LIMITS OF SOLUTIONS OF A PERTURBED LINEAR DIFFERENTIAL EQUATION

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Abstract

Using interesting techniques, an existence result for the problem $\ddot{x} + 2f(t)\dot{x} + x + g(t,x) = 0$, $\lim\limits_{t\to+\infty} x(t) = \lim\limits_{t\to+\infty} \dot{x}(t) = 0$, is given in [2]. This note treates the same problem via Schauder-Tychonoff and Banach theorems.

AMS Mathematics Subject Classification (2000): 34B40, 34D05, 34D10.

Key words and phrases: Boundary value problems on infinite interval, Asymptotic properties, Perturbation.

1. Introduction

This note is devoted to the existence of the solutions for the boundary value problem

$$\ddot{x} + 2f(t)\dot{x} + x + g(t,x) = 0, \tag{1.1}$$

$$x(+\infty) = \dot{x}(+\infty) = 0, \tag{1.2}$$

where

$$x(+\infty) := \lim_{t\to+\infty} x(t), \ \dot{x}(+\infty) := \lim_{t\to+\infty} \dot{x}(t).$$

The equation (1.1) has been considered by different authors (see e.g. [2], [3]).

In [2] T.A. Burton and T. Furumochi research the asymptotic stability for the equation (1.1); more precisely, in certain hypotheses (enumerated below) one proves that for initial data small enough, the equation (1.1) admits solutions defined on $\mathbb{R}_+ = [0, +\infty)$ fulfilling the conditions

$$x(+\infty) = 0, \ \dot{x}(+\infty) = 0. \tag{1.3}$$

This result is established by using Schauder's fixed point theorem to an adequate operator H, built in the Banach space

$$C := \left\{ z : \mathbb{R}_+ \to \mathbb{R}^2, \ z \text{ continuous and bounded} \right\},$$

endowed with the usual norm $\|z\|_\infty := \sup\limits_{t \in \mathbb{R}_+} |z(t)|$, where $|\cdot|$ represents a nom in $\mathbb{R}^2$.

To build the operator H one changes firstly the equation (1.1) to a system

$$\dot{z} = A(t) z + F(t, z), \tag{1.4}$$

which is a perturbed system for

$$\dot{z} = A(t) z. \tag{1.5}$$

(here A is a quadratic matrix 2×2, $z = \begin{pmatrix} x \\ y \end{pmatrix}$ and F is a function with values in $\mathbb{R}^2$; the expressions of A and F will be given in the next section)

It is known that if (1.5) is uniformly asymptotically stable, then for "small" perturbations F (1.4) is uniformly asymptotically stable; in the case considered by T. A. Burton and T. Furumochi, the system (1.5) is only asymptotically stable and it is not uniformly asymptotically stable, which praises again the importance of the obtained result.

Let $Z = Z(t)$ be the fundamental matrix of the system (1.5) which is principal in 0; as is well known, if $w : \mathbb{R}_+ \to \mathbb{R}^2$ is a continuous function, then the solutions of the system (1.4) equipped with the condition

$$z(0) = z_0 \tag{1.6}$$

coincide with the fixed points of the operator H defined by

$$(Hw)(t) := Z(t) z_0 + \int_0^t Z(t) Z^{-1}(s) F(s, w(s)) \, ds. \tag{1.7}$$

The problem of the existence of solutions for the system (1.4) for which $z(+\infty)$ does exist (in particular, $z(+\infty) = 0$) has been treated in the general case in [2], if the system (1.5) is uniformly asymptotically stable. In this case, the operator H is considered in C_l, the subspace of C, where

$$C_l := \{x \in C, \ (\exists) \ x(+\infty)\}.$$

By using this space one gets an advantage since an efficient compactness criterion holds (see [1]), which in C it doesn't happen; from this criterion it can be easily proved that H is a compact operator in C_l. In the case when (1.5) is only asymptotically stable the operator H is not compact in C_l. To obviate this difficulty, T.A. Burton and T. Furumochi use the another (equivalent) variant of Schauder's theorem, by identifying a compact convex nonempty subset S of C i.e.

$$S := \{z \in C, \ |z(t)| \leq q(t), \text{ on } \mathbb{R}_+\},$$

where S is equi-continuous and $q : \mathbb{R}_+ \to \mathbb{R}$ is a positive continuous function with $\lim\limits_{t\to+\infty} q(t) = 0$. By showing that $HS \subset S$ and H is continuous, one can apply Schauder's theorem.

In the present paper we show firstly that for initial data small enough the equation (1.1) admits solutions defined on $\mathbb{R}_+$; next we prove that each such a solution fulfills (1.2). To this aim we use, as in [2] an equivalent equation of type (1.6). The admitted hypotheses will be analogous with the ones from [2].

Unlike T.A. Burton and T. Furumochi, we shall apply Schauder-Tychonoff' fixed point theorem in the Fréchet space

$$C_c := \left\{z : \mathbb{R}_+ \to \mathbb{R}^2, \ z \text{ continuous}\right\},$$

endowed with a family of seminorms as chosen as to determine the convergence on compact subsets of $\mathbb{R}_+$ with the usual topology.

Furthermore we shall indicate the possibility to apply Banach's fixed point theorem in C_c.

2. Convergent solutions

Let $f : \mathbb{R}_+ \to (0, +\infty)$ and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be two continuous functions, where $\mathbb{R}_+ = [0, +\infty)$. Consider the boundary value problem (1.1), (1.2).

Admit the following hypotheses:

(i) f is of class $C^1(\mathbb{R}_+)$, $f(+\infty) = 0$ and $\int_0^{+\infty} f(s)\,ds = +\infty$, where $\int_0^{+\infty} f(s)\,ds := \lim\limits_{t\to+\infty} \int_0^t f(s)\,ds$;

(ii) there exists $K \in (0,1)$ such that

$$\left|f'(t) + f^2(t)\right| \le Kf(t)\,,\ t \in \mathbb{R}_+; \tag{2.1}$$

(iii) there exists $M > 0$ and $\alpha > 1$ such that

$$|g(t,x)| \le Mf(t)\,|x|^{\alpha}\,,\ t \in \mathbb{R}_+, x \in \mathbb{R}. \tag{2.2}$$

The main result is contained in the following theorem.

Theorem 2.1 *Under hypotheses (i), (ii) and (iii) there exists an $a > 0$ such that every solution x of the equation* (1.1) *with $|x(0)| < a$ is defined on $\mathbb{R}_+$ and satisfies the condition* (1.2).

The proof will be made in more steps and it needs some preliminary notations.

Firstly, as in [2] one changes the equation (1.1) to a system

$$z' = A(t)\,z + B(t)\,z + F(t,z)\,, \tag{2.3}$$

where $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{cases} x = y - f(t)\,x \\ y = (f'(t) + f^2(t) - 1)\,x - f(t)\,y - g(t,x) \end{cases}$,

$A(t) = \begin{pmatrix} -f(t) & 1 \\ -1 & -f(t) \end{pmatrix}$, $B(t) = \begin{pmatrix} 0 & 0 \\ f'(t) + f^2(t) & 0 \end{pmatrix}$ and

$F(t,z) = \begin{pmatrix} 0 \\ -g(t,x) \end{pmatrix}$.

Attach to system (2.3) the initial condition

$$z(0) = z_0. \tag{2.4}$$

Denote by Z the fundamental matrix of the system

$$z' = A(t)\,z, \tag{2.5}$$

which is principal in 0. An easy computation shows that

$$Z(t) = \exp\left(-\int_0^t f(s)\,ds\right)\cdot\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},\ t \in \mathbb{R}_+. \tag{2.6}$$

For $z = (x, y) \in \mathbb{R}^2$ set $|z| := \max\{|x|, |y|\}$ and for a matrix $U = (u_{ij})_{i,j\in\overline{1,2}}$ set

$$|U| := \max_{i\in\overline{1,2}} \sum_{j=1}^{2} |u_{ij}|.$$

Consider as fundamental the space

$$C_c := \left\{z : \mathbb{R}_+ \to \mathbb{R}^2,\ z \text{ continuous}\right\}.$$

C_c is a Fréchet space (i.e. a complete metrizable real linear space) with respect to the family of seminorms

$$|z|_n := \sup_{t\in[0,n]} \{|z(t)|\}. \tag{2.7}$$

We mention that the topology defined by the family of seminorms (2.7) is the topology of the convergence on compact subsets of $\mathbb{R}_+$; in addition, a family $A \subset C_c$ is relatively compact if and only if it is equi-continuous and uniformly bounded on compacts subsets of $\mathbb{R}_+$ (Ascoli-Arzela' theorem).

Define in C_c the operator

$$(Hw)(t) := Z(t) z_0 + \int_0^t Z(t) Z^{-1}(s) [B(s) w(s) + F(s, w(s))] ds, \tag{2.8}$$

$w \in C_c$.

It is obvious that the set of solutions for the problem $(2.3), (2.4)$ coincides with the set of fixed points of H.

Set

$$B_\rho := \{z \in C_c,\ \ |z(t)| \le \rho\},$$

where $\rho > 0$ is a fixed number; obviously, B_ρ is a closed bounded convex nonempty subset of C_c.

Lemma 2.1 *There exists a number $h > 0$ such that for every $\rho \in (0, h)$ there exists a number $a > 0$ with the property for every z_0 with $|z_0| \in (0, a)$,*

$$HB_\rho \subset B_\rho. \tag{2.9}$$

Proof. Let $w \in B_\rho$ and $z = Hw$; therefore z is given by (2.8).

We have the following estimates:

$$|Z(t) z_0| \le e^{-\int_0^t f(s)ds} |z_0|, \tag{2.10}$$

$$\left|\int_0^t Z(t)\,Z^{-1}(s)\,B(s)\,w(s)\,ds\right| \le K\int_0^t e^{-\int_s^t f(u)du} f(s)\,|w(s)|\,ds, \qquad (2.11)$$

$$\left|\int_0^t Z(t)\,Z^{-1}(s)\,F(s,w(s))\,ds\right| \le M\int_0^t e^{-\int_s^t f(u)du} f(s)\,|w(s)|^{\alpha}\,ds, \qquad (2.12)$$

which are proved in [2].

By substituting in (2.11) and (2.12) the inequality $|w(s)| \le \rho$, $s \in \mathbb{R}_+$ one gets

$$|z(t)| \le |z_0| + \rho\left[K + M\rho^{\alpha-1}\right]. \qquad (2.13)$$

Indeed, one has successively

$$\begin{aligned}
|z(t)| &\le |z_0| + K\int_0^t e^{-\int_s^t f(u)du} f(s)\,\rho ds + \\
&\quad + M\int_0^t e^{-\int_s^t f(u)du} f(s)\,\rho^{\alpha} ds \\
&\le |z_0| + (K\rho + M\rho^{\alpha})\int_0^t \left(-e^{-\int_s^t f(u)du}\right)'(s)\,ds = \\
&= |z_0| + (K\rho + M\rho^{\alpha})\left(1 - e^{-\int_0^t f(u)du}\right) \le |z_0| + (K\rho + M\rho^{\alpha}).
\end{aligned}$$

Let $h := \left(\frac{1-K}{M}\right)^{\frac{1}{\alpha-1}}$; then, if $\rho < h$ one has $K + M\rho^{\alpha-1} < 1$.

Let $\rho \in (0,h)$ be arbitrary; set

$$a := \rho\left[1 - \left(K + M\rho^{\alpha-1}\right)\right]. \qquad (2.14)$$

Obviously, $a > 0$; in addition it results that

$$(|z_0| < a) \Longrightarrow (|(Hw)(t)| \le \rho), \qquad (2.15)$$

which ends the proof. $\square$

Lemma 2.2 *If z is a solution of the problem* $(2.3), (2.4)$ *defined on* $\mathbb{R}_+$, *then for $|z_0|$ small enough,* $z(+\infty) = 0$.

Proof. Let $z = \begin{pmatrix} x \\ y \end{pmatrix}$. Then one has

$$\begin{aligned}
|z(t)| &\le |z_0|\,e^{-\int_0^t f(s)ds} + \int_0^t e^{-\int_s^t f(u)du}\left|f'(s) + f^2(s)\right|\,|x(s)|\,ds + \\
&\quad + \int_0^t e^{-\int_s^t f(u)du}\,|g(s,x(s))|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \; e^{-\int_0^t f(s)ds}\left|z_0\right| + \\
&\quad + \int_0^t e^{-\int_s^t f(u)du}\left[Kf(s)\left|x(s)\right| + Mf(s)\left|x(s)\right|^{\alpha}\right]ds \\
&: \; = r(t)\,,\ t \in \mathbb{R}_+.
\end{aligned}
$$

Evaluating $r'(t)$ one finds

$$
\begin{aligned}
r'(t) &= -\left|z_0\right| e^{-\int_0^t f(s)ds} f(t) + \left[Kf(t)\left|x(t)\right| + Mf(t)\left|x(t)\right|^{\alpha}\right] - \\
&\quad - \int_0^t e^{-\int_s^t f(u)du} f(t)\left[Kf(s)\left|x(s)\right| + Mf(s)\left|x(s)\right|^{\alpha}\right]ds \\
&= -\left|z_0\right| e^{-\int_0^t f(s)ds} f(t) + \left[Kf(t)\left|x(t)\right| + Mf(t)\left|x(t)\right|^{\alpha}\right] - \\
&\quad - f(t)\left[r(t) - \left|z_0\right| e^{-\int_0^t f(s)ds}\right] \\
&= \left[Kf(t)\left|x(t)\right| + Mf(t)\left|x(t)\right|^{\alpha}\right] - f(t)\,r(t)\,.
\end{aligned}
$$

Since $0 \leq \left|x(t)\right| \leq \left|z(t)\right| \leq r(t)$, one obtains

$$
r'(t) \leq f(t)\left[(K-1) + Mr(t)^{\alpha-1}\right] r(t)\,,\ r(0) = \left|z_0\right|.
$$

By Ciaplyghin's lemma we get $r(t) \leq q(t)\,,\ t \in \mathbb{R}_+$, where $q = q(t)$ is the unique solution of the (Bernoulli) problem

$$
\left\{
\begin{array}{l}
q' = f(t)\left[(K-1) + q^{\alpha-1}\right] q \\
q(0) = \left|z_0\right|
\end{array}
\right. ,
$$

i.e.

$$
\begin{aligned}
q(t) &= \Bigg[\left|z_0\right|^{1-\alpha} e^{(1-\alpha)(K-1)\int_0^t f(s)ds} + \\
&\quad + M(1-\alpha)\, e^{(1-\alpha)(K-1)\int_0^t f(s)ds}\,\cdot \\
&\quad \cdot \int_0^t f(s)\, e^{-(1-\alpha)(K-1)\int_0^s f(u)du} ds\Bigg]^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

We have an equivalent form of $q(t)$,

$$
q(t) = \left[e^{(1-\alpha)(K-1)\int_0^t f(s)ds}\left(\left|z_0\right|^{1-\alpha} - \frac{M}{1-K}\right) + \frac{M}{1-K}\right]^{\frac{1}{1-\alpha}}.
$$

Therefore, for $|z_0| \in \left(0, \left(\frac{1-K}{M}\right)^{\frac{1}{\alpha-1}}\right)$, by hypotheses (i), (ii) and (iii) we find

$$\lim_{t\to+\infty} q(t) = 0.$$

Hence,

$$z(+\infty) = \lim_{t\to+\infty} z(t) = 0.$$

The proof is now complete. $\square$

To prove Theorem 2.1 it remains to show that for every z_0 with $|z_0|$ small enough, the problem $(2.3), (2.4)$ admits solutions on $\mathbb{R}_+$. To this aim we use the following theorem.

Theorem (SCHAUDER-TYCHONOFF) *Let E be a Fréchet space, $S \subset E$ be a closed bounded convex nonempty subset of E and $H : S \to S$ be a continuous operator. If HS is relatively compact in E, then H admits fixed points.*

Setting $E = C_c$, H given by (2.8), and $S = B_\rho$ it remains to prove the continuity of H and the relatively compactness of HS.

Let $w_n \in B_\rho$ such that $w_m \to w$ in C_c, as $m \to \infty$; that means $(\forall)\ \epsilon > 0$, $(\exists)\ m_0 = m_0(\epsilon)$, $(\forall)\ m > m_0$, $(\forall)\ t \in [0,n]$, $|w_m(t) - w(t)| < \epsilon$.

But

$$\begin{aligned}
|(Hw)(t) - (Hw_m)(t)| &\leq \left|\int_0^n Z(t) Z^{-1}(s) B(s) [w(s) - w_m(s)]\, ds\right| + \\
&\quad + \left|\int_0^n Z(t) Z^{-1}(s) [F(s, w(s)) - F(s, w_m(s))] ds\right| \\
&\leq \alpha_n \int_0^n |w(s) - w_m(s)|\, ds + \\
&\quad + \beta_n \int_0^n |F(s, w(s)) - F(s, w_m(s))|\, ds,
\end{aligned}$$

where

$$\alpha_n = \sup_{t,s\in[0,n]} \left|Z(t) Z^{-1}(s) B(s)\right| \text{ and } \beta_n = \sup_{t,s\in[0,n]} \left|Z(t) Z^{-1}(s)\right|.$$

Since $F(t,z)$ is uniformly continuous for $t \in [0,n]$ and $|z| \leq \rho$, it follows that the sequence $F(t, w_m(t))$ converges uniformly on $[0,n]$ to $F(t, w(t))$, which finally proves the continuity of H.

Let us show that HB_ρ is relatively compact; from $HB_\rho \subset B_\rho$ it results that HB_ρ is unifomly bounded in C_c.

Let $w \in B_\rho$ be arbitrary; since $z = Hw \in B_\rho$ and

$$z' = A(t)\,z + B(t)\,w + F(t,w)$$

one gets

$$|z'(t)| \le \gamma_n \rho + \delta_n,\ t \in [0,n],$$

where

$$\gamma_n := \sup_{t\in[0,n]} \{|A(t)|\} + \sup_{t\in[0,n]} \{|B(t)|\}$$

and

$$\delta_n := \sup_{t\in[0,n],\ w\le\rho} \{|F(t,w)|\}.$$

So, having the family of derivatives uniformly bounded, HB_ρ is equicontinuous on the compact subsets of $\mathbb{R}_+$. The proof is now complete. $\square$

3. The case when H is contractant

In [2] the mentioned result is obtained by admitting in addition a supplementary hypothesis i.e.

(iv) for every $\delta > 0$ there exists $L(\delta) > 0$ such that

$$|g(t,x) - g(t,y)| \le L(\delta)\, f(t)\, |x-y|,\ \ |x|, |y| < \delta,\ t \in \mathbb{R}_+,$$

and $L(\delta)$ is a continuous and increasing function.

If we admit a Lipschitz condition, then one can prove that the problem $(2.3), (2.4)$ admits an unique solution which is convergent to zero for every z_0 with $|z_0| < a$. Obviously, we can separately prove the uniqueness, but we prefer to obtain simultaneously the existence and the uniqueness, by admitting a weaker condition than (iv).

To this aim, we shall use the Banach's fixed point theorem which in Fréchet spaces has the following statement.

Theorem (BANACH) *Let E be a Fréchet space endowed with a family of seminorms $|\cdot|_n$ and let $S \subset E$ be a closed subset of E. Let $H : S \to S$ be an operator fulfilling the following condition: for every $n \in \mathbb{N}^*$ there exists a positive number $L_n \in [0,1)$ such that for every $x, y \in S$,*

$$|Hx - Hy|_n \le L_n\, |x-y|_n. \tag{3.1}$$

Then H admits an unique fixed point.

To apply the stated Banach's theorem we shall consider in the space C_c another family of seminorms, topologically equivalent with (2.9), i.e.

$$|z|_n := \sup_{t\in[0,n]} \left\{ |z(t)| e^{-\lambda_n |t|} \right\}, \tag{3.2}$$

where $\lambda_n > 0$ are arbitrary numbers.

Theorem 3.1 *Suppose that the hypotheses (i), (ii) and (iii) are fulfilled and in addition for every $n \in \mathbb{N}^*$ there exists $L_n \in [0,1)$ such that for every $x_1, x_2 \in \mathbb{R}_+$ with $|x_i| \le \rho$, $(\rho < a)$*

$$|g(t,x_1) - g(t,x_2)| \le L_n |x_1 - x_2|, \; t \in [0,n]. \tag{3.3}$$

Then the problem $(2.3), (2.4)$ *admits an unique solution which is convergent to zero.*

Proof. Consider the same operator H with $HB_\rho \subset B_\rho$. Let $z_i = (x_i, y_i) \in B_\rho$, $i \in \overline{1,2}$; since $|x_i - y_i| \le |z_1 - z_2|$, $i \in \overline{1,2}$ we have

$$|(Hz_1)(t) - (Hz_2)(t)| \le \int_0^t \left| Z(t) Z^{-1}(s) \right| [|B(s)| + L_n] |z_1(s) - z_2(s)| \, ds,$$

for every $t \in [0,n]$.

Therefore, by setting

$$\mu_n := \sup_{t,s\in[0,n]} \left| Z(t) Z^{-1}(s) \right| [|B(s)| + L_n],$$

we obtain

$$\begin{aligned} |(Hz_1)(t) - (Hz_2)(t)| &\le \mu_n \int_0^t |z_1(s) - z_2(s)| e^{-\lambda_n |s|} e^{\lambda_n |s|} ds \\ &\le \mu_n |z_1 - z_2|_n \frac{1}{\lambda_n} \left(e^{\lambda_n |t|} - 1 \right) \le \\ &\le \frac{\mu_n}{\lambda_n} |z_1 - z_2|_n e^{\lambda_n |t|}. \end{aligned}$$

By multiplying this last inequality with $e^{-\lambda_n |t|}$ and by passing to sup as $t \in [0,n]$ it results

$$|Hz_1 - Hz_2|_n \le \frac{\mu_n}{\lambda_n} |z_1 - z_2|_n.$$

Hence, by taking $\lambda_n > \mu_n$ the hypotheses of Banach's theorem are fulfilled. □

4. Examples

As the authors mention in [2] an example of functions f and g could be $f(t) = \frac{1}{t+1}$, $g(t,x) = \frac{x^2}{t+1}$. Another examples could be: $f(t) = \frac{1}{\ln(t+2)}$, $g(t,x) = \frac{x^2}{\ln(t+2)}$ or $f(t) = \frac{1}{(t+2)\ln(t+2)}$, $g(t,x) = \frac{x^2}{(t+2)\ln(t+2)}$. Then, the hypotheses (i), (ii), (iii) and (iv) are fulfilled, with $K = \frac{3}{4}$, $\alpha = 2$, $M = 1$, $L(\delta) = 2\delta$.

References

[1] Avramescu, C., Sur l'existence des solutions convergentes des systèmes d'équations différentielles ordinaires, *Ann. Mat. Purra ed Appl.*, (IV), Vol. LXXXI, 1969, pp. 147-168.

[2] Burton, T.A. and Furumochi, T., A note on stability by Schauder's theorem, *Funkcialaj Ekvacioj,* 44(2001), pp. 73-82.

[3] Hatvani, L., Integral conditions on the asymptotic stability for the damped linear oscillator with small damping, *Proc. Amer. Math. Soc.,* 124(1996), pp. 415-422.

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