ON THE ASYMPTOTIC NATURE OF A CLASS OF SECOND ORDER NONLINEAR SYSTEMS

Juan E. Nápoles Valdés

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Abstract

In this paper, we obtain some results on the nonoscillatory behaviour of the system (1), which contains as particular cases, some well known systems. By negation, oscillation criteria are derived for these systems. In the last section we present some examples and remarks, and various well known oscillation criteria are obtained.

Universidad de la Cuenca del Plata
Lavalle 50
(3400) Corrientes
Argentina

Universidad Tecnológica Nacional
French 414, U.D.B. Matemáticas
(3500) Resistencia, Chaco
Argentina

e-mail: idic@ucp.edu.ar and matbasicas@frre.utn.edu.ar

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1 Introduction and physical motivations

We are concerned with the oscillatory behaviour of solutions of the following second order nonlinear differential system:

$$\begin{aligned} x' &= a(t)x + b(t)f(y), \\ y' &= -c(t)g(x) + d(t)y, \end{aligned} \tag{1}$$

where the functions a, b, c, d of the independent variable t are real-valued and continuous on $[t_0, +\infty)$, for some $t_0 \geq 0$ with $b(t) > 0$. The functions f and g are also real-valued continuous functions on R such that:

i) $g'(x) > 0$ for all $x \in R$ and $xg(x) > 0$ for all $x \neq 0$.
ii) $yf(y) > 0$ for all $y \neq 0$.

Further conditions will be imposed in the appropriate moments.

A solution $(x(t), y(t))$ of (1) is said to be continuable if it exists on some interval $[t_0, +\infty)$. A continuable solution is said to be oscillatory if one (or both) of its components has an infinite number of zeros with ∞ as the only accumulation point. The system (1) is said to be oscillatory if all continuable solutions $(x(t), y(t))$ are oscillatory.

That the oscillatory nature of the equation:

$$y" + q(t)y = 0, t \in [0, \infty) \tag{2}$$

and the existence of solutions of Riccati equations:

$$r'(t) = r^2(t) + q(t),\ t \in [a, \infty),\ a > 0, \tag{3}$$

are closely related is well known. Many important results in the oscillation theory of (2) are in fact established by studying (3), see [17-18] and [21]. Particularly useful in those studies is the theory of differential and integral inequalities (see [13] and [27]). The present work supports this view point.

Kwong and Wong (see [20]) have studied the oscillatory nature of the system:

$$\begin{aligned} x' &= a_1(t)f(y), \\ y' &= -a_2(t)g(x), \end{aligned} \tag{4}$$

which includes the classical Emden-Fowler systems:

$$\begin{aligned} x' &= a_1(t)\,|y|^{\lambda}\,sgny, \\ y' &= -a_2(t)\,|x|^{\nu}\,sgnx, \end{aligned} \tag{5}$$

studied by Mirzov in the papers [27-29]. Further details can be found in [17].

In [7] Elbert studied some nonlinear system of the type:

$$\begin{aligned} x' &= a(t)y + b(t)y^{\overset{*}{\frac{1}{n}}}, \\ y' &= -c(t)x^{\overset{*}{n}} + d(t)y, \end{aligned} \tag{6}$$

where the number n is positive and the star above the exponent denotes the power function preserves the sign of function, for example, $x^{\overset{*}{n}} = |x|^n\,sgnx$. It is clear that system (6) is an Emden-Fowler type system (5).

A particular case of system (5), the Emden-Fowler equation:

$$\frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right) = t^{\sigma}u^{p}, \tag{7}$$

has received a good deal of attention, being both a physically important and a mathematically significant nonlinear differential equation (see [4], [16], [25], [26], [37] and [42]). This equation is familiar in the context of the theory of diffusion and reaction [11] as governing the concentration u of a substance which disappears by an pth order isothermal reaction at each point x of a slab of catalyst. When such an equation is normalized, in the special case $\rho + \sigma = 0$, making $\phi x = \frac{t^{1-\rho}}{1-\rho}$ $(\rho \neq 1)$, or $\phi x = \ln t$, $\rho = 1$, $u(x)$ is the concentration as a fraction of the concentration outside the slab and x the distance from the central plane as a fraction of the half thickness of the slab, the parameter ϕ^2 may be interpreted as the ratio of the characteristic diffusion rate. It is known in the chemical engineering literature as the Thiele modulus.

Consider the boundary condition:

$$u = \frac{du}{dx} = 0, x = x_0.$$

In this context there is also an important functional of the function

$$\eta = \int_0^1 u^p(x)dx = \frac{1}{\phi^2}\left(\frac{du}{dx}\right)_{x=1}.$$

Physically, it represents the ratio of the total reaction rate to the maximum possible reaction rate and it is known as the effectiveness factor. It is a function of the parameter ϕ and its behavior as this goes to zero or to infinity is significant. The study of the Emden-Fowler equation for this case may seem rather special, but the asymptotic results are much more important.

The particular case of equation (7)

$$u'' + x^\sigma u^n = 0,$$

in which σ and n can take different numerical values, occurs in astrophysics. This was studied by Emden [9] and others in their research on polytropic gas spheres. That equation is a generalization of the Thomas-Fermi equation of atomic physics.

The physical origin of the problem will now be discussed briefly. The researches of Lord Kelvin [15] on convective equilibrium led to subsequent studies in this field by Lane, Emden and others. Lane [22] was interested in the density and the temperature in the solar atmosphere, considered as a configuration under its own gravitation. Ritter [41] independently conducted investigations into the nature of the equilibrium of stellar configurations. Emden [9] systematized earlier work and added a number of important contributions to the theory. He considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Fowler [10] presented a much more rigorous mathematical treatment of the theory and obtained the asymptotic behavior of the solutions.

The goal of this work is to obtain some results on the nonoscillatory behaviour of the system (1), which contains as particular cases, the systems (2), (5), (6) and (7). By negation, oscillation criteria are derived. The method used contains the Hartman's method applied to the linear second order differential equation (see [12, Ch XI]). In section 3 we present some examples and remarks, and various well known oscillation criteria are obtained.

2 The system (1).

First we generalize the Riccati equations to the system (1).

Let the system (1) be nonoscillatory and the interval $[t_1, +\infty)$ be a disconjugacy interval (see [2] or [12]) and $(x(t), y(t))$ be a solution of (1) such that $x(t)\neq 0$ for $t\geq t_1$. Let the function $r = r(t)$ be defined by:

$$r = \frac{f(y)}{g(x)}, \tag{8}$$

then r is continuous and satisfies the generalized Riccati equation:

$$r' + p(t)r^2 + q(t)r + s(t) = 0, \tag{9}$$

where $p(t) = b(t)g'(x(t))$, $s(t) = c(t)f'(y(t))$ and $q(t) = a(t)\frac{g'(x(t))}{g(x(t))}x(t) - d(t)\frac{f'(y(t))}{f(y(t))}y(t)$.

This easily follows by differentiating (8) and making use of (1).

For convenience, we introduce the following function:

$$\lambda^*(t) = \exp\left(\int_{t_1}^{t} q(s)ds\right). \tag{10}$$

Thus we can define the set A of the admissible pairs (λ, μ) (see [7]) of the functions $\lambda(t)$, $\mu(t)$ by the following restrictions:

6a) $\lambda(t)$, $\mu(t)$ are continuous, positive and λ^* is continuously differentiable on $[t_1, +\infty)$,

6b) $\int_{t_1}^{\infty} \frac{\lambda(t)}{\mu(t)} \left| \frac{(\lambda^*(t))'}{\lambda^*(t)} - \frac{\lambda'(t)}{\lambda(t)} \right|^2 dt < \infty$,

6c) $\lim\limits_{T\to\infty} \int_{t_1}^{T} \mu(t)dt = \infty$,

6d) $\limsup\limits_{T\longrightarrow\infty} \frac{\int_{t_1}^{T} \frac{\mu^2(t)\lambda(t)}{p(t)}}{\left(\int_{t_1}^{T} \mu(t)dt\right)^2} < \infty$.

Clearly, the existence of the set A depends heavily on the coefficients a, b, d of the system (1) and we will suppose that it is nonempty, moreover, for the sake of convenience, there are functions μ such that $(\lambda^*, \mu) \in A$.

Regarding the fourth coefficient c, we will research the behaviour of the function $H(T)$ defined by

$$H(T) = \frac{\int_{t_1}^{T} \mu(t)\left(\int_{t_t}^{t} \lambda(u)s(u)du\right)dt}{\int_{t_1}^{T} \mu(t)dt}. \tag{11}$$

This function can be considered as the quasi average of the function $h(t) = \int_{t_1}^{t} \lambda(u)s(u)du$. It is clear that if the relation $\underset{t\to\infty}{Lim}\ h(t) = \overline{C}$ holds where $\overline{C}$ may be finite or infinite then $\underset{t\to\infty}{Lim}\ H(t) = \overline{C}$. This property will be called the averaging property of $H(T)$.

In this paper, we will use the well known inequality:

$$2\,|uv| \leq |u|^2 + |v|^2 . \tag{12}$$

We now state the following results related to the function $H(T)$.

Lemma 1. Let the system (1) be nonoscillatory and let $(x(t), y(t))$ be a solution such that $x(t) \neq 0$ on $[t_1, \infty)$ with some $t_1 \geq t_0$. Let the function $r(t)$ be given by (8). If for some function $\lambda = \lambda(t)$ of a pair $(\lambda, \mu) \in A$ the inequality:

$$\int_{t_1}^{\infty} p(s)r^2(s)\lambda(s)ds < \infty, \tag{13}$$

holds, then the function $H(T)$ defined by (11), corresponding μ, is bounded on $[t_1, \infty)$. If $\lambda = \lambda^*$ then $\underset{t\to\infty}{Lim}\ H(t) = \overline{C}$ exists and is finite.

Proof. Multiplying (9) by λ and integrating from t_1 to t, we obtain:

$$\begin{aligned} r(t)\lambda(t) + \int_{t_1}^{t} r(s)\left[\lambda(s)\frac{(\lambda^*(s))'}{\lambda^*(s)} - \lambda'(s)\right] ds + & \\ \int_{t_1}^{t} p(s)r^2(s)\lambda(s)ds + \int_{t_1}^{t} \lambda(u)s(u)du - r(t_1)\lambda(t_1) &= 0, \end{aligned} \tag{14}$$

since $q(s) = \frac{(\lambda^*(s))'}{\lambda^*(s)}$. Putting:

$$\begin{aligned} u(t) &= (2(1-\varepsilon)p(t)\lambda(t))^{\frac{1}{2}} r(t) \\ & \text{and} \\ v(t) &= \left[\lambda(t)\frac{(\lambda^*(t))'}{\lambda^*(t)} - \lambda'(t)\right] (2(1-\varepsilon)p(t)\lambda(t))^{-\frac{1}{2}}, \end{aligned} \tag{15}$$

we deduce from (12), with $0 < \varepsilon < 1$, that:

$$\left| r(t)\left[\lambda(t)\frac{(\lambda^*(t))'}{\lambda^*(t)} - \lambda'(t)\right]\right| \leq (1-\varepsilon)p(t)\lambda(t)\,|r(t)|^2 + \gamma(\varepsilon)\frac{\left|\lambda(t)\frac{(\lambda^*(t))'}{\lambda^*(t)} - \lambda'(t)\right|^2}{p(t)\lambda(t)},$$

where $\gamma(\varepsilon)=\frac{1}{4(1-\varepsilon)}$. Hence:

$$\left|r(t)\lambda(t)+\int_{t_1}^{t}p(s)r^2(s)\lambda(s)ds+\int_{t_1}^{t}\lambda(u)s(u)du-r(t_1)\lambda(t_1)\right|\le$$

$$\le\int_{t_1}^{t}|r(s)|\left|\lambda(s)\frac{(\lambda^*(s))'}{\lambda^*(s)}-\lambda'(s)\right|ds\le \tag{16}$$

$$\le(1-\varepsilon)\int_{t_1}^{t}p(s)\lambda(s)\left|r(s)\right|^2ds+\gamma(\varepsilon)\int_{t_1}^{t}\frac{\left|\lambda(s)\frac{(\lambda^*(s))'}{\lambda^*(s)}-\lambda'(s)\right|^2}{p(s)\lambda(s)}ds.$$

From this inequality it follows that:

$$r(t)\lambda(t)+\varepsilon\int_{t_1}^{t}p(s)r^2(s)\lambda(s)ds+\int_{t_1}^{t}\lambda(u)s(u)du-r(t_1)\lambda(t_1)\le$$

$$\tag{17}$$

$$\le\gamma(\varepsilon)\int_{t_1}^{t}\frac{\left|\lambda(s)\frac{(\lambda^*(s))'}{\lambda^*(s)}-\lambda'(s)\right|^2ds}{p(s)\lambda(s)}.$$

Using (6b) and (13):

$$\int_{t_1}^{t}\lambda(u)s(u)du-C_1\le\lambda(t)\left|r(t)\right|, \tag{18}$$

where $C_1=r(t_1)\lambda(t_1)+\gamma(\varepsilon)\int_{t_1}^{\infty}\frac{\left|\lambda(s)\frac{(\lambda^*(s))'}{\lambda^*(s)}-\lambda'(s)\right|^2}{p(s)\lambda(s)}ds$. Multiplying (18) by μ, integrating over $[t_1,T]$ and using the definition of function H, we obtain:

$$H(T)-C_1\le\frac{\int_{t_1}^{T}\lambda(t)\mu(t)\left|r(t)\right|dt}{\int_{t_1}^{T}\mu(t)dt}:=L(T). \tag{19}$$

From this we can derive two relations for the function $L(T)$. The first is a simple consequence of the Holder inequality:

$$0\le L(T)\le\left[\frac{\int_{T_1}^{T}\frac{\lambda(t)\mu^2(t)}{p(t)}dt}{\left(\int_{t_1}^{T}\mu(t)dt\right)^2}\right]^{\frac{1}{2}}\left[\int_{T_1}^{T}p(t)\lambda(t)r^2(t)dt\right]^{\frac{1}{2}}. \tag{20}$$

Let T_1 be an arbitrary number such that $T_1 > t_1$. Using again the Holder inequality we get the second relation:

$$L(T) \leq \frac{\int_{t_1}^{T} \lambda(t)\mu(t)\,|r(t)|\,dt}{\int_{t_1}^{T} \mu(t)dt} + \left[\frac{\int_{t_1}^{T} \frac{\lambda(t)\mu^2(t)}{p(t)}dt}{\left(\int_{t_1}^{T} \mu(t)dt\right)^2}\right]^{\frac{1}{2}} \left[\int_{t_1}^{T} p(t)\lambda(t)r^2(t)dt\right]^{\frac{1}{2}}.$$

From (6c)-(6d) we obtain:

$$\limsup_{T\longrightarrow\infty} L(T) \leq \left[\sup_{T>T_1} \frac{\int_{t_1}^{\infty} \frac{\lambda(t)\mu^2(t)}{p(t)}dt}{\left(\int_{t_1}^{T} \mu(t)dt\right)^2}\right]^{\frac{1}{2}} \left[\int_{T_1}^{\infty} p(t)\lambda(t)r^2(t)dt\right]^{\frac{1}{2}},$$

from this, (13) and the second relation for $L(T)$ we have, by letting $T_1 \longrightarrow \infty$:

$$\underset{T\to\infty}{Lim}\ L(T) = 0. \tag{21}$$

Then it follows by (19) that $\limsup\limits_{T\longrightarrow\infty} H(T) \leq C_1$. It is easy to obtain the formulation of a lower estimate for $H(T)$ using the second inequality involved in (12) and we leave that to the reader. So, we have that $H(T)-C_2 \geq -L(T)$, where:

$$C_2 = r(t_1)\lambda(t_1) - (2-\varepsilon)\int_{t_1}^{\infty} p(s)\lambda(s)r(s)^2ds - \gamma(\varepsilon)\int_{t_1}^{\infty} \frac{\left|\lambda(s)\frac{(\lambda^*(s))'}{\lambda^*(s)} - \lambda'(s)\right|^2}{p(s)\lambda(s)}ds,$$

using again (13) we obtain $\liminf\limits_{T\longrightarrow\infty} H(T) \geq C_2$. This, together with the above relations, proves the first part of the lemma.

If $\lambda = \lambda^*$, from (14) we have:

$$0 = r(t)\lambda^*(t) - \int_{t_1}^{\infty} p(s)r^2(s)\lambda^*(s)ds + \int_{t_1}^{t} \lambda^*(u)s(u)du - C, \tag{22}$$

with $C = r(t_1)\lambda^*(t_1) - \int_{t_1}^{\infty} p(s)r^2(s)\lambda^*(s)ds$. (22) after multiplying by μ and integrating between t_1 and T ($T > t_1$) yields:

$$|H(T) - C| \leq \frac{\int_{t_1}^{T} \mu(t)\int_{t_1}^{\infty} p(s)\lambda^*(s)r^2(s)dsdt}{\int_{t_1}^{T} \mu(t)dt} + \frac{\int_{t_1}^{T} \mu(t)\lambda^*(t)\,|r(t)|\,dt}{\int_{t_1}^{T} \mu(t)dt},$$

(13) implies that $\int_{t_1}^{\infty} p(t)\lambda^*(t)r^2(t)dt$ tends to zero as $t_1 \longrightarrow \infty$. Hence the first term of the right hand side tends to zero, while the second term is $L(T)$, which tends to zero using (20) and (21), therefore $\underset{T\to\infty}{Lim}\ H(T) = C$. Thus, the proof is complete.■

Let the functions $S(T)$ and $M(T)$ be introduced for $T > t_1$ by

$$\begin{aligned} S(T) &= \int_{t_1}^{T} \mu(t) \left(\int_{t_1}^{t} p(s)\lambda(s)r^2(s)ds \right) dt, \\ M(T) &= \int_{t_1}^{T} \mu(t)dt. \end{aligned} \tag{23}$$

By (6c), $\underset{T\to\infty}{Lim}\ M(T) = \infty$. We assume that:

$$\underset{T\to\infty}{Lim} \int_{t_1}^{T} p(t)\lambda(t)r^2(t)dt = \infty. \tag{24}$$

Using the averaging property of the function $H(T)$ we have:

$$\underset{T\to\infty}{Lim} \frac{S(T)}{M(T)} = \infty. \tag{25}$$

and

$$\underset{T\to\infty}{Lim}\ S(T) = \infty. \tag{26}$$

But, by the inequality (6b) we can write (17) as

$$r(t)\lambda(t) + \varepsilon \int_{t_1}^{t} p(s)r^2(s)\lambda(s)ds \leq C_1 - \int_{t_1}^{t} \lambda(u)s(u)du,$$

where the constant is the same as in (18). Multiplying this last inequality by μ and using (11) after integration we have:

$$\frac{\int_{t_1}^{T} \lambda(t)\mu(t)r(t)dt}{M(T)} + \varepsilon \frac{S(T)}{M(T)} \leq C_1 - H(T). \tag{27}$$

Putting:

$$\liminf_{T\longrightarrow\infty} H(T) > -\infty, \tag{28}$$

we have that the right hand side of (27) is bounded from above, then from (25) it will be less than $\frac{\varepsilon}{2}\frac{S(T)}{M(T)}$ for $T > T_1$ with some T_1 sufficiently large. Consequently we obtain:

$$\varepsilon\frac{S(T)}{M(T)} < \frac{\int_{t_1}^{T}\lambda(t)\mu(t)r(t)dt}{M(T)} = L(T),\ T > T_1, \tag{29}$$

with $L(T)$ as in (19). From (23) we deduce:

$$S'(T) = \mu(T)\int_{t_1}^{T} p(t)\lambda(t)r^2(t)dt, M' = \mu,$$

the estimate (20) with (19) implies:

$$L^2 \leq \left(\frac{\int_{t_1}^{T}\frac{\lambda(t)\mu^2(t)}{p(t)}dt}{M^2(T)}\right)^{\frac{1}{2}}\frac{S'(T)}{M'(T)}, \tag{30}$$

from (6d) we have sufficiently large T_1 and N such that:

$$\left(\frac{\int_{t_1}^{T}\frac{\lambda(t)\mu^2(t)}{p(t)}dt}{M^2(T)}\right)^{\frac{1}{2}} < N,\ T > T_1.$$

Combining this with (29) and (30) we get that:

$$\gamma_1 M'M^{-2} < N^{\frac{1}{2}}S'S^{-2},\ T > T_1, \tag{31}$$

where γ_1 is a positive constant depending only on N.

We have, by (26) and (6c) that $\gamma_1 M^{\frac{1}{2}} < N^{\frac{1}{2}}S^{\frac{1}{2}}$, hence $\frac{S}{M} < N\gamma_1^{-2}$ for $T > T_1$ which contradicts (24), hence the relation (13) is valid. This shows that we can apply the lemma and we can obtain, under simple conditions, the following result.

Theorem 1. Let us suppose that system (1) be nonoscillatory and disconjugate on $[t_1, \infty)$, and the pair of the functions (λ, μ) be admissible for (1). If the function $H(T)$ defined by (7) fullfils the inequality (28), then the relation (13) is valid and the function $H(T)$ is bounded on $[t_1, \infty)$. Moreover in the case $\lambda = \lambda^*$, the limit

$$\underset{t\to\infty}{Lim}\, H(t) = C, \tag{32}$$

with C finite, holds.

Remark 1. It is clear that the above theorem is a conversion of Lemma 1, under suitable assumptions.

In the next result, we formulate a sufficient criterion for oscillation of the solutions of the system (1).

Theorem 2. Let (λ, μ) be an admissible pair for the system (1). If for some $t_1 > t_0$ the relation $\underset{t\to\infty}{Lim}\ H(t) = \infty$ holds then the system (1) is oscillatory. Also, if for an admissible pair (λ^*, μ) the relations $\limsup\limits_{T\to\infty} H(T) > \liminf\limits_{T\to\infty} H(T) > -\infty$ hold then the system (1) is oscillatory.

Proof. Assuming the opposite, suppose that the system (1) is nonoscillatory. By the assumptions on $H(T)$ the condition (28) is fulfilled, hence Theorem 1 is valid. Thus the limit of the function $H(T)$, if any, had to be finite. This is the desired contradiction. This completes the proof.■

Remark 2. It is not difficult to show that the limits here are independent of the choice of the value t_1.

Theorem 2 may be simplified by the following:

Corollary. Let λ be a function such that there exists at least one function μ satisfying $(\lambda, \mu) \in A$. If the relation:

$$\underset{T\to\infty}{Lim} \int_{t_1}^{T} \lambda(t)s(t)dt = \infty, \tag{33}$$

holds for some $t_1 \geq t_0$ then system (1) is oscillatory.

Proof. We consider the function $H(T)$ for $T > t_1$. From definition of $H(T)$ the limit in (33) yields the same limit for $H(T)$, i.e., $\underset{t\to\infty}{Lim}\ H(t) = \infty$. Theorem 2 implies that the system (1) can be only oscillatory.■

Under stronger restrictions on the pairs (λ, μ) can be established a more stringent criterion for nonoscillation, thus we have:

Theorem 3. Let us suppose that the system (1) be nonoscillatory and disconjugate on $[t_1, \infty)$. Let (λ^*, μ) be a pair of functions satisfying the conditions (6a), (6c) and

$$\limsup_{t\longrightarrow\infty} \frac{\frac{\lambda^*(t)}{p(t)}}{\int_{t_1}^{t} \mu(s)ds} < \infty. \tag{34}$$

Moreover let the relation (28) be valid. Then the relation (32) holds and

$$\underset{T\to\infty}{Lim} \frac{\int_{t_1}^{T} \mu(t) \left| C - \int_{t_1}^{t} \lambda^*(u)s(u)du \right|^2 dt}{\int_{t_1}^{T} \mu(t)dt} = 0. \tag{35}$$

Proof. We show that the pair (λ^*, μ) under restrictions of Theorem 3 is admissible, i.e., it fulfills (6d), too. We have, from (34), for sufficiently large N and T_1 that:

$$\frac{\frac{\lambda^*(t)\mu(t)}{p(t)}}{M(t)} < N,\, t > T_1, \tag{36}$$

where M is defined by (23). Since $M'(t) = \mu(t)$, we have that:

$$\frac{\lambda^*(t)\mu^2(t)}{p(t)} < NM(t)M'(t),\, t > T_1,$$

putting $K(T) = \int_{t_1}^{T} \frac{\lambda^*(t)\mu^2(t)}{p(t)} dt$ we have by integration:

$$K(T) - K(T_1) < N\frac{M^2(T) - M^2(T_1)}{2},\, T > T_1.$$

From here we obtain $\limsup\limits_{T\longrightarrow\infty} \frac{K(T)}{M^2(T)} \leq \frac{N}{2}$ in other words, the relation (6d) holds.

Thus the pair (λ^*, μ) is admissible and the conditions of Theorem 1 are satisfied, therefore the relation (13) holds and $\underset{T\to\infty}{Lim}\ H(T) = C$ (with C finite). Repeating the proof of the lemma and rewriting (22) in the form:

$$\left| C - \int_{t_1}^{t} \lambda^*(u)s(u)du \right|^2 = \left| r(t)\lambda^*(t) - \int_{t_1}^{\infty} p(s)\lambda^*(s)r(s)ds \right|^2.$$

We have $\left| C - \int_{t_1}^{t} \lambda^*(u)s(u)du \right|^2 \leq 2\left\{ r^2(t)\left(\lambda^*(t)\right)^2 + \left(\int_{t_1}^{\infty} p(s)\lambda^*(s)r(s)ds \right)^2 \right\}$ and then

$$0 \leq \frac{\int_{t_1}^{T} \mu(t) \left| C - \int_{t_1}^{t} \lambda^*(u)s(u)du \right|^2 dt}{M(T)} \leq 2\frac{\int_{t_1}^{T} \mu(t)r^2(t)\left(\lambda^*(t)\right)^2 dt}{M(T)} \tag{37}$$

$$+2\frac{\int_{t_1}^{T}\mu(t)\left(\int_{t}^{\infty}b(s)r^2(s)ds\right)^2dt}{M(T)}=M_1(T)+M_2(T).$$

From the averaging property of function $H(T)$, M_2 tends to zero as $T\to\infty$. Let T_1 be as large as in (36), then we have for all $T\geq T_2>T_1$:

$$\begin{aligned}\frac{M_1(T)}{2} &= \frac{\int_{t_1}^{T_2}\mu(t)r^2(t)\left(\lambda^*(t)\right)^2dt+\int_{T_2}^{T}\frac{p(t)\lambda^*(t)r^2(t)(\lambda^*(t))\mu(t)}{p(t)}dt}{M(T)}< \\ &< \frac{\int_{t_1}^{T_2}\mu(t)r^2(t)\left(\lambda^*(t)\right)^2dt+NM(T)\int_{T_2}^{T}p(t)\lambda^*(t)r^2(t)dt}{M(T)},\end{aligned}$$

therefore:

$$\limsup_{T\longrightarrow\infty}\frac{M_1(T)}{2}\leq N\int_{T_2}^{\infty}p(t)\lambda^*(t)r^2(t)dt,\ T_2>T_1.$$

Hence by (13), $\underset{T\to\infty}{Lim}\ M_1(T)=0=\underset{T\to\infty}{Lim}\ M_2(T)$. Thus (37) implies the desired conclusion. This completes the proof.■

In the next theorem, we obtain a companion criterion for oscillation of system (1).

Theorem 4. Let (λ^*,μ) be an admissible pair for the system (1) satisfying the relation (34). If the function $H(T)$ defined by (11) satisfies the relation (32) and

$$\limsup_{T\longrightarrow\infty}\frac{\int_{t_1}^{T}\mu(t)\left|C-\int_{t_1}^{t}\lambda^*(u)s(u)du\right|^2dt}{M(T)}>0,$$

then system (1) is oscillatory.

The proof of this last result is omitted because it is based on ideas of the proof of Theorem 2.

Another nonoscillation criterion can be established if the relation (6b) is omitted. But it is necessary to defin the set $\overline{A}$ of the pairs (λ,μ) by the conditions (6a), (6c) and (6d).

Hence the requirement (6b) is dropped and therefore $A\subset\overline{A}$. Similarly, let

$$\overline{H(T)}=\frac{\int_{t_1}^{T}\mu(t)\left(\int_{t_1}^{t}\lambda(u)s(u)du-\gamma(\varepsilon)\frac{\left|\lambda(t)\frac{(\lambda^*(t))'}{\lambda^*(t)}-\lambda'(t)\right|^2}{p(t)\lambda(t)}\right)dt}{\int_{t_1}^{T}\mu(t)dt}. \tag{38}$$

Thus, we can rewrite (17) as:

$$r(t)\lambda(t)+\int_{t_1}^{t} p(s)r^2(s)\lambda(s)ds - r(t_1)\lambda(t_1)+$$

$$+\int_{t_1}^{T}\left(\lambda(t)s(t)-\gamma(\varepsilon)\frac{\left|\lambda(s)\frac{(\lambda^*(s))'}{\lambda^*(s)}-\lambda'(s)\right|^2}{p(t)\lambda(t)}\right)dt\le 0,$$

hence by (38):

$$\overline{H(T)}-r(t_1)\lambda(t_1)\le\frac{\int_{t_1}^{T}\lambda(t)\mu(t)\left|r(t)\right|dt}{\int_{t_1}^{T}\mu(t)dt}=L(T). \tag{39}$$

By (6c)-(6d) and (13) the relation (21) is true, so the function $\overline{H(T)}$ is bounded from above. We suppose that the relation (13) is not true, then the functions $S(T)$, $M(T)$ given by (23) satisfy the relations in (25), (26). From (39) we have:

$$\frac{\int_{t_1}^{T}\lambda(t)\mu(t)r(t)dt}{M(T)}+\varepsilon\frac{S(T)}{M(T)}\le\overline{H(T)}-r(t_1)\lambda(t_1).$$

But the right hand side is bounded from above, hence we have for sufficiently large T_1 the relation (29) and in the same way we would have the boundedness of the quotient $\frac{S}{M}$ for large $T{\geq}T_1$, but this contradicts (25). Again the inequality (13) holds and according to the above formulae, the function $\overline{H(T)}$ is bounded above. Hence we have the following:

Theorem 5. Let us suppose that system (1) be nonoscillatory and let $(x(t),y(t))$ be a solution such that $x(t){\neq}0$ on $[t_1,\infty)$. Let r be as above. If for the function λ of a pair $(\lambda,\mu)\in\overline{A}$ the inequality (13) holds then with the corresponding μ the function $\overline{H(T)}$ is bounded from above. On the other hand if $\overline{H(T)}$ in (38) is bounded from below and system (1) is nonoscillatory then the inequality (9) holds again and, consequently, $\overline{H(T)}$ is bounded from above.

The next result is obtained as a consequence of this theorem.

Theorem 6. Let the pair $(\lambda, \mu) \in \overline{A}$. If for some $t_1 \geq t_0$ and $0 < \varepsilon < 1$ the relation:

$$\underset{T\to\infty}{Lim} \frac{\int_{t_1}^{T} \mu(t) \left(\int_{t_1}^{t} \lambda(u)s(u)du - \gamma(\varepsilon) \frac{\left| \lambda(t) \frac{(\lambda^*(t))'}{\lambda^*(t)} - \lambda'(t) \right|^2}{(p(t)\lambda(t))^n} \right) dt}{\int_{t_1}^{T} \mu(t)dt} = \infty,$$

holds, then system (1) is oscillatory.

Remark 3. The results obtained are consistent with the well known oscillatory case $x' = by$, $y' = -cx$ (b, c positive constants). It is enough take the pair $\lambda(t) = [b(t-t_1)]^{\alpha}$, $\alpha < 1$ and $\mu(t) = bt$.

Remark 4. Elbert [7] gave information on the oscillatory nature of equation (E), i.e., system (1) with $f(y) = y^{\overset{*}{\frac{1}{n}}}$ and $g(x) = x^{\overset{*}{n}}$. So, our results contain those given in that paper. In particular, the Examples 1, 2 and 3 are still valid.

Methodological Remark. From Theorem 2 and (11), we can obtain various well known integral criteria for oscillation of some class of differential equations of second order, rewritten in the Riccati form (9). The following results are devoted to clarify that.

In [46] the author gave the following oscillation result for equation

$$x'' + p(t)x'(t) + q(t)x(t) = 0, \tag{40}$$

where p and q are continuous on $[t_0, \infty)$, $t_0 > 0$, and p and q are allowed to take on negative values for arbitrarily large t.

[**46, Theorem**]. If there exist $\alpha \in (1, \infty)$ and $\beta \in [0, 1)$ such that

$$\limsup_{t\longrightarrow\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-s)^{\alpha} s^{\beta} q(s)ds = \infty, \tag{41}$$

$$\limsup_{t\longrightarrow\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} \left[(t-s)p(s)s + \alpha s - \beta(t-s)\right]^2 (t-s)^{\alpha-2} s^{\beta-2} ds < \infty, \tag{42}$$

then (40) is oscillatory.

From the definition of $H(t)$ and our Theorem 2, taking $\mu(t) = \alpha(t-t_0)^{\alpha-1}$, we obtain the desired conclusion without making use of (42).

In [5] it is studied the equation

$$(r(t)x'(t))' + h(t)f(x(t))x'(t) + y(t, x(t)) = H(t, x(t), x'(t)) \tag{43}$$

where $f : R \to R$, $r, h : [t_0, \infty) \to R$, $t_0 \geq 0$ and $\Psi : [t_0, \infty) \mathrm{x} R \longrightarrow R$, $H : [t_0, \infty) \mathrm{x} R \mathrm{x} R \to R$ are continuous functions, r$(t) > 0$ for $t \geq t_0$. For all $x \neq 0$ and for $t \in [t_0, \infty)$ we assume that there exist continuous functions $g : R \to R$ and $p, q : [t_0, \infty) \to R$ such that

$$xg(x) > 0, g'(x) \geq k > 0, x \neq 0; \frac{\Psi(t, x)}{g(x)} \geq q(t), \frac{H(t, x, x')}{g(x)} \leq p(t), x \neq 0. \tag{44}$$

And the authors considered the equation (43) of sublinear type, e.g., satisfying

$$0 < \int_0^{\varepsilon} \frac{du}{g(u)} < \infty, 0 < \int_0^{-\varepsilon} \frac{du}{g(u)} < \infty, \varepsilon > 0. \tag{45}$$

The main result of that paper is the following result.

[5, Theorem 1]. Suppose (44) and (45) hold. Furthermore, assume that

$$f(x) \geq -c, c > 0, x \in R,$$

$$0 < r(t) \leq a, \ a > 0, \ t \in [t_0, \infty),$$

there exists a continuously differentiable function $\rho(t)$ on $[t_0, \infty)$ such that

$$\rho(t) > 0, \ \rho'(t) \geq 0, \ \rho''(t) \leq 0, \ [t_0, \infty),$$

and

$$\gamma(t) = \rho'(t)r(t) + c\rho(t)h(t) \geq 0, \gamma'(t) \leq 0, \ t \geq t_0,$$

$$\liminf_{t \longrightarrow \infty} \int_{t_0}^{t} \rho(s)(q(s) - p(s))ds > -\infty \tag{46}$$

$$\liminf_{t \longrightarrow \infty} \left(\int_{t_0}^{t} \frac{ds}{\rho(s)} \right)^{-1} \int_{t_0}^{t} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(u)(q(u) - p(u))duds = \infty. \tag{47}$$

Then equation (43) is oscillatory.

We easily obtain the assumption (47) from the definition of $H(t)$ and making use of Theorem 2. Let us note that assumption (46) holds.

3 Some particular cases and related results.

We present here some illustrative examples to show how well-known oscillation criteria for different equations can be obtained using Corollary 1.

Example 1. Kwong has shown in [18] that for the equation $x" + q(t)x = 0$, a sufficient condition for oscillation is that $\int_0^\infty \overline{Q}(t)dt = \infty$ for some $\gamma > 1$ and where $\overline{Q}(t) = \min(Q_+, 1) = \min(\max(Q(t), 0), 1)$ with $Q(t) = \int_0^t q(s)ds$. In this case $\lambda^*(t) \equiv 1$, and we choose $\lambda(t) = \mu(t) \equiv 1$. It is not difficult to show that they form an admissible pair for this equation if $Q(\infty) = \infty$. Therefore, we have a criterion comparable with the Corollary 5 of this paper.

Example 2. For the equation

$$x" + a(t)g(x) = 0, \tag{48}$$

studied by Burton and Grimmer in [1], we know that if $a(t) > 0$ for $t \in [0, \infty)$ and g satisfies the condition (ii) of section 1, a necessary and sufficient condition for the oscillation of this equation is that:

$$\int_{t_1}^{\infty} a(t)g[\pm k(t - t_1)]dt = \pm\infty, \tag{49}$$

for some $k > 0$ and all t_1. Also in this case $\lambda^*(t) \equiv 1$. Let $\lambda(t) \equiv 1$, $\mu(t) = g[k(t - t_1)]$ with $k > 0$ and $t_1 \geq 0$. Then the pair $(1, a(t)g[k(t - t_1)])$ is admissible for this equation if

$$\int_{t_1}^{\infty} a(t)dt = \infty, \tag{50}$$

which coincides with the sufficiency of the above result.

Another admissible pair is $\lambda = t^\alpha (\alpha < 1)$ and $\mu(t) \equiv 1$, under the same condition (53).

On the other hand, under assumption (50), the class of equations (48) is not very large, but if this condition is not fulfilled, we can exhibit equations that have nonoscillatory solutions. For example, the equation

$$x" + (kt^\lambda sin\ t)\,|x|^\gamma\, sgn\ x = 0,\ t > 0,$$

(see [30]) where k, γ and $\gamma > 0$ are constants, has a nonoscillatory solution if and only if

λ<-1 for γ>1,
λ<-1, k arbitrary and γ=1,
λ=-1, $|k| \leq 2^{-\frac{1}{2}}$ and γ=1,
λ<-γ for 0<γ<1.

Further details can be found in [30-34].

Example 3. The Corollaries 1 and 3 of [19] for the equation

$$y"(t) + a(t)\,|y(t)|^{\gamma}\, sgn\ y(t) = 0\ (\gamma > 1),$$

can be obtained choosing the following pairs $\lambda(t)\equiv 1$, $\mu(t) = \int_{t_1}^{t} a(s)ds$, under condition (37) (Corollary 1) and $\lambda(t) = \phi(t)$, $\mu(t)\equiv 1$, with ϕ some positive nondecreasing function of class $C^1[0,\infty)$ satisfying:

$$\int_0^{\infty} \frac{(\phi'(t))^2}{\phi(t)} dt < \infty, \quad \underset{T\to\infty}{Lim} \int_0^T \phi(s)a(s)ds = \infty,$$

for integrable coefficient case, $\int^{\infty} a(t)dt < \infty$, (Corollary 3).

Remark 5. Using these examples and Theorems 2 and 4, it is easy to see how to obtain the oscillation results of Kwong and Zettl [21] (Theorems 4 and 7 and Corollaries 9 and 10), Wong [45], Yan [47] (see example 2 and, mainly, final remark), Kwong and Wong [19] (Theorems 1, 2 and 3), Lewis and Wright [23] (see example 1 of this work with $m = n\equiv 1$) and Repilado and Ruiz [40] (Theorem 1, also cf. Example 1 above). The details are lengthy but essentially routine, therefore they are left to the reader.

Remark 6. Our results are consistent with several earlier results on the oscillatory nature of second order nonlinear differential equation closed to system (1). We consider the equation (see [24]):

$$[r(t)\Phi(u'(t))]' + c(t)\Phi(u(t)) = 0.$$

Making $v(t) = r(t)\Phi(u(t))$ we obtain the Emden-Fowler type system:

$$u' = \Phi^{-1}\left(\frac{v(t)}{r(t)}\right), v' = -c(t)\Phi(u(t)). \tag{51}$$

If in (51) we make $r(t)\equiv 1$ and $\Phi(s) = s$, then that system reduces the linear equation (L). Many criteria for oscillation of (2) have been found which involve the behaviour of the integral of $c(t)$. It is easily seen from

Theorem 2 (or Theorem 4) we can obtain special cases of results of [3], [8], [24], [43] and results of Wintner [44] and Kamenev [14].

Remark 7. The above remark is still valid if we consider the equation $(p(t)x'(t))' + q(t)x(t) = 0$ (see [36]).

Remark 8. In [6] the authors studied the second order nonlinear differential equation:

$$(r(t)f(x'))' + p(t)f(g(x), r(t)f(x')) + q(t)g(x) = 0,$$

under suitable assumptions. An admissible pair is $(1, \rho(t)q(t))$, where ρ is a positive and differentiable function defined on $[t_0, \infty)$. It is clear that Theorem 3 of [6] can be obtained from Theorem 4 under milder conditions.

Remark 9. From results of [38], [39] and ideas presented here, we can obtain generalizations to bidimensional system:

$$x' = \alpha(y) - \beta(y)f(x),\ y' = -a(t)g(x), \tag{52}$$

(which contain the classical Liénard equation). This is not a trivial problem. The resolution implies obtaining results similar to Theorems 2 and 4 for completing the study of oscillatory nature of solutions of (52).

Remark 10. In [35] the author studied the equation

$$(p(t)\varphi(x'(t)))' + f(t, x(t), x'(t)) = 0,$$

which is equivalent to system:

$$x' = \varphi^{-1}\left(\frac{y}{p(t)}\right),\ y' = -f\left(t, x, \varphi^{-1}\left(\frac{y}{p(t)}\right)\right),$$

a system of type (1) with $a{\equiv}0$, $b{\equiv}1$ and $-c(t)g(x)+d(t)y = -f\left(t, x, \varphi^{-1}\left(\frac{y}{p(t)}\right)\right)$. From that paper and ideas used here, arises the following open problem:

Under which conditions can we obtain analogous results to Theorems 2 and 4, valid for the equation $(p(t)\varphi(x'(t)))' + f(t, x(t), x'(t)) = 0$ *?*

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