u^m(1,s)ds \\ &= k'(t)u_0(1) + \int_0^t k'(t-s)u_t^m(1,s)ds. \qquad (2.10)\end{aligned}$$

Substituting the inequality (2.9) and identity (2.10) into (2.8) and using Lemma 2.2 we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \{|u_{tt}^m|^2 + \mu(x,t)|u_{xt}^m|^2\}dx &\le \frac{1}{2}\int_0^1 \mu_t(x,t)|u_{xt}^m|^2 dx + c\{|u_t^m(1,t)|^2 \\ &\quad + k^2(t)|u_0(1)|^2 + k(0)|k'|\Box u^m(1,t) \\ &\quad + k(0)k(t)|u^m(1,t)|^2\} + c\int_0^1 |u_{tt}^m|^2 dx \\ &\quad - \int_0^1 h'(u^m)u_t^m u_{tt}^m dx - \frac{\tau}{2}|u_{tt}^m(1,t)|^2 \\ &\quad + \frac{\tau}{2}k'(t)|u_t^m(1,t)|^2 - \frac{\tau}{2}k''\Box u_t^m(1,t) \\ &\quad + \frac{\tau}{2}\frac{d}{dt}\left[k'\Box u_t^m(1,t) - k(t)|u_t^m(1,t)|^2\right] \\ &\quad + \frac{1}{2}\int_0^1 |u_{xx}^m|^2 dx. \qquad (2.11)\end{aligned}$$

From the condition (1.6) and from the Sobolev imbedding we have

$$\int_0^1 h'(u^m)u_t^m u_{tt}^m dx \le c\left[\int_0^1 (1+|u_x^m|^2)dx\right]^{\frac{\rho-1}{2}} \left[\int_0^1 |u_{xt}^m|^2 dx\right]^{\frac{1}{2}} \left[\int_0^1 |u_{tt}^m|^2 dx\right]^{\frac{1}{2}}.$$

Taking into account the first estimate (2.7) we conclude that

$$\int_0^1 h'(u^m)u_t^m u_{tt}^m dx \le c\left\{\int_0^1 |u_{xt}^m|^2 dx + \int_0^1 |u_{tt}^m|^2 dx\right\}. \tag{2.12}$$

From the approximate system (2.5) we get

$$\int_0^1 \mu|u_{xx}^m|^2 dx = \int_0^1 u_{tt}^m u_{xx}^m dx - \int_0^1 \mu_x u_x^m u_{xx}^m dx + \int_0^1 h(u^m)u_{xx}^m dx.$$

Using the elementary inequality, the Sobolev's imbedding and the hypothesis (1.6) we obtain

$$\frac{1}{2}\int_0^1 |u_{xx}^m|^2 dx \le c\left\{\int_0^1 |u_{tt}^m|^2 dx + \int_0^1 |u_x^m|^2 dx\right\}. \tag{2.13}$$

Denoting by

$$E_0(t,u^m) = \frac{1}{2}\int_0^1 \{|u_t^m|^2 + \mu(x,t)|\frac{\partial u^m}{\partial x}|^2\}dx + \frac{\tau}{2}k(t)|u^m(1,t)|^2 - \frac{\tau}{2}k'\Box u^m(1,t)$$

we find by (2.11)

$$\begin{aligned} \frac{d}{dt}E_0(t,u_t^m) \quad \le \quad & \frac{1}{2}\int_0^1 \mu_t(x,t)|u_{xt}^m|^2 dx + c\{|u_t^m(1,t)|^2 \\ & + k^2(t)|u_0(1)|^2 + k(0)|k'|\Box u^m(1,t) \\ & + k(0)k(t)|u^m(1,t)|^2\} + c\int_0^1 |u_{tt}^m|^2 dx \\ & - \int_0^1 h'(u^m)u_t^m u_{tt}^m dx - \frac{\tau}{2}|u_{tt}^m(1,t)|^2 \\ & + \frac{\tau}{2}k'(t)|u_t^m(1,t)|^2 - \frac{\tau}{2}k''\Box u_t^m(1,t) \\ & + \frac{1}{2}\int_0^1 |u_{xx}^m|^2 dx. \end{aligned} \tag{2.14}$$

Substituting the inequalities (2.12) and (2.13) into (2.14) and using Poincaré's inequality , the hypothesis (2.3), the first estimate (2.7) and applying Gronwall's inequality we conclude that

$$E_0(t,u_t^m) \le c, \quad \forall t \in [0,T], \quad \forall m \in \mathbb{N}. \tag{2.15}$$

The rest of the proof is a matter of routine. ■

3. Exponential Decay

In this section we shall study the asymptotic behavior of the solutions of system (1.1)-(1.3) when the resolvent kernel k is exponentially decreasing, that is, there exist positive constants b_1, b_2 such that:

$$k(0) > 0, \quad k'(t) \le -b_1 k(t), \quad k''(t) \ge -b_2 k'(t). \tag{3.1}$$

Note that this conditions implies that

$$k(t) \leq k(0)e^{-b_1 t}.$$

The point of departure of this study is to establish some inequalities given in the next lemmas.

Lemma 3.1 *Any strong solution of system (1.1)-(1.3) satisfies*

$$\begin{aligned}\frac{d}{dt}E(t) &\leq -\frac{\tau}{2}|u_t(1,t)|^2 + \frac{\tau}{2}k^2(t)|u_0(1)|^2 + \frac{\tau}{2}k'(t)|u(1,t)|^2 \\ &\quad -\frac{\tau}{2}k''(t)\Box u(1,t) + \frac{1}{2}\int_0^1 \mu_t(x,t)|u_x|^2 dx.\end{aligned}$$

Proof. Multiplying (1.1) by u_t and integrating over $[0,1]$ we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [|u_t|^2 + \mu(x,t)|u_x|^2 + 2H(u)]dx = \mu(1,t)u_x(1,t)u_t(1,t) + \frac{1}{2}\int_0^1 \mu_t(x,t)|u_x|^2 dx. \quad (3.2)$$

From equality (2.1) we have

$$\mu(1,t)u_x(1,t)u_t(1,t) = -\tau|u_t(1,t)|^2 - \tau k(0)u(1,t)u_t(1,t) + \tau k(t)u_0(1)u_t(1,t) - k' * u(1,t)u_t(1,t).$$

Using Lemma 2.2 we get

$$k' * u(1,t)u_t(1,t) = -\frac{1}{2}k'(t)|u(1,t)|^2 + \frac{1}{2}k''\Box u(1,t) - \frac{1}{2}\frac{d}{dt}\left[k'\Box u(1,t) - (k(t)-k(0))|u(1,t)|^2\right]$$

Substituting the two above equalities into (3.2), our conclusion follows. ■

Let us define the functional

$$\psi(t) = \int_0^1 (xu_x + (\frac{1}{2}-\theta)u)u_t dx + \frac{\tau}{2}|u(1,t)|^2,$$

where θ is a positive number. The following Lemma plays an important role for the construction of the Lyapunov functional.

Lemma 3.2 *Any strong solution of system (1.1)-(1.3) satisfies*

$$\begin{aligned}\frac{d}{dt}\psi(t) &\leq -\theta\int_0^1 |u_t|^2 dx - (\frac{1}{2}-\theta)\int_0^1 \mu|u_x|^2 dx - (\frac{\delta}{2} - \theta(2+\delta))\int_0^1 H(u)dx \\ &\quad + c|u_t(1,t)|^2 + (c + \frac{\tau}{2\epsilon})k(t)^2|u_0(1)|^2 + ck(0)k(t)|u(1,t)|^2 \\ &\quad + (c + \frac{\tau}{2\epsilon})k(0)|k'|\Box u(1,t) + \frac{1}{2}\int_0^1 x\mu_x(x,t)|u_x|^2 dx\end{aligned}$$

where $\epsilon = \frac{\mu_0}{\widetilde{c}\tau}$, $\widetilde{c}$ *is a positive constant and* $0 < \theta < min(\frac{1}{2}, \frac{\delta}{2(2+\delta)})$.

Proof. Using the equation (1.1) we get

$$\begin{aligned}\frac{d}{dt}\int_0^1 u_t(xu_x + (\frac{1}{2}-\theta)u)dx &\leq \frac{1}{2}|u_t(1,t)|^2 - \theta\int_0^1 |u_t|^2 dx + \frac{1}{2}\mu(1,t)|u_x(1,t)|^2 \\ &\quad -(1-\theta)\int_0^1 \mu|u_x|^2 dx + (\frac{1}{2}-\theta)\mu(1,t)u_x(1,t)u(1,t) \\ &\quad + \int_0^1 H(u)dx - (\frac{1}{2}-\theta)\int_0^1 uh(u)dx. \quad (3.3)\end{aligned}$$

Using the hypotheses (1.5) we have

$$-(\frac{1}{2}-\theta)\int_0^1 uh(u)dx \leq -(2+\delta)(\frac{1}{2}-\theta)\int_0^1 H(u)dx. \tag{3.4}$$

Note that

$$\begin{aligned}-k(0)u(1,t)-k'*u(1,t) &= -\int_0^t k'(t-s)[u(1,s)-u(1,t)]ds - k(t)u(1,t)\\ &\leq (\int_0^t |k'(s)|ds)^{\frac{1}{2}}[|k'|\Box u(1,t)]^{\frac{1}{2}} + k(t)|u(1,t)|\\ &\leq |k(t)-k(0)|^{\frac{1}{2}}[|k'|\Box u(1,t)]^{\frac{1}{2}} + k(t)|u(1,t)|. \end{aligned} \tag{3.5}$$

From equality (2.1), from the hypothesis for the function μ and from the inequality (3.5) we have

$$\begin{aligned}\mu(1,t)|u_x(1,t)|^2 &\leq c\{|u_t(1,t)|^2 + k^2(t)|u_0(1)|^2\\ &+k(0)|k'(t)|\Box u(1,t) + k(0)k(t)|u(1,t)|^2\}.\end{aligned} \tag{3.6}$$

From equality (2.1) we have

$$\begin{aligned}\mu(1,t)u_x(1,t)u(1,t) &= -\frac{\tau}{2}\frac{d}{dt}|u(1,t)|^2 - k(0)|u(1,t)|^2\\ &+k(t)u_0(1)u(1,t) - k'*u(1,t)u(1,t).\end{aligned}$$

Using (3.5), Poincare's inequality and the elementary inequality $2ab \leq a^2+b^2$ we conclude that

$$\begin{aligned}\mu(1,t)u_x(1,t)u(1,t) &\leq -\frac{\tau}{2}\frac{d}{dt}|u(1,t)|^2 + \frac{\tau}{2\epsilon}k^2(t)|u_0(1)|^2\\ &+\frac{\tau k(0)}{2\epsilon}|k'|\Box u(1,t) + ck(0)k(t)|u(1,t)|^2\\ &+\frac{\widetilde{c}\epsilon\tau}{2}\int_0^1 |u_x|^2dx\end{aligned} \tag{3.7}$$

where $\widetilde{c}$ is a fixed positive constant. Substituting (3.6)-(3.7) into (3.3) and fixing $\epsilon = \frac{\mu_0}{\widetilde{c}\tau}$ follows the conclusion of Lemma. ■

Let us introduce the functional

$$\mathcal{L}(t) = NE(t) + \psi(t), \tag{3.8}$$

with $N > 0$. It is not difficult to see that $\mathcal{L}(t)$ verifies

$$q_0E(t) \leq \mathcal{L}(t) \leq q_1E(t), \tag{3.9}$$

where q_0 and q_1 are positive constants. We will show later that the functional $\mathcal{L}$ satisfies the inequality of the following Lemma.

Lemma 3.3 *Let f be a real positive function of class C^1. If there exist positive constants γ_0, γ_1 and c_0 such that*

$$f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and c such that

$$f(t) \leq (f(0) + c)e^{-\gamma t}.$$

Proof. First, let us suppose that $\gamma_0 < \gamma_1$. Define $F(t)$ by

$$F(t) := f(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$

Then

$$F'(t) = f'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \leq -\gamma_0 F(t).$$

Integrating from 0 to t we arrive at

$$F(t) \leq F(0) e^{-\gamma_0 t} \quad \Rightarrow \quad f(t) \leq (f(0) + \frac{c_0}{\gamma_1 - \gamma_0}) e^{-\gamma_0 t}.$$

Now, we shall assume that $\gamma_0 \geq \gamma_1$. Under these conditions we get

$$f'(t) \leq -\gamma_1 f(t) + c_0 e^{-\gamma_1 t} \quad \Rightarrow \quad (e^{\gamma_1 t} f(t))' \leq c_0.$$

Integrating from 0 to t we obtain

$$f(t) \leq (f(0) + c_0 t) e^{-\gamma_1 t}.$$

Since $t \leq (\gamma_1 - \epsilon) e^{(\gamma_1 - \epsilon)t}$ for any $0 < \epsilon < \gamma_1$ we conclude that

$$f(t) \leq (f(0) + c_0(\gamma_1 - \epsilon)) e^{-\epsilon t}.$$

This completes the present proof. ■

Finally, we shall show the main result of this section.

Theorem 3.1 *Let us suppose that the initial data $(u_0, u_1) \in V \times L^2(0,1)$ and that the resolvent k satisfies the conditions (3.1). Then there exist positive constants α_1 and γ_1 such that*

$$E(t) \leq \alpha_1 e^{-\gamma_1 t} E(0), \quad \forall t \geq 0.$$

Proof. We will suppose that the initial data $(u_0, u_1) \in H^2(0,1) \cap V \times V$ and satisfies (2.3); our conclusion will follow by standard density arguments. Using the Lemmas 3.1 and 3.2 we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -\frac{\tau}{2} N |u_t(1,t)|^2 + \frac{\tau}{2} N k^2(t) |u_0(1)|^2 + \frac{\tau}{2} N k'(t) |u(1,t)|^2 \\ & -\frac{\tau}{2} N k'' \Box u(1,t) - \theta \int_0^1 |u_t|^2 dx - (\frac{1}{2} - \theta) \int_0^1 \mu(x,t) |u_x|^2 dx \\ & -(\frac{\delta}{2} - \theta(2+\delta)) \int_0^1 H(u)) dx + c|u_t(1,t)|^2 + ck^2(t)|u_0(1)|^2 \\ & + ck(0)k(t)|u(1,t)|^2 + ck(0)|k'|\Box u(1,t). \qquad (3.10)\end{aligned}$$

Then, choosing N large enough we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -q_2 E(t) + ck^2(t)E(0),$$

where $q_2 > 0$ is a small constant. Here we used the assumptions (3.1) in order to conclude the following estimates

$$-\frac{\tau}{2}k''\Box u(1,t) \leq c_1 k'\Box u(1,t),$$
$$\frac{\tau}{2}k'|u(1,t)|^2 \leq -c_1 k|u(1,t)|^2$$

for the corresponding two terms appearing in the inequality (3.8). Using (3.9) we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{q_2}{q_1}\mathcal{L}(t) + ck^2(t)E(0).$$

Using the exponential decay of k and Lemma 3.3 we conclude

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + c\}e^{-\gamma_1 t}$$

for all $t \geq 0$, where $\gamma_1 = min(\gamma_0, \frac{q_2}{q_1})$. Use of (3.9) now completes the proof. ■

4. Polynomial rate of decay

Here, our attention will be focused on the uniform rate of decay when the resolvent k decays polynomially as $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Therefore, we will assume that the resolvent kernel k satisfies

$$k(0) > 0, \quad k'(t) \leq -b_1[k]^{1+\frac{1}{p}}, \quad k''(t) \geq b_2[-k'(t)]^{1+\frac{1}{p+1}}, \tag{4.1}$$

for some $p > 1$ and some positive constants b_1 and b_2. The following lemmas will play an important role in the sequel.

Lemma 4.1 *Let m and h be integrable functions, and let $0 \leq r < 1$ and $q > 0$. Then, for $t \geq 0$:*

$$\int_0^t |m(t-s)h(s)|ds \leq (\int_0^t |m(t-s)|^{1+\frac{1-r}{q}}|h(s)|ds)^{\frac{q}{q+1}}(\int_0^t |m(t-s)|^r|h(s)|ds)^{\frac{1}{q+1}}.$$

Proof. Let

$$v(s) := |m(t-s)|^{1-\frac{r}{q+1}}|h(s)|^{\frac{q}{q+1}}, \quad w(s) := |m(t-s)|^{\frac{r}{q+1}}|h(s)|^{\frac{1}{q+1}}.$$

Then using Hölder's inequality with $\delta = \frac{q+1}{q}$ for v and $\delta^* = q+1$ for w we arrive at the conclusion. ■

Lemma 4.2 *Let $p > 1$, $0 \le r < 1$ and $t \ge 0$. Then for $r > 0$,*

$$
\begin{aligned}
&(|k'|\Box u(1,t))^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}}\\
&\le 2^{\frac{1}{(1-r)(p+1)}}\Big(\int_0^t |k'(s)|^r ds ||u||^2_{L^\infty((0,T),H^1(0,1))}\Big)^{\frac{1}{(1-r)(p+1)}}(|k'|^{1+\frac{1}{p+1}}\Box u(1,t)),
\end{aligned}
$$

and for $r = 0$

$$
\begin{aligned}
&(|k'|\Box u(1,t))^{\frac{p+2}{p+1}}\\
&\le 2\Big(\int_0^t ||u(.,s)||^2_{H^1(0,1)} ds + t||u(.,s)||^2_{H^1(0,1)}\Big)^{\frac{1}{p+1}}(|k'|^{1+\frac{1}{p+1}}\Box u(1,t)).
\end{aligned}
$$

Proof. The above inequalitiy is a immediate consequence of Lemma 4.1 with

$$m(s) := |k'(s)|, \quad h(s) := |u(x,t) - u(x,s)|^2, \quad q := (1-r)(p+1)$$

and t fixed. ■

Lemma 4.3 *Let $f \ge 0$ be a differentiable function satisfying*

$$f'(t) \le \frac{-\bar{c}_1}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} + \frac{\bar{c}_2}{(1+t)^\beta} f(0), \quad \textit{for} \quad t \ge 0,$$

for some positive constants $\bar{c}_1, \bar{c}_2$, α and β such that

$$\beta \ge \alpha + 1.$$

Then there exists a constant $\bar{c}_3 > 0$ such that

$$f(t) \le \frac{\bar{c}_3}{(1+t)^\alpha} f(0), \quad \textit{for} \quad t \ge 0.$$

Proof. Let us denote by

$$F(t) = f(t) + \frac{2\bar{c}_2}{\alpha}(1+t)^{-\alpha} f(0).$$

Differentiating this function we have

$$F'(t) = f'(t) - 2\bar{c}_2(1+t)^{(-\alpha+1)} f(0).$$

From hypothesis on f' and observing that $\beta \ge \alpha + 1$ we get

$$
\begin{aligned}
F'(t) \quad &\le \quad \frac{-\bar{c}_1}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} - \bar{c}_2(1+t)^{-(\alpha+1)} f(0)\\
&\le \frac{-c}{f(0)^{\frac{1}{\alpha}}}\left[f(t)^{1+\frac{1}{\alpha}} + \left(\frac{f(0)}{(1+t)^\alpha}\right)^{1+\frac{1}{\alpha}}\right].
\end{aligned}
$$

Noting that

$$F(t)^{1+\frac{1}{\alpha}} \leq f(t)^{1+\frac{1}{\alpha}} + \left(\frac{f(0)}{(1+t)^\alpha}\right)^{1+\frac{1}{\alpha}}$$

we obtain

$$F'(t) \leq -\frac{c}{f(0)^{\frac{1}{\alpha}}} F(t)^{1+\frac{1}{\alpha}}.$$

Integrating the last inequality from 0 to t, it follows

$$F(t) \leq \frac{cF(0)}{(1+ct)^\alpha} \leq \frac{c}{(1+t)^\alpha} f(0).$$

Therefore

$$f(t) \leq \frac{\bar{c}_3}{(1+t)^\alpha} f(0).$$

This complete the present proof. $\blacksquare$

Theorem 4.1 *Let us suppose that the initial data $(u_0, u_1) \in V \times L^2(0,1)$ and that the resolvent k satisfies the conditions (4.1). Then there is a positive constant c for which we have*

$$E(t) \leq \frac{c}{(1+t)^{p+1}} E(0).$$

Proof. We will suppose that the initial data $(u_0, u_1) \in H^2(0,1) \cap V \times V$ and satisfies (2.4); our conclusion will follow by standard density arguments. We define the functional $\mathcal{L}$ as in (3.8) and we have the equivalence to the energy term E as give in (3.9) again. The negative term

$$-ck(t)|u(1,t)|^2$$

can be obtained from Lemma 3.1 and from the estimate

$$k(t)|u(1,t)|^2 \leq c \int_0^1 \mu(x,t)|u_x|^2 dx.$$

From the Lemmas 3.1 and 3.2, using the properties of k'' from the assumptions (4.1) for the term

$$-\frac{\tau}{2} k'' \Box u(1,t)$$

we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) \quad &\leq \quad -c_1\Big(\int_0^1 |u_t|^2 + \mu(x,t)|u_x|^2 + H(u)dx \\ &\qquad +k(t)|u(1,t)|^2 + N(-k')^{1+\frac{1}{p+1}} \Box u(1,t)\Big) + c_2 k^2(t) E(0). \end{aligned} \tag{4.2}$$

Let us fix $0 < r < 1$ such that

$$\frac{1}{p+1} < r < \frac{p}{p+1}.$$

Under this condition

$$\int_0^\infty |k'|^r \leq c\int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty.$$

From the Lemma 4.2 we get

$$\begin{aligned}
&(-k')^{1+\frac{1}{p+1}}\Box u(1,t)\\
&\geq \frac{c}{E(0)^{\frac{1}{(1-r)(p+1)}}}((-k')\Box u(1,t))^{1+\frac{1}{(1-r)(p+1)}}
\end{aligned}$$

(with $c = c(r)$). On the other hand since the energy is bounded we have

$$\begin{aligned}
&(k(t)|u(1,t)|^2 + \int_0^1 |u_t|^2 + \mu(x,t)|u_x|^2 + H(u)dx)^{1+\frac{1}{(1-r)(p+1)}}\\
&\leq cE(0)^{\frac{1}{(1-r)(p+1)}}(k(t)|u(1,t)|^2 + \int_0^1 |u_t|^2 + \mu(x,t)|u_x|^2 + H(u)dx).
\end{aligned}$$

From inequality (4.2) using the last two inequalities we conclude:

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \quad\leq\quad & -\frac{c}{E(0)^{\frac{1}{(1-r)(p+1)}}}[(k(t)|u(1,t)|^2 + \int_0^1 |u_t|^2 + \mu(x,t)|u_x|^2\\
&+H(u)dx)^{1+\frac{1}{(1-r)(p+1)}} + ((-k')\Box u(1,t))^{1+\frac{1}{(1-r)(p+1)}}] + ck^2(t)E(0).
\end{aligned}$$

This implies, observing (3.9)

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + ck^2(t)E(0). \tag{4.3}$$

Applying the Lemma 4.3 with $f = \mathcal{L}$ and $\beta = 2p$ we have:

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0). \tag{4.4}$$

Since $(1-r)(p+1) > 1$ we obtain from the inequality (4.4)

$$\int_0^\infty E(s)ds \leq c\int_0^\infty \mathcal{L}(s)ds \leq c\mathcal{L}(0), \tag{4.5}$$

$$t||u(.,t)||^2_{H^1(0,1)} \leq ct\mathcal{L}(t) \leq c\mathcal{L}(0), \tag{4.6}$$

$$\int_0^t ||u(.,s)||^2_{H^1(0,1)} \leq c\int_0^\infty \mathcal{L}(t)dt \leq c\mathcal{L}(0). \tag{4.7}$$

With the estimates (4.5)-(4.7) and using Lemma 4.2 (case $r = 0$) we obtain

$$(-k')^{1+\frac{1}{p+1}}\Box u(1,t) \geq \frac{c}{E(0)^{\frac{1}{p+1}}}((-k')\Box u(1,t))^{1+\frac{1}{p+1}}.$$

Hence, with the same arguments as in the derivation of (4.3), we have

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + ck^2(t)E(0).$$

Applying the Lemma 4.3, we obtain

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{p+1}}\mathcal{L}(0),$$

and hence by (3.9) we conclude

$$E(t) \leq \frac{c}{(1+t)^{p+1}}E(0),$$

which completes the present proof. ■

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