GLOBAL EXISTENCE OF SOLUTIONS FOR REACTION-DIFFUSION SYSTEMS WITH A FULL MATRIX OF DIFFUSION COEFFICIENTS AND NONHOMOGENEOUS BOUNDARY CONDITIONS.

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ABSTRACT. In this article, we generalize the results obtained in [16] concerning uniform bounds and so global existence of solutions for reaction-diffusion systems with a full matrix of diffusion coefficients satisfying a balance law and with homogeneous Neumann boundary conditions. Our techniques are based on invariant regions and Lyapunov functional methods. We demonstrate that our results remain valid for nonhomogeneous boundary conditions and with out balance law's condition. The nonlinear reaction term has been supposed to be of polynomial growth.

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1. INTRODUCTION

We consider the following reaction-diffusion system

$$\frac{\partial u}{\partial t}\text{-}a\Delta u\text{-}b\Delta v = f(u,v) \quad \text{in } \mathbb{R}^+ \times \Omega \tag{1.1}$$

$$\frac{\partial v}{\partial t}\text{-}c\Delta u\text{-}a\Delta v = g(u,v) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$

with the boundary conditions

$$\lambda u + (1-\lambda)\frac{\partial u}{\partial \eta} = \beta_1 \text{ and } \lambda v + (1-\lambda)\frac{\partial v}{\partial \eta} = \beta_2 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x) \quad \text{in } \Omega, \tag{1.4}$$

where

$$(i)\; 0 < \lambda < 1 \text{ and } \beta_i \in \mathbb{R},\; i = 1, 2,$$

(Robin nonhomogeneous boundary conditions) or homogeneous Neumann boundary conditions

$$(ii)\; \lambda = \beta_i = 0,\; i = 1, 2,$$

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or homogeneous Dirichlet boundary conditions

$$(iii)\ 1-\lambda=\beta_i=0,\ i=1,2.$$

Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$, $\dfrac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$, $\sigma,\ \rho,\ a$, b and c are positive constants satisfying the condition $2a>(b+c)$ which reflects the parabolicity of the system. The initial data are assumed to be in the following region

$$\Sigma=\begin{cases}\left\{(u_0,v_0)\in IR^2 \text{ such that } v_0\geq\mu\left|u_0\right|\right\} \text{ if } \beta_2\geq\mu\left|\beta_1\right|,\\ \left\{(u_0,v_0)\in IR^2 \text{ such that } \mu u_0\geq\left|v_0\right|\right\} \text{ if } \beta_1\geq\mu\left|\beta_2\right|,\end{cases}$$

where

$$\mu=\sqrt{\frac{c}{b}}.$$

One will treat the first case, the second will be discussed at the last section. We suppose that the reaction terms f and g are continuously differentiable, polynomially bounded on Σ, $(f(r,s),g(r,s))$ is in Σ for all (r,s) in $\partial\Sigma$ (we say that (f,g) points into Σ on $\partial\Sigma$); that is

$$g(s,\mu s)\geq\mu f(s,\mu s) \text{ and } g(-s,\mu s)\geq-\mu f(-s,\mu s) \text{ for all } s\geq0 \tag{1.5}$$

and for positive constants $C<\mu$ sufficiently close to μ, we have

$$Cf(u,v)+g(u,v)\leq C_1\left(u+v+1\right) \text{ for all } u \text{ and } v \text{ in } \Sigma \tag{1.6}$$

where $C_1$is a positive constant.

The system (1.1)-(1.2) may be regarded as a perturbation of the simple and trivial case where $b=c=0$; for which nonnegative solutions exist globally in time. Always in this case with homogeneous Neumann boundary conditions but when the coefficient of $-\Delta u$ in the first equation is different of the one of $-\Delta v$ in the second one (diagonal case), N. Alikakos [1] established global existence and L^∞-bounds of solutions for positive initial data in the case

$$g(u,v)=-f(u,v)=-uv^\beta \tag{*}$$

where $1<\beta<\frac{(n+2)}{n}$. The reactions given by (*) satisfy in fact a condition analogous to (1.5) and form a special case; since (f,g) point into Σ on $\partial\Sigma$ by taking $\Sigma=\mathbb{R}^+\times\mathbb{R}^+$. K. Masuda [20] showed that solutions to this system exist globally for every $\beta>1$ and converge to a constant vector as $t\to+\infty$. A. Haraux and A. Youkana [6] have generalized the method of K. Masuda to handle nonlinearities $uF(v)$ that are from a particular case of our one; since they took also $\Sigma=\mathbb{R}^+\times\mathbb{R}^+$. Recently S. Kouachi and A. Youkana [18] have generalized the method of A. Haraux and A. Youkana by adding $-c\Delta v$ to the left-hand side of the diagonal case and by taking nonlinearities $f(u,v)$ of a weak exponential growth. J. I. Kanel and M. Kirane [10] have proved global existence , in the case $g(u,v)=-\ f(u,v)=-uv^n$

and n is an odd integer, under an embarrassing condition which can be written in our case as

$$|b-c| < C_p,$$

where C_p contains a constant from an estimate of Solonnikov. Recently they ameliorate their results in [11] to obtain global existence under the conditions

$$b < \left(\frac{a^2}{a^2+c^2}\right) c$$

and

$$|F(v)| \leq C_F (1+|v|^{1+\alpha},$$

where α and C_F are positive constants with $\alpha < 1$ sufficiently small and $g(u,v) = -f(u,v) = -uF(v)$. All techniques used by authors cited above showed their limitations because some are based on the embedding theorem of Sobolev as Alikakos [1], Hollis-Martin-Pierre [8], ... another as Kanel-Kirane [11] use a properties of the Neumann function for the heat equation for which one of it's restriction the coefficient of $-\Delta u$ in equation (1.1) must be larger than the one of $-\Delta v$ in equation (1.2) whereas it isn't the case of problem (1.1)-(1.4).

This article is a continuation of [16] where the reaction terms satisfied the condition $\sigma g + \rho f \equiv 0$ with σ and ρ was any positive constants, and the function $g(u,v)$ positive and polynomially bounded. In that article we have considered the homogeneous Neumann boundary conditions and established global existence of solutions with initial data in the invariant region Σ considered here.

The components $u(t,x)$ and $v(t,x)$ represent either chemical concentrations or biological population densities and system (1.1)-(1.2) is a mathematical model describing various chemical and biological phenomena (see E. L. Cussler [2], P. L. Garcia-Ybarra and P. Clavin [4], S. R. De Groot and P. Mazur [5], J. Jorne [9], J. S. Kirkaldy [14], A. I. Lee and J. M. Hill [19] and J. Savchik, B. Changs and H. Rabitz [22].

2. EXISTENCE.

In this section, we prove that if (f,g) points into Σ on $\partial\Sigma$, then Σ is an invariant region for problem (1.1)-(1.4), i.e. the solution remains in Σ for any initial data in Σ. At this stage and once the invariant regions are constructed, both problems of local and global existence become easier to be established: For the first problem we demonstrate that system (1.1)-(1.2) with boundary conditions (1.3) and initial data in Σ is equivalent to a problem for which local existence over the whole time interval $[0, T_{\max}[$ can be obtained by known procedure and for the second, since we use usual techniques based on Lyapunov functionals which are not directly applicable to problem (1.1)-(1.4) and need invariant regions (see M. Kirane and S. Kouachi [12], [13], S. Kouachi [15]and [16] and S. Kouachi and A. Youkana [18]).

2.1. **Invariant regions.** The main result of this subsection is the following

Proposition 1. *Suppose that (f,g) points into Σ on $\partial\Sigma$, then for any (u_0, v_0) in Σ the solution $(u(t,.), v(t,.))$ of the problem (1.1)-(1.4) remains in Σ for all t in $[0, T^*[$.*

Proof. The proof is similar to that in S. Kouachi [16].
Multiplying equation (1.1) through by $\sqrt{c}$ and equation (1.2) by $\sqrt{b}$, subtracting the resulting equations one time and adding them an other time we get

$$\frac{\partial w}{\partial t} - \lambda_1 \Delta w = F(w,z) \text{ in }]0,T^*[\times \Omega \tag{2.2}$$

$$\frac{\partial z}{\partial t} - \lambda_2 \Delta z = G(w,z) \text{ in }]0,T^*[\times \Omega, \tag{2.3}$$

with the boundary conditions

$$\lambda w + (1-\lambda)\frac{\partial w}{\partial \eta} = \rho_1 \text{ and } \lambda z + (1-\lambda)\frac{\partial z}{\partial \eta} = \rho_2 \text{ on }]0,T^*[\times \partial\Omega \ , \tag{2.4}$$

and the initial data

$$w(0,x) = w_0(x), \qquad z(0,x) = z_0(x) \qquad \text{in } \Omega, \tag{2.5}$$

where

$$w(t,x) = \sqrt{c}u(t,x) + \sqrt{b}v(t,x) \text{ and } z(t,x) = -\sqrt{c}u(t,x) + \sqrt{b}v(t,x), \tag{2.6}$$

for all (t,x) in $]0,T^*[\times \Omega$,

$$F(w,z) = \sqrt{b}g + \sqrt{c}f \text{ and } G(w,z) = \sqrt{b}g - \sqrt{c}f \quad \text{for all } (u,v) \text{ in } \Sigma, \tag{2.7}$$

$$\lambda_1 = a + \sqrt{bc} \text{ and } \lambda_2 = a - \sqrt{bc}$$

and

$$\rho_1 = \sqrt{c}\beta_1 + \sqrt{b}\beta_2 \text{ and } \rho_2 = -\sqrt{c}\beta_1 + \sqrt{b}\beta_2$$

First, let's notice that the condition of parabolicity of the system (1.1)-(1.2) implies the one of the (2.2)-(2.3) system; since $2a > (b+c) \Rightarrow a - \sqrt{bc} > 0$.
Now, it suffices to prove that the region

$$\left\{(w_0, z_0) \in IR^2 \text{ such that } w_0 \geq 0, z_0 \geq 0\right\} = IR^+ \times IR^+,$$

is invariant for system (2.2)-(2.3). Since, from (1.5) we have $F(0,z) = \sqrt{b}g(-v,\mu v) + \sqrt{c}f(-v,\mu v) \geq 0$ for all $z \geq 0$ and all $v \geq 0$, then $w(t,x) \geq 0$ for all $(t,x) \in]0,T^*[\times \Omega$, thanks to the invariant region's method (see Smoller [23]) and because $G(w,0) = \sqrt{b}g(v,\mu v) - \sqrt{c}f(v,\mu v) \geq 0$ for all $w \geq 0$ and all $v \geq 0$, then $z(t,x) \geq 0$ for all $(t,x) \in]0,T^*[\times \Omega$, then Σ is an invariant region for the system (1.1)-(1.3). ∎

Then system (1.1)-(1.2) with boundary conditions (1.3) and initial data in Σ is equivalent to system (2.2)-(2.3) with boundary conditions (2.4) and positive initial data (2.5). As it has been mentioned at the beginning of this section and since ρ_1 and ρ_2 are positive, for any initial data in $\mathbb{C}\left(\overline{\Omega}\right)$ or $\mathbb{L}^p(\Omega)$, $p \in (1,+\infty)$; local existence and uniqueness of solutions to the initial value problem (2.2)-(2.5) and consequently those of problem (1.1)-(1.4) follow from the basic existence theory for abstract semilinear differential equations (see A. Friedman [3], D. Henry [7] and Pazy [21]). The solutions are classical on $]0,T^*[$, where T^* denotes the eventual blowing-up time in $\mathbb{L}^{\infty}(\Omega)$. The local solution is continued globally by a priori estimates.

Once invariant regions are constructed, one can apply Lyapunov technique and establish global existence of unique solutions for (1.1)-(1.4).

2.2. **Global existence.** As the determinant of the linear algebraic system (2.6), with regard to variables u and v, is different from zero, then to prove global existence of solutions of problem (1.1)-(1.4) comes back in even to prove it for problem (2.2)-(2.5). To this subject, it is well known that (see Henry [7]) it suffices to derive an uniform estimate of $\|F(w,z)\|_p$ and $\|G(w,z)\|_p$ on $[0,T^*[$ for some $p > N/2$, where $\|.\|_p$denotes the usual norms in spaces $\mathbb{L}^p(\Omega)$ defined by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_{\Omega} |u(x)|^p \, dx, \ 1 \leq p < \infty \text{ and } \|u\|_{\infty} = \underset{x\in\Omega}{ess\sup} |u(x)| .$$

Let us define, for all positive integer n, the finite sequence

$$\theta_i = \theta^{i^2}, \ i = 0, 1, ..., n \tag{2.8}$$

where θ is a positive constant sufficiently large such that

$$\theta > \frac{a}{\sqrt{a^2 - bc}}. \tag{2.9}$$

The main result and in some sense the heart of the paper is:

Theorem 2. *Let $(w(t,.), z(t,.))$ be any positive solution of the problem (2.2)-(2.5) and let the functional*

$$t \longrightarrow L(t) = \int_{\Omega} H_n\left(w(t,x), z(t,x)\right) dx, \tag{2.10}$$

where

$$H_n(w,z) = \sum_{i=0}^{n} C_n^i \theta_i w^i z^{n-i}. \tag{2.11}$$

Then the functional L is uniformly bounded on the interval $[0, T^*]$.

Proof. Differentiating L with respect to t yields

$$\begin{aligned} L'(t) &= \int_{\Omega} \left[\frac{\partial H_n}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial H_n}{\partial z}\frac{\partial z}{\partial t}\right] dx \\ &= \int_{\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial w}\Delta w + \lambda_2 \frac{\partial H_n}{\partial z}\Delta z\right) dx + \int_{\Omega} \left(f\frac{\partial H_n}{\partial w} + \frac{\partial H_n}{\partial z} g\right) dx \\ &= I + J. \end{aligned}$$

By simple use of Green's formula we have

$$I = I_1 + I_2,$$

where

$$I_1 = \int_{\partial\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial w} \frac{\partial w}{\partial \eta} + \lambda_2 \frac{\partial H_n}{\partial z} \frac{\partial z}{\partial \eta} \right) dx \tag{2.12}$$

and

$$I_2 = -\int_{\Omega} \left(\lambda_1 \frac{\partial^2 H_n}{\partial w^2} \left|\nabla w\right|^2 + 2a \frac{\partial^2 H_n}{\partial w \partial z} \nabla w \nabla z + \lambda_2 \frac{\partial^2 H_n}{\partial z^2} \left|\nabla z\right|^2 \right) dx. \tag{2.13}$$

First, let's calculate the first and second partial derivatives of H_n with respect to w and z: We have

$$\frac{\partial H_n}{\partial w} = \sum_{i=1}^{n} \left(iC_n^i \theta_i w^{i-1} z^{n-i} \right) \text{ and } \frac{\partial H_n}{\partial z} = \sum_{i=0}^{n-1} \left((n-i) C_n^i \theta_i w^i z^{n-i-1} \right).$$

Using the formula

$$iC_n^i = nC_{n-1}^{i-1}, \text{ for all } i = 1, ..., n \tag{2.14}$$

and changing the index i by $i-1$, we get

$$\frac{\partial H_n}{\partial w} = (n-1) \sum_{i=0}^{n-1} \left(C_{n-1}^i \theta_{i+1} w^i z^{n-1-i} \right). \tag{2.15}$$

For $\frac{\partial H_n}{\partial z}$, using (2.14) and the fact that

$$C_n^i = C_n^{n-i}, \text{ for all } i = 0, ..., n, \tag{2.16}$$

we get

$$\frac{\partial H_n}{\partial z} = (n-1) \sum_{i=0}^{n-1} \left(C_{n-1}^i \theta_i w^i z^{n-1-i} \right). \tag{2.17}$$

While applying formulas (2.15) and (2.17) to $\frac{\partial H_n}{\partial w}$ and (2.17) to $\frac{\partial H_n}{\partial z}$, we deduce by analogy

$$\frac{\partial^2 H_n}{\partial w^2} = (n-1)(n-2) \sum_{i=0}^{n-2} \left(C_{n-2}^i \theta_{i+2} w^i z^{n-2-i} \right), \tag{2.18}$$

$$\frac{\partial^2 H_n}{\partial w \partial z} = (n-1)(n-2) \sum_{i=0}^{n-2} \left(C_{n-2}^i \theta_{i+1} w^i z^{n-2-i} \right) \tag{2.19}$$

and

$$\frac{\partial^2 H_n}{\partial z^2} = (n-1)(n-2) \sum_{i=0}^{n-2} \left(C_{n-2}^i \theta_i w^i z^{n-2-i} \right). \tag{2.20}$$

Now we claim that there exists a positive constant C_2 independent of $t \in [0, T_{\max}[$ such that

$$I_1 \leq C_2 \text{ for all } t \in [0, T_{\max}[. \tag{2.21}$$

To see this, we follow the same reasoning as in S. Kouachi [18]:
(i) If $0 < \lambda < 1$, using the boundary conditions (1.4) we get

$$I_1 = \int\limits_{\partial\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial w} (\gamma_1 - \alpha w) + \lambda_2 \frac{\partial H_n}{\partial z} (\gamma_2 - \alpha z) \right) dx,$$

where $\alpha = \dfrac{\lambda}{1-\lambda}$ and $\gamma_i = \dfrac{\rho_i}{1-\lambda}$, $i = 1,\ 2$. Since

$$\begin{aligned} H(w,z) &= \lambda_1 \frac{\partial H_n}{\partial w} (\gamma_1 - \alpha w) + \lambda_2 \frac{\partial H_n}{\partial z} (\gamma_2 - \alpha z) \\ &= P_{n-1}(w,z) - Q_n(w,z), \end{aligned}$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n and since the solution is positive, then

$$(**) \qquad \limsup_{(|w|+|z|)\to+\infty} H(w,z) = -\infty,$$

which prove that H is uniformly bounded on $\mathbb{R}_+^2$ and consequently (2.21).
(ii) If $\lambda = 0$, then $I_1 = 0$ on $[0, T_{\max}[$.
(iii) The case of homogeneous Dirichlet conditions is trivial; since in this case the positivity of the solution on $[0, T_{\max}[\times\Omega$ implies $\dfrac{\partial w}{\partial \eta} \leq 0$ and $\dfrac{\partial z}{\partial \eta} \leq 0$ on $[0, T_{\max}[\times\partial\Omega$. Consequently one gets again (2.21) with $C_2 = 0$.

$$I_2 = -n(n-1)\sum_{i=0}^{n-2} C_{n-2}^i w^i z^{n-2-i} \int\limits_{\Omega} \left(\lambda_1\theta_{i+2} \left|\nabla w\right|^2 + 2a\theta_{i+1}\nabla w \nabla z + \lambda_2\theta_i \left|\nabla z\right|^2 \right) dx.$$

Using (2.8) and (2.9) we deduce that the quadratic forms (with respect to ∇w and ∇z) are positive since

$$(2.22) \qquad (a\theta_{i+1})^2 - \lambda_1\lambda_2\theta_i\theta_{i+2} = \theta^{2(i+1)^2} \left(a^2 - (a^2 - bc)\theta^2\right) < 0,\ i = 0, ..., n-2.$$

Then

$$(2.23) \qquad I_2 \leq 0.$$

Using (2.15) and (2.17) we get

$$J = n\sum_{i=0}^{n-1} C_{n-1}^i w^i z^{n-1-i} \int\limits_{\Omega} \left[(\theta_{i+1} F(w,z) + \theta_i G(w,z)) \right] dx.$$

Using the expressions (2.7), we obtain

$$J = n\sum_{i=0}^{n-1} C_{n-1}^i w^i z^{n-1-i} \int\limits_{\Omega} \left[\left(\sqrt{c}(\theta_{i+1} - \theta_i) f + \sqrt{b}(\theta_{i+1} + \theta_i) g \right) \right] dx,$$

which can be written

$$J = n\sum_{i=0}^{n-1} C_{n-1}^i (\theta_{i+1} + \theta_i) w^i z^{n-1-i} \int\limits_{\Omega} \left[\sqrt{c} \left(\frac{\left(\frac{\theta_{i+1}}{\theta_i}\right) - 1}{\left(\frac{\theta_{i+1}}{\theta_i}\right) + 1} \right) f + \sqrt{b} g \right] dx.$$

Since the function $x \to \frac{x-1}{x+1}$ is increasing with $\lim\limits_{x\to+\infty}\left(\frac{x-1}{x+1}\right) = 1$ and since the sequence $\left\{\frac{\theta_{i+1}}{\theta_i}\right\}_{0\le i\le n}$ is increasing by choosing θ sufficiently large with $\lim\limits_{i\to+\infty}\left(\frac{\theta_{i+1}}{\theta_i}\right) = +\infty$, we get by using condition (1.6) and relation (2.6) successively

$$J \le C_3 \int\limits_\Omega \left[\sum_{i=1}^{p} (w+z+1)\, C_{p-1}^{i-1} w^{i-1} z^{p-i}\right] dx.$$

Following the same reasoning as in S. Kouachi [15] we get

$$J \le C_4 L(t) \text{ on } [0, T^*].$$

Then we have

$$\overset{\bullet}{L}(t) \le C_5 L(t) + C_6 L^{(p-1)/p}(t) \text{ on } [0, T^*].$$

While putting

$$Z = L^{1/p},$$

one gets

$$p\overset{\bullet}{Z} \le C_5 Z + C_6.$$

The resolution of this linear differential inequality gives the uniform boundedness of the functional L on the interval $[0, T^*]$, what finishes at the same time our reasoning and the proof of the theorem.

Corollary 3. *Suppose that the functions $f(r,s)$ and $g(r,s)$ are continuously differentiable on Σ, point into Σ on $\partial\Sigma$ and satisfy condition (1.5). Then all solutions of (1.1)-(1.4) with initial data in Σ and uniformly bounded on Ω are in $\mathbb{L}^\infty(0, T^*; \mathbb{L}^p(\Omega))$ for all $p \ge 1$.*

Proof. The proof is an immediate consequence of theorem 2, expressions (2.6) and the trivial inequalities

$$\int\limits_\Omega (w(t,x) + z(t,x))^p \, dx \le L(t) \text{ on } [0, T^*[.$$

Proposition 4. *Under hypothesis of corollary 3 if the reactions $f(r,s)$ and $g(r,s)$ are polynomially bounded, then all solutions of (1.1)-(1.3) with initial data in Σ and uniformly bounded on Ω are global in time.*

Proof. As it has been mentioned above; it suffices to derive an uniform estimate of $\|F(w,z)\|_p$ and $\|G(w,z)\|_p$ on $[0, T^*[$ for some $p > n/2$. Since the functions $f(u,v)$ and $g(u,v)$ are polynomially bounded on Σ, then using relations (2.6) and (2.7) we get that $F(w,z)$ and $G(w,z)$ are too and the proof becomes an immediate consequence of corollary 2.3.

3. Final remarks

If $\beta_1 \geq \mu |\beta_2|$, all previous results remain valid in the region

$$\Sigma = \left\{ (u_0, v_0) \in IR^2 \text{ such that } u_0 \geq \sqrt{\frac{b}{c}} |v_0| \right\}.$$

In this case, system (2.2)-(2.3) becomes

$$\frac{\partial w}{\partial t} - \lambda_2 \Delta w = F(w, z) \text{ in }]0, T^*[\times \Omega, \tag{$2.2'$}$$

$$\frac{\partial z}{\partial t} - \lambda_1 \Delta z = G(w, z) \text{ in }]0, T^*[\times \Omega, \tag{$2.3'$}$$

where

$$w = \sqrt{c}u - \sqrt{b}v \text{ and } z = \sqrt{c}u + \sqrt{b}v \text{ on }]0, T^*[\times \Omega \tag{$2.6'$}$$

and

$$F(w, z) = (\sqrt{c}f - \sqrt{b}g)(u, v) \text{ and } G(w, z) = (\sqrt{c}f(u, v) + \sqrt{b}g)(u, v), \tag{$2.7'$}$$

with the boundary conditions

$$\lambda w + (1 - \lambda)\frac{\partial w}{\partial \eta} = \rho_1 \text{ and } \lambda z + (1 - \lambda)\frac{\partial z}{\partial \eta} = \rho_2 \text{ on }]0, T^*[\times \partial\Omega, \tag{$2.4'$}$$

where

$$\rho_1 = \sqrt{c}\beta_1 - \sqrt{b}\beta_2 \geq 0 \text{ and } \rho_2 = \sqrt{c}\beta_1 + \sqrt{b}\beta_2 \geq 0.$$

The conditions (1.5) and (1.6) become respectively

$$\mu f(s, \mu s) \geq g(s, \mu s) \text{ and } \mu f(s, -\mu s) \geq -g(s, -\mu s) \text{ for all } s \geq 0 \tag{$1.5'$}$$

and

$$Cf(u, v) + g(u, v) \leq C_1' (u + v + 1) \quad \text{for all } (u, v) \text{ in } \Sigma, \tag{$1.6'$}$$

for positive constants $C > \mu$ sufficiently close to μ where C_1' is a positive constant.

Remark 1. *The case where boundary condition imposed to u are of different type to those imposed to v remains open for some technical difficulties.*

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