The Uniqueness of the Periodic Solution for A Class of Differential Equations[1]

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Abstract. In this paper we are concerned with a class of nonlinear differential equations and obtaining the sufficient conditions for the uniqueness of the periodic solution by using Brouwer's fixed point theory and the Sturm Theorem.
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1 Introduction

This paper is concerned with the uniqueness of periodic solutions for the following differential equations

$$\mu(t, x, x')x' = f(t, x) \tag{1}$$

where $(t, x) \in [0, 2\pi]\times \mathbb{R}$ and $\mu(t, x, z) \in C'([0, 2\pi] \times \mathbb{R} \times \mathbb{R})$, $f \in C^2\big([0, 2\pi] \times \mathbb{R}\big)$, and $\mu(t, x, x')$, $f(t, x)$ are 2π-periodic functions with respect to t.

It is easy to see that equation (1) is more general than the classical ordinary differential equation

$$x' = f(t, x) \quad \text{for all} \quad (t, x) \in [0, 2\pi] \times \mathbb{R} \tag{2}$$

During the past three decades, with the use of topological degree theory, general critical point theory, fixed point theory, boundary value condition theory and cross-ratio method, some profound results on the existence and the number of periodic solutions for equation (2) have been presented (see references [1-15] and the reference therein). But none of these papers are concerned with the uniqueness of the periodic solutions for equation (1). However, when does the equations (1) or (2) have a unique 2π-periodic solution ?

In the present paper, using the Brouwer's fixed pointed theorem, the Sturm Theorem, and some results of the optimal control theory method, we are trying to obtain two theorems for the sufficient conditions which guarantee that equation (1) has a unique 2π-periodic solution.

Consider the following conditions

(H1): Suppose that $f(t, x) = [x - x_0(t)] \cdot G(t, x)$. Here $f(t, x)$ and $x_0(t)$ are 2π-periodic continuous functions, and $x_0(t)$ separates the domain $0 \leq$

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$t \leq 2\pi$ into two parts, denoted by Ω_1 and Ω_2 (we assume that Ω_1 is above $x = x_0(t)$ and Ω_2 is below $x = x_0(t)$). Suppose that there exist two sets S_1: $\{(t,x)|x_1 \leq x \leq x_2,\ 0 \leq t \leq 2\pi\}$ and S_2: $\{(t,x)|x_3 \leq x \leq x_4,\ 0 \leq t \leq 2\pi\}$ in the domains Ω_1 and Ω_2, respectively, such that $G(t,x)$ has the same sign for all (t,x) in S_1 and S_2. Here we assume that either $x = x_1$ or $x = x_4$ doesn't intersect with $x = x_0(t)$.

(H2): Suppose that $f_{tx}, f_{xx} : \mathbb{R}^2 \to \mathbb{R}$ and $[\frac{f(t,x)}{\mu(t,x,z)}]_x : \mathbb{R}^3 \to \mathbb{R}$ exist, and are 2π-periodic continuous with respect to t, where f_{tx} denotes the partial derivative with respect to t and x, and $[\frac{f(t,x)}{\mu(t,x,z)}]_x$ denotes the derivative of the quotient $\frac{f(t,x)}{\mu(t,x,z)}$ with respect to x. Suppose that there exist two positive real numbers L and M, one non-negative integer N, and two non-negative continuous functions $u_1(t)$ and $u_2(t)$, such that

$$L \leq \mu(t,x,x') \leq M$$

$$-u_1(t) \leq f_{tx} + f_{xx}\frac{f(t,x)}{\mu(t,x,z)} + f_x[\frac{f(t,x)}{\mu(t,x,z)}]_x \leq -u_2(t)$$

and

$$(N+1)^2 \gneqq \frac{u_1(t)}{L} \geq \frac{u_2(t)}{M} \gneqq N^2$$

where $\gneqq$ indicates "greater than or equal to" but not identically equal.

(H3): Suppose that f_x, f_{tx}, and f_{xx} are continuous and 2π-periodic with respect to t. Let $(k-1)^2 < A < k^2 < B < \infty$, where k is the minimal positive integer suiting the inequality. Assume that there exists a $\beta(x) \in C[0,2\pi]$ such that

$$-A \geq f_{tx} + f_{xx} \cdot f + (f_x)^2 \geq -\beta(x) \geq -B, \qquad \int_0^{2\pi} \beta(t)dt < 2\pi A + 2(B-A)\alpha_k$$

where α_k is the minimal positive root of

$$\sqrt{A}\cot\big(\sqrt{A}\frac{\pi - x}{2k}\big) = \sqrt{B}\tan\big(\sqrt{B}\frac{x}{2k}\big)$$

Our main results are the following theorems:

Theorem 1: If H(1) and H(2) hold, then equation (1) has a unique 2π-periodic solution.

Theorem 2: If H(1) and H(3) hold, then equation (2) has a unique 2π-periodic solution.

2 The Proof of Main Results

Proof of Theorem 1. Consider the case $G(t,x) < 0$ for all (t,x) in S_1 and S_2. Obviously, in the domain Ω_1, we have $f(t,x(t)) < 0$ for any $x(t) > x_0(t)$.

In the domain Ω_2, we have $f(t,x(t)) > 0$ for any $x(t) < x_0(t)$. In the compact set $[0,2\pi]\times[x_1,x_2]$, $f(t,x)$ has the maximal value, denoted by m_1, $(m_1 < 0)$. Hence, we can choose some negative number k_1 such that $k_1 > \frac{m_1}{L}$ and the whole segment l_1: $x_1(t) = k_1 t + x_2$ is inside the set S_1 whenever $0 \le t \le 2\pi$. Similarly, in the compact set $[0,2\pi]\times[x_3,x_4]$, $f(t,x)$ has the minimal value, denoted by m_2, $(m_2 > 0)$. Thus, we also can choose some positive number k_2 such that $k_2 < \frac{m_2}{M}$ and the whole segment l_2: $x_2(t) = k_2 t + x_3$ is inside the set S_2 whenever $0 \le t \le 2\pi$.

$$\begin{cases} x_2 > x_1 & & \\ \frac{dx_1(t)}{dt} & = & k_1 \\ & \geq & \frac{1}{\mu\big((t,x_1(t),x_1'(t)\big)} f[t,x_1(t)] \qquad t\in[0,2\pi] \end{cases}$$

and

$$\begin{cases} x_4 > x_3 & & \\ \frac{dx_4(t)}{dt} & = & k_2 \\ & \leq & \frac{1}{\mu\big((t,x_2(t),x_2'(t)\big)} f[t,x_2(t)] \qquad t\in[0,2\pi]. \end{cases}$$

Letting $L_1 = x_3, L_2 = x_2, L_3 = x_4$, and $L_4 = x_1$, clearly, we can see that $L_1 < L_2$ and $L_3 < L_4$.

Define an operator T_1: $C[L_1,L_2]\to C[L_3,L_4]$:

$$\forall x(0)\in C[L_1,L_2], \qquad T_1x(0) = x(2\pi).$$

Since $C[L_3,L_4]\subseteq C[L_1,L_2]$, we get that T_1 is continuous and maps $x(t)$ from $C[L_1,L_2]$ to $C[L_3,L_4]$. Consequently,

$$T_1\big(C[L_1,L_2]\big)\subseteq C[L_1,L_2].$$

By the Brouwer's fixed point theorem, we obtain that T_1 has at least one fixed point in $C[L_1,L_2]$. That is, equation (1) has at least one 2π-periodic solution.

In the case $G(t,x) > 0$ for all (t,x) in S_1 and S_2, the proof is similar to the above. In the domain Ω_1, we have $f(t,x(t)) > 0$ for any $x(t) > x_0(t)$. In the domain Ω_2, we have $f(t,x(t)) < 0$ for any $x(t) < x_0(t)$. In the compact set $[0,2\pi]\times[x_1,x_2]$, $f(t,x)$ has the minimal value, denoted by n_1, $(n_1 > 0)$. Hence, we can choose some positive number such that $k_3 < \frac{n_1}{M}$ and the whole segment l_3: $x_3(t) = k_3 t + x_1$ is inside the set S_1 whenever $0 \le t \le 2\pi$. Similarly, in the compact set $[0,2\pi]\times[x_3,x_4]$, $f(t,x)$ has the maximal value, denoted by n_2, $(n_2 < 0)$. Thus, we also can choose some negative number k_4 such that $k_4 > \frac{n_2}{L}$ and the whole segment l_4: $x_4(t) = k_4 t + x_4$ is inside the set S_2 whenever $0 \le t \le 2\pi$.

$$\begin{cases} x_2 > x_1 & & \\ \frac{dx_3(t)}{dt} & = & k_3 \\ & \leq & \frac{1}{\mu\big((t,x_1(t),x_1'(t)\big)} f[t,x_1(t)] \qquad t\in[0,2\pi] \end{cases}$$

and

$$\begin{cases} x_4 > x_3 \\ \frac{dx_4(t)}{dt} & = & k_4 \\ & \geq & \frac{1}{\mu\big((t,x_2(t),x_2'(t)\big)} f[t, x_2(t)] \qquad t \in [0, 2\pi]. \end{cases}$$

Letting $L_1 = x_3, L_2 = x_2, L_3 = x_4$, and $L_4 = x_1$, clearly, we can see that $L_1 < L_2$ and $L_3 < L_4$.

Define an operator T_2: $C[L_1, L_2] \to C[L_3, L_4]$:

$$\forall x(2\pi) \in C[L_1, L_2], \qquad T_2 x(2\pi) = x(0).$$

Since $C[L_3, L_4] \subseteq C[L_1, L_2]$, we have that T_2 is continuous and maps $x(t)$ from $C[L_1, L_2]$ to $C[L_3, L_4]$. Consequently,

$$T_2\big(C[L_1, L_2]\big) \subseteq C[L_1, L_2].$$

By the Brouwer's fixed point theorem, we obtain that T_2 has at least one fixed point in $C[L_1, L_2]$. That is, equation (1) has at least one 2π-periodic solution.

Differentiating both sides of equation (1) with respect to t, we have

$$\big(\mu(t, x, x')x'\big)' = F(t, x)$$

where $F(t, x) = f_t(t, x) + f_x(t, x) \cdot \frac{f(t,x)}{\mu(t,x,x')}$. Here $f_t(t, x)$ denotes the partial derivative with respect to t, and $F_x(t, x)$ denotes the partial derivative with respect to x. By recalling the assumption (H2), we know that $-u_1(t) \leq F_x(t, x) \leq -u_2(t)$.

Define an operator T: $C^1_{2\pi}[0, 2\pi] \to C^1_{2\pi}[0, 2\pi]$, where $C^1_{2\pi}[0, 2\pi]$ denotes the set of all the 2π-periodic continuous differentiable functions in $\big([0, 2\pi] \times \mathbb{R}\big)$. For any $\omega \in C^1_{2\pi}[0, 2\pi]$, assume that $T_\omega = T_\omega(t)$ is a 2π-periodic solution of the equation

$$\big(\mu(t, \omega, \omega')x'\big)' = \int_0^1 F_x(t, \theta\omega) d\theta x + f(t, 0). \tag{3}$$

Next, we prove that equation (3) has at most one 2π-periodic solution under the conditions (H2). Suppose that $T_y = T_y(t)$ is another 2π-periodic solution of equation (3). Then $b(t) = T_\omega(t) - T_y(t)$ must be a 2π-periodic solution of

$$\big(\mu(t, \omega, \omega')x'\big)' = \int_0^1 F_x(t, \theta\omega) d\theta x. \tag{4}$$

Compare equation (4) with

$$(Mx')' = -B(t)x. \tag{5}$$

Letting $dt=\mu(t,\omega,\omega')d\tau$, $t(\tau)=\int_0^\tau \mu[t(s),\omega(t(s)),\omega'(t(s))]ds$, then equations (5) and (4) are equivalent to the following systems, respectively,

$$\begin{cases} \frac{dx}{d\tau}=\frac{\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]}{M}x_1 \\ \frac{dx_1}{d\tau}=-\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]+B(t)x \end{cases} \tag{6}$$

and

$$\begin{cases} \frac{dx}{d\tau}=x_1 \\ \frac{dx_1}{d\tau}=\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]\cdot\int_0^1 F_x(t,\theta\omega)d\theta x. \end{cases} \tag{7}$$

Assume that (6) has the solution of the form

$$\begin{pmatrix} x \\ x_1 \end{pmatrix}=\begin{pmatrix} \cos X(\tau) & \sin X(\tau) \\ -\sin X(\tau) & \cos X(\tau) \end{pmatrix}\cdot\begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix} \tag{8}$$

where $X(\tau)$ is to be determined, and $Y(\tau), C(\tau)$ are as follows

$$Y(\tau)=\begin{pmatrix} \cos X(\tau) & \sin X(\tau) \\ -\sin X(\tau) & \cos X(\tau) \end{pmatrix} \qquad C(\tau)=\begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix}$$

$$\begin{aligned} & X'(\tau)\cdot\begin{pmatrix} -\sin X(\tau) & \cos X(\tau) \\ -\cos X(\tau) & -\sin X(\tau) \end{pmatrix}\cdot C(\tau)+Y(\tau)C_1'(\tau) \\ = & \begin{pmatrix} 0 & \frac{\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]}{M} \\ -\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]B(t) & 0 \end{pmatrix}\cdot Y(\tau)C(\tau). \end{aligned}$$

Simplifying the above, we have

$$\begin{aligned} C'(\tau) \;=\; & \left(\begin{pmatrix} E_{11} & \frac{\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]}{M}-\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]\cdot\int_0^1 F_x(t,\theta\omega)d\theta\sin^2 X \\ E_{21} & E_{22} \end{pmatrix}\right. \\ & \left.+X'(\tau)\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\cdot C(\tau) \end{aligned}$$

where

$$\left.\begin{aligned} E_{11} &= \mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]B(t(\tau))\sin X\cos X-\tfrac{\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))])\sin X\cos X}{M} \\ E_{21} &= -\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]B(t(\tau))\cos^2 X-\tfrac{\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))])\sin^2 X}{M}+X' \\ E_{22} &= -\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]B(t(\tau))\cos X\sin X+\tfrac{\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))])\sin X\cos X}{M} \end{aligned}\right\}.$$

Let $X_1(\tau)$ denote the solution of the following initial value problem

$$\begin{cases} X_1'=\frac{\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]}{M}\cos^2 X_1+\mu[t(\tau),\omega(t(\tau)),\omega'(t(\tau))]B(t(\tau))\sin^2 X \\ X_1(0)=0. \end{cases} \tag{9}$$

Similarly, assume that (7) has the solution of the form (8), then

$$C'(\tau) = \left(\begin{pmatrix} * & \cos^2 X - \mu[t(\tau), \omega(t(\tau)), \omega'(t(\tau))] \sin^2 X \int_0^1 F_x(t, \theta\omega) d\theta \\ * & * \end{pmatrix} \right.$$
$$\left. + X'(\tau) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \cdot C(\tau).$$

Letting $X_2(\tau)$ denote the solution of the following initial value problem

$$\begin{cases} X_2' = \cos^2 X_2 - \mu[t(\tau), \omega(t(\tau)), \omega'(t(\tau))] \sin^2 X_2 \int_0^1 F_x(t, \theta\omega) d\theta \\ X_2(0) = 0. \end{cases} \tag{10}$$

Note that for any $\tau_0 \in \mathbb{R}^+ \setminus \{0\}$, by the Sturm comparison theorem for the first order ordinary differential equations, we know that $X_1(\tau_0) < X_2(\tau_0)$. From (6) and (7), if $\tau_1, \tau_2 \in \mathbb{R}^+ \setminus \{0\}$ are zeros of the nontrivial solutions $x = x(t), x_1 = x_1(t)$ of equations (6) and (7), respectively, and satisfy the initial value problems

$$x(0) = 0, \qquad x_1(0) = 0$$

then

$$X_1(\tau_1), X_2(\tau_2) \in \{n\pi, n = 0, 1, 2, \cdots\}. \tag{11}$$

On the other hand, if τ_1, $\tau_2 \in \mathbb{R}^+ \setminus \{0\}$ satisfy (11), then (6),(7) must have the nontrivial solutions $x = x(t)$ and $x_1 = x_1(t)$, such that $x(0) = x(t(\tau_1)) = 0$ and $x_1(0) = x_1(t(\tau_2)) = 0$, respectively.

(I). In the case $N \geq 1$, notice that $X_2[\tau(2\pi)] \geq X_1[\tau(2\pi)]$. By using the Sturm comparison theorem, we obtain that $X_1[\tau(2\pi)] > 2N\pi$. Thus, $X_2[\tau(2\pi)] > 2N\pi$. Again by the Sturm comparison theorem, we obtain that every nontrivial solution of equation (4) must have at least $2N$ zeros in the open interval $(0, 2\pi)$. Without the loss of generality, we assume that $b(0) = b(2\pi) = 0$.

Compare equation (4) with

$$(Lx')' = -u_2(t)x.$$

By using the similar arguments as the above, and letting $X_3(\tau) and X_4(\tau)$ denote the solutions of the following initial value problems, respectively, we have

$$\begin{cases} X_3' = \frac{L}{\mu[t(\tau), \omega(t(\tau)), \omega'(t(\tau))]} \cos^2 X_3 - L \sin^2 X_3 \int_0^1 F_x[t(\tau), \theta\omega] d\theta \\ X_3(0) = 0 \end{cases}$$

and

$$\begin{cases} X_4' = \cos^2 X_4 + L u_1(t(\tau)) \sin^2 X_4 \\ X_4(0) = 0. \end{cases}$$

Notice that $X_4[\tau(2\pi)] \geq X_3[\tau(2\pi)]$. Also by using the Sturm comparison theorem, we obtain that $X_4[\tau(2\pi)] < 2(N+1)\pi$. Thus,

$$X_3[\tau(2\pi)] < 2(N+1)\pi. \tag{12}$$

Since $b(t)$ is 2π-periodic, it is impossible for $b(t)$ to have an odd number of zeros in $[0, 2\pi]$. By our previous hypothesis $b(0) = b(2\pi) = 0$ and the above conclusion that $b(t)$ has at least $2N$ zeros in $(0, 2\pi)$, we can conclude that $b(t)$ has at least $2(N+1)$ zeros in $[0, 2\pi]$. Hence, $X_3[\tau(2\pi)] \geq 2(N+1)\pi$. This yields a contradiction with (12).

(II). In the case $N = 0$, we can conclude that $b(t)$ has zeros in $\mathbb{R}$ by the Sturm comparison theorem. Without the loss of generality, we assume that $b(0) = 0$. Due to the periodicity, $b(t)$ has at least two zeros in $(0, 2\pi)$. The rest of the proof is similar to that of the case (I), so it is omitted.

¿From the above proof, we know that equation (1) has at least a 2π-periodic solution. However, we know that every 2π-periodic solution of equation (1) must be 2π-periodic solution of equation $(\mu(t, x, x')x')' = F(t, x)$. Therefore, we can conclude that under the assumptions (H1) and (H2) equation (1) has a unique 2π-periodic solution. The proof is complete.

Proof of Theorem 2. It is well-known that the following result of the optimal control theory can be widely applied. For the detailed proof, we may go back to [18]. Some interesting applications of the optimal control theory method to several boundary value problems for ordinary differential equations can be found in [16-19]. Let $(k-1)^2 < A < k^2 < B$, where k is the minimal positive integer suiting the inequality. Suppose that $u \in L[0, 2\pi]$ satisfying

$$A \leq u(t) \leq B \qquad and \int_0^{2\pi} u(t)dt < 2A\pi + 2(B-A)\alpha_k$$

where $\alpha_k = \alpha(\frac{1}{2k})$, the minimal positive root of

$$\sqrt{A}\cot\left(\sqrt{A}\lambda(\pi - x)\right) = \sqrt{B}\tan(\sqrt{B}\lambda x)$$

for $\lambda = \frac{1}{2k}$. Then the periodic boundary value problem

$$\begin{cases} y'' + u(t)y = f(t) \\ u(t) = u(t + 2\pi) \\ y(0) = y(2\pi), y'(0) = y'(2\pi) \end{cases}$$

has a unique 2π-periodic solution for each 2π-periodic function $f \in L[0, 2\pi]$.

To prove that equation (2) has at least one 2π-periodic solution, we can use the same arguments as that of theorem 1. To prove the uniqueness, by differentiating both sides of equation (2) with respect to t, we have

$$x'' = F(t, x) \tag{13}$$

where $F(t,x) = f_t + f \cdot f_x$ and $F_x(t,x) = f_{tx} + f_{xx} \cdot f + (f_x)^2$. Let $X_1(t)$ and $X_2(t)$ be any two 2π-periodic solutions of equation (13). Then $b(t) = X_1(t) - X_2(t)$ is a 2π-periodic solution of

$$x'' = \int_0^1 F_x[t, x_2(t) + \theta x_1(t)] d\theta x. \tag{14}$$

¿From the assumption, we see

$$A \leq -\int_0^1 F_x[t, x_2(t) + \theta x_1(t)] d\theta x \leq \beta(x).$$

By using the above result of optimal control theory, $b(t) \equiv 0$ for all $t \in \mathbb{R}$. Therefore, equation (2) has a unique 2π-periodic solution. The proof is complete.

3 Conclusion

¿From Section 2, we can see that using the Sturm Theorem as well as the Brouwer's fixed pointed theorem is really an effective approach for equations (1) and (2). It is easily noted that even when $a(t,x,x') = 1$, our conditions are different from all those in the previous references [1-8]. We also can use this method to study the sublinear Duffing equations investigated in [20].

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