ON A FIXED POINT THEOREM OF KRASNOSELKII-SCHAEFER TYPE

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Abstract

In this paper a variant of a fixed point theorem to Krasnoselskii-Schaefer type is proved and it is further applied to certain nonlinear integral equation of mixed type for proving the existence of the solution.

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1 Introduction

Nonlinear integral equations have been studied in the literature by several authors since long time. See for example, Miller et al. [7], Corduneanu [3] and the references given therein. Nonlinear integral equations of mixed type have been discused in Krasnoselskii [6], Nashed and Wong [8] and Dhage [4] etc. It is known that the integral equations of mixed type arise as an inversion of the initial and boundary value problems of perturbed differential equations. This and other like facts entail the importance of the study of integral equations of mixed type and in the present study we shall prove the existence result for a certain nonlinear integral equations of mixed type via fixed point method. In particular, given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, the set of all real numbers, we study the following nonlinear functional integral equation (in short FIE) of mixed type

$$x(t) = q(t) + \int_0^{\mu(t)} v(t,s)x(\theta(s))ds + \int_0^{\sigma(t)} k(t,s)g(s, x(\eta(s)))ds, \ \ t \in J, \tag{1}$$

where $q : J \to \mathbb{R}$, $v, k : J \times J \to \mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R}$ and $\mu, \theta, \sigma, \eta : J \to J$.

The special cases of the FIE (1) have been studied in the literature for various aspects of the solution. The topological fixed point theorem such as Krasnoselskii [6] is generally employed for proving the existence result for the integral equations of mixed type. In this paper the existence theorem for the FIE (1) is obtained via a new fixed point technique developed in the following section.

2 Abstract result

Let X be a Banach space. A mapping $A : X \to X$ is called a nonlinear contraction if there exists a continuous nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Ax - Ay\| \leq \phi(\|x - y\|)$$

for all $x, y \in X$, where $\phi(r) < r$ for $r > 0$. In particular if $\phi(r) = \alpha r, 0 < \alpha < 1$, A is called a contraction on X, with contraction constant α. Then we have the following fixed point theorem, given in Boyd and Wong [1], which is useful for proving the existence and uniqueness theorems for nonlinear differential and integral equations.

Theorem 2.1 *Let S be a closed convex and bounded subset of a Banach space X and let $A : S \to S$ be a nonlinear contraction. Then A has a unique fixed point x^* and the sequence $\{A^n x\}$ of successive iterations converges to x^* for each $x \in X$.*

The operator $T : X \to X$ is called compact if $T(X)$ is a compact subset of X. Similarly $T : X \to X$ is called totally bounded if $T(S)$ is a totally bounded set in X, for every bounded subset S of X. Finally a completely continuous operator $T : X \to X$ is one which is continuous and totally bounded.

A fixed point theorem of Schaefer [9], concerning the completely continuous operator is

Theorem 2.2 *Let $T : X \to X$ be a completely continuous operator. Then either*

i) the operator equation $x = \lambda T x$ has a solution for $\lambda = 1$, or

ii) the set $\mathcal{E} = \{u \in X : u = \lambda T u\}$ is unbounded for $\lambda \in (0, 1)$.

In a recent paper, Burton and Kirk [2] combined Theorem 2.1 and Theorem 2.2 and proved the following fixed point theorem:

Theorem 2.3 *Let $A, B : X \to X$ be two operators satisfying:*

(a) A is contraction, and

(b) B is completely continuous.

Then either

(i) the operator equation $Ax + Bx = x$ has a solution, or

(ii) the set $\mathcal{E} = \left\{ u \in X |\ \lambda A\left(\frac{u}{\lambda}\right) + \lambda B u = u \right\}$ is unbounded for $\lambda \in (0, 1)$.

We note that Theorem 2.3 is useful for proving existence theorems for nonlinear integral equations of mixed type. See Burton and Kirk [2]. Sometime it is possible the operator A is not contraction, but some iterates A^p of A, for some $p \in N$, is a contraction. Then in this situation Theorem 2.3 is not useful for applications. This necessitates to search for a new fixed point theorem of Nashed-Wong-Schaefer type, which is the main motivation of the present work.

Theorem 2.4 *Let $A, B : X \to X$ be two operators such that*

(A) A is linear and bounded, and there exists a $p \in \mathbb{N}$ such that A^p is a nonlinear contraction, and

(B) B is completely continuous.

Then either

(i) the operator equation $Ax + \lambda Bx = x$ has a solution for $\lambda = 1$ or

(ii) the set $\mathcal{E} = \{u \in X |\ Au + \lambda Bu = u,\ 0 < \lambda < 1\}$ is unbounded.

Proof. Define a mapping T on X by

$$Tx = (I - A)^{-1}Bx.$$

The operator equation $\lambda(I - A)^{-1}Bx = x$ is equivivalent to the operator $Ax + \lambda Bx = x$ for each $\lambda \in [0, 1]$. Therefore the conclusion of the theorem immediately follows by an application of Theorem 2.2, if we show that the operator T is well defined and completely continuous on X.

We have

$$(I - A)^{-1} = I + A + A^2 + \ldots + A^{p-1} + \ldots = (I - A^p)^{-1}\left(\sum_{j=0}^{p-1} A^j\right).$$

Since A^p is a nonlinear contraction, the operator $(I - A^p)^{-1}$ exists on X in view of Theorem 2.1. Again since A is linear and bounded, $\left(\sum_{j=0}^{p-1} A^j\right)$ is a linear and bounded operator on X into X. Hence the composition $(I - A^p)^{-1}\left(\sum_{j=0}^{p-1} A^j\right)$ exists and so T is well defined and maps X into itself. The linearity together with the boundedness of A implies the continuity of A and consequently the continuity of $A^j, j = 1, 2, \ldots$. Therefore the operator $(I - A)^{-1}$ is continuous on X. Hence the operator T, which is a composition of a continuous and a completely continuous operator, is completely continuous on X into itself. The desired conclusion follows by an application of Theorem 2.2.

3 Existence results

We shall seek the existence of the solution to the FIE (1) in the space $BM(J, \mathbb{R})$ of all bounded and measurable real-valued functions on J. Define a norm $\|\cdot\|$ in $BM(J, \mathbb{R})$ by

$$\|x\|_{BM} = \max_{t \in J} |x(t)|.$$

Clearly $BM(J, \mathbb{R})$ becomes a Banach space with respect to this maximum norm. By $L^1(J, \mathbb{R})$ we denote the space of all Lebesque measurable functions on J with the usual norm $\|\cdot\|_{L^1}$ given by $\|x\|_{L^1} = \displaystyle\int_0^1 |x(t)|dt$.

We need the following definition in the sequel.

Definition 3.1 *A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to satisfy Caratheodory's conditions or simply is called L^1-Caratheodory if:*

(i) $t \to \beta(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \to \beta(t, x)$ is continuous almost everywhere for $t \in J$, and

(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x)| \leq h_r(t), \quad \text{a.e. } t \in J,$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

We consider the following assumptions in the sequel:

(H_0) The functions $\mu, \theta, \sigma, \eta : J \to J$ are continuous.

(H_1) The function $q : J \to \mathbb{R}$ is continuous.

(H_2) The functions $v, k : J \times J \to \mathbb{R}$ are continuous.

(H_3) The function $g(t, x)$ is L^1-Caratheodory.

(H_4) There exists a function $\phi \in L^1(J, \mathbb{R})$ and a continuous and nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$|g(t, x)| \leq \phi(t)\psi(|x|), \quad \text{a.e. } t \in J,$$

for all $x \in \mathbb{R}$.

Before stating the main existence result, we consider the FIE,

$$x(t) = \lambda q(t) + \int_0^{\mu(t)} v(t, s)x(\theta(s))ds + \lambda \int_0^{\sigma(t)} k(t, s)g(s, x(\eta(s)))ds, \ \ t \in J, \tag{2}$$

where $0 < \lambda < 1$.

Theorem 3.1 *Assume that the hypotheses* (H_0)*-*(H_4) *hold. Suppose also that* $\mu(t) \leq t, \theta(t) \leq t, \eta(t) \leq t$ *and* $\sigma(t) \leq t$ *for all* $t \in J$ *and*

$$\int_{\|q\|_{BM}}^{\infty} \frac{ds}{s + \psi(s)} > C,$$

where $C = max\{V, K\|\phi\|_{L^1}\}$, *and* $V = \max_{t,s \in J} |v(t,s)|, K = \max_{t,s\in J} |k(t,s)|$. *Then the FIE (1) has a solution on* J.

Proof. Define the operators $A, B : BM(J, \mathbb{R}) \to BM(J, \mathbb{R})$ by

$$Ax(t) = \int_0^{\mu(t)} v(t,s)x(\theta(s))ds, \quad t \in J, \tag{3}$$

and

$$Bx(t) = q(t) + \int_0^{\sigma(t)} k(t,s)g(s, x(\eta(s)))ds, \quad t \in J. \tag{4}$$

Then the problem of finding the solution of FIE (1) is just reduced to finding the solution of the operator equation $Ax(t) + Bx(t) = x(t),\ t \in J$, which is the same as $Ax(t) + \lambda Bx(t) = x(t),\ t \in J$, when $\lambda = 1$. We shall show that the operators A and B satisfy all the conditions of Theorem 2.4. Obviously the operator A is linear and bounded with $\|A\| = V$. We show that A^n is a contraction for large value of n. Let $x, y \in BM(J, \mathbb{R})$. Then

$$\begin{aligned}
|Ax(t) - Ay(t)| &= \left| \int_0^{\mu(t)} v(t,s)x(\theta(s))ds - \int_0^{\mu(t)} v(t,s)y(\theta(s))ds \right| \\
&= \left| \int_0^{\mu(t)} v(t,s)[x(\theta(s)) - y(\theta(s))]ds \right| \\
&\leq \int_0^{\mu(t)} |v(t,s)||x(\theta(s)) - y(\theta(s))|ds \\
&\leq \int_0^t V\|x - y\|_{BM} ds \\
&\leq V\|x - y\|_{BM}.
\end{aligned}$$

Taking the maximum over t,

$$\|Ax - Ay\|_{BM} \leq V\|x - y\|_{BM}.$$

Again

$$\begin{aligned}
|A^2x(t) - A^2y(t)| &\leq \int_0^{\mu(t)} V \left(\int_0^{\mu(s)} |v(s,\tau)|\|x - y\|_{BM} d\tau \right) ds \\
&\leq \int_0^t V \left(\int_0^s |v(s,\tau)|\|x - y\|_{BM} d\tau \right) ds \\
&\leq \frac{V^2}{2!}\|x - y\|_{BM}.
\end{aligned}$$

In general for any $n \in \mathbb{N}$,

$$|A^n x(t) - A^n y(t)| \leq \frac{V^n}{n!} \|x - y\|_{BM}$$

or

$$\|A^n x - A^n y\|_{BM} \leq \frac{V^n}{n!} \|x - y\|_{BM}.$$

Since $\lim\limits_{n\to\infty} \dfrac{V^n}{n!} = 0$, there exists a $p \in \mathbb{N}$ such that $\dfrac{V^p}{p!} < 1$ and consequently A^p is a contraction on $BM(J, \mathbb{R})$.

Next we show that B is a completely continuous operator on $BM(J, \mathbb{R})$. Since σ and η are continuous, it follows using the standard arguments as in Granas et al. [5] that B is a continuous operator on $BM(J, \mathbb{R})$. Let $\{x_n\}$ be a bounded sequence in $BM(J, \mathbb{R})$ such that $\|x_n\|_{BM} \leq r$ for some $r > 0$ and for every $n \in \mathbb{N}$. Then by (H_3),

$$\begin{aligned}
\|Bx_n\|_{BM} &\leq \max_{t\in J} |q(t)| + \max_{t\in J} \int_0^t |k(t,s)||g(s, x_n(\eta(s)))| ds \\
&\leq \|q\|_{BM} + K \int_0^1 h_r(s) ds \\
&= \|q\|_{BM} + K \|h_r\|_{L^1}.
\end{aligned}$$

This shows that $\{Bx_n : n \in \mathbb{N}\}$ is a uniformly bounded set in $BM(J, \mathbb{R})$. Next we show that $\{Bx_n : n \in \mathbb{N}\}$ is equicontinuous set. Let $t, \tau \in J$. Then we have,

$$\begin{aligned}
|Bx_n(t) \;&-\; Bx_n(\tau)| \\
&\leq \left| \int_0^{\sigma(t)} k(t,s) g(s, x_n(\eta(s))) ds - \int_0^{\sigma(\tau)} k(\tau,s) g(s, x_n(\eta(s))) ds \right| \\
&\quad + |q(t) - q(\tau)| \\
&\leq \left| \int_0^{\sigma(t)} k(t,s) g(s, x_n(\eta(s))) ds - \int_0^{\sigma(t)} k(\tau,s) g(s, x_n(\eta(s))) ds \right| \\
&\quad + \left| \int_0^{\sigma(t)} k(\tau,s) g(s, x_n(\eta(s))) ds - \int_0^{\sigma(\tau)} k(\tau,s) g(s, x_n(\eta(s))) ds \right| \\
&\quad + |q(t) - q(\tau)| \\
&\leq \left| \int_0^{\sigma(t)} [k(t,s) - k(\tau,s)] g(s, x_n(\eta(s))) ds \right| \\
&\quad + \left| \int_{\sigma(\tau)}^{\sigma(t)} k(\tau,s) g(s, x_n(\eta(s))) ds \right| + |q(t) - q(\tau)| \\
&\leq \int_0^1 |k(t,s) - k(\tau,s)| h_r(s) ds + K |p(t) - p(\tau)| + |q(t) - q(\tau)|
\end{aligned}$$

$$= \quad \|h_r\|_{L^1} \int_0^1 |k(t,s) - k(\tau,s)|ds + K|p(t) - p(\tau)| + |q(t) - q(\tau)|,$$

where $p(t) = \displaystyle\int_0^{\sigma(t)} h_r(s)ds$.

Since p, q and k are uniformly continuous functions, it follows that

$$|Bx_n(t) - Bx_n(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

Hence $\{Bx_n : n \in \mathbb{N}\}$ is equicontinuous and consequently $\{Bx_n : n \in \mathbb{N}\}$ is compact by Arzela-Ascoli theorem. Hence B is a completely continuous operator on $BM(J, \mathbb{R})$. Thus all the conditions of Theorem 2.4 are satisfied. Hence an application of it yields that either conclusion (i) or conclusion (ii) holds. This further implies that either FIE (1) has a solution or the set $\mathcal{E}$ of all solutions of FIE (2) is unbounded. We will show that condition (ii) is not possible. If x is any solution to FIE (2) then we have,

$$x(t) = \lambda q(t) + \int_0^{\mu(t)} v(t,s)x(\theta(s))ds + \lambda \int_0^{\sigma(t)} k(t,s)g(s, x(\eta(s)))ds, \;\; t \in J,$$

for $\lambda \in (0,1)$. Then

$$\begin{aligned} |x(t)| &\leq |q(t)| + \int_0^{\mu(t)} |v(t,s)||x(\theta(s))|ds + \int_0^{\sigma(t)} |k(t,s)||g(s, x(\eta(s)))|ds \\ &\leq \|q\|_{BM} + \int_0^{\mu(t)} V|x(\theta(s))|ds + \int_0^{\sigma(t)} K\phi(s)\psi(|x(\eta(s))|)ds. \end{aligned}$$

Let $w(t) = \max\limits_{s \in [0,t]} |x(s)| = |x(t^*)|$ for some $t^* \in [0,t]$. Obviously $|x(t)| \leq w(t)$, for each $t \in J$. From the above inequality one has

$$\begin{aligned} w(t) = |x(t^*)| &\leq \|q\|_{BM} + \int_0^{\mu(t^*)} V|x(\theta(s))|ds + \int_0^{\sigma(t^*)} K\phi(s)\psi(|x(\eta(s))|)ds \\ &\leq \|q\|_{BM} + \int_0^{t^*} Vw(s)ds + \int_0^{t^*} K\phi(s)\psi(w(s))ds \\ &\leq \|q\|_{BM} + V\int_0^{t} w(s)ds + K\|\phi\|_{L^1} \int_0^{t} \psi(w(s))ds \\ &\leq \|q\|_{BM} + C\int_0^{t} [w(s) + \psi(w(s))]ds, \end{aligned}$$

where $C = max\{V, K\|\phi\|_{L^1}\}$.

Put $u(t)=\|q\|_{BM}+C\displaystyle\int_0^t[w(s)+\psi(w(s))]ds,\ \ t\in J$. Then $u(0)=\|q\|_{BM},\ \ w(t)\leq u(t),\ \ t\in J$ and

$$u'(t)=C[w(t)+\psi(w(t))]\leq C[u(t)+\psi(u(t))],$$

or

$$\frac{u'(t)}{u(t)+\psi(u(t))}\leq C,\quad t\in J.$$

On integration of this inequality w.r.t. t from 0 to t yields

$$\int_0^t\frac{u'(s)}{u(s)+\psi(u(s))}ds\leq\int_0^t Cds\leq C.$$

Now change of the variable in the above inequality yields,

$$\int_{\|q\|_{BM}}^{u(t)}\frac{ds}{s+\psi(s)}\leq C<\int_{\|q\|_{BM}}^{\infty}\frac{ds}{s+\psi(s)}.$$

From the above inequality it follows that there exists a constant $M>0$ such that $u(t)\leq M$ for $t\in J$. This further implies that $|x(t)|\leq w(t)\leq u(t)\leq M$, for $t\in J$, and therefore

$$\|x\|=\max_{t\in J}|x(t)|\leq M.$$

Thus the conclusion (ii) of Theorem 2.4 does not hold and so the conclusion (i) holds. Consequently the FIE (1) has a solution on J. This completes the proof. ■

As an application, we consider the initial value problem (in short IVP) of first order ordinary functional differential equation

$$x'(t)=\alpha(t)x(\theta(t))+g(t,x(\eta(t))),\quad a.e.\ t\in J \tag{5}$$

$$x(0)=x_0\in\mathbb{R} \tag{6}$$

where $\alpha:J\to\mathbb{R}$ is continuous and $g:J\times\mathbb{R}\to\mathbb{R}$.

By a solution of the IVP (5)–(6) we mean a function $x\in AC(J,\mathbb{R})$ which satisfies the equations (5)–(6), where $AC(J,\mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

Theorem 3.2 *Assume that the hypotheses* $(H_3)-(H_4)$ *hold. Suppose also that* $\theta(t)\leq t$ *and* $\eta(t)\leq t$, *for all* $t\in J$ *and*

$$\int_{|x_0|}^{\infty}\frac{ds}{s+\psi(s)}>C,$$

where $C=max\{\sup_{t\in J}|\alpha(t)|,\|\phi\|_{L^1}\}$. *Then the IVP (5)–(6) has a solution on J.*

Proof. The IVP (5)–(6) is equivalent to to the integral equation

$$x(t) = x_0 + \int_0^t \alpha(s)x(\theta(s))ds + \int_0^t g(s, x(\eta(s)))ds, \quad t \in J.$$

Take $q(t) = x_0, \mu(t) = \theta(t) = \sigma(t) = \eta(t) = t$ for all $t \in J$, $v(t,s) = \alpha(t)$, and $k(t,s) = 1$ for all $t, s \in J$. Then all the conditions of Theorem 3.1 are satisfied and hence an application of it yields the desired conclusion, since $AC(J, \mathbb{R}) \subseteq BM(J, \mathbb{R})$. The proof is complete. ■

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