PERIODIC SOLUTIONS OF NEUTRAL DUFFING EQUATIONS

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Abstract. We consider the following neutral delay Duffing equation

$$ax''(t) + bx'(t) + cx(t) + g(x(t-\tau_1),\ x'(t-\tau_2),\ x''(t-\tau_3)) = p(t) = p(t+2\pi),$$

where a, b and c are constants, τ_i, $i = 1, 2, 3$, are nonnegative constants, $g : R \times R \times R \to R$ is continuous, and $p(t)$ is a continuous 2π-periodic function. In this paper, combining the Brouwer degree theory with a continuation theorem based on Mawhin's coincidence degree, we obtain a sufficient condition for the existence of 2π-periodic solution of above equation.

Key words: Periodic solution, Duffing equation, Brouwer degree, coincidence degree.

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1. Introduction

On the existence problem of periodic solutions for the Duffing equations

$$x''(t) + g(x) = p(t) = p(t+2\pi), \tag{1.1}$$

so far there has been a wide literature since the interest in studying Eq.(1.1) comes from different sources. Under the conditions which exclude the resonance cases,

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many results have been obtained [1,2,3,4]. At resonance, many authors have paid much attention to the problem in recent years. [5] and [6] resolved the existence problem of 2π-periodic solutions of Eq.(1.1) under some different conditions, respectively.

On the other hand, a few papers have appeared[7,8,9,10,11,12] which dealt with the existence problem of periodic solutions to the delay Duffing equations such as

$$x''(t) + g(x(t-\tau)) = p(t) = p(t+2\pi). \tag{1.2}$$

Under some conditions which exclude the resonance cases, some results have been obtained[13,14,15].

Next, [17] discussed the Duffing equations of the form

$$x''(t) + m^2x(t) + g(x(t-\tau)) = p(t) = p(t+2\pi), \tag{1.3}$$

where m is a positive integer, and proved the existence of 2π-periodic solutions of Eq.(1.3) under some conditions.

Jack Hale [21] and [22] put forward the Euler's equations which are of the form

$$x''(t) = f(t, x(t), x(t-r), x'(t), x'(t-r), x''(t-r)),$$

where r is a positive constant.

Motivated by above papers, in the present paper, we consider the neutral Duffing equations of the form

$$ax''(t)+bx'(t)+cx(t)+g(x(t-\tau_1),\ x'(t-\tau_2),\ x''(t-\tau_3)) = p(t) = p(t+2\pi), \tag{1.4}$$

where a, b, c are constants, τ_1, τ_2, τ_3 are nonnegative constants, $g : R\times R \times R \to R$ is continuous, and p(t) is a continuous 2π-periodic function.

To the best of our knowledge, in this direction, few papers can be found in the literature. In this paper, combining the Brouwer degree theory with a continuation theorem based on Mawhin's coincidence degree[16], we obtain a sufficient condition for the existence of 2π-periodic solution of Eq.(1.4).

2. Existence of a Periodic Solution

In order to obtain the existence of a periodic solution of Eq. (1.4), we first make the following preparations.

Let X and Z be two Banach spaces. Consider an operator equation

$$Lx = \lambda Nx,$$

where L: Dom $L \cap X \to Z$ is a linear operator and $\lambda \in [0, 1]$ a parameter. Let P and Q denote two projectors such that

$$P : \mathrm{Dom}\, L \cap X \to \mathrm{Ker}\, L \quad \text{and} \quad Q : Z \to Z/\mathrm{Im}\, L.$$

In the sequel, we will use the following result of Mawhin[16].

LEMMA 2.1. *Let X and Z be two Banach spaces and L a Fredholm mapping of index 0. Assume that $N : \overline{\Omega} \to Z$ is L-compact on $\overline{\Omega}$ with Ω open bounded in X. Furthermore suppose*

(a). *For each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap$ Dom L*

$$Lx \neq \lambda Nx.$$

(b). *For each $x \in \partial\Omega \cap$ Ker L,*

$$QNx \neq 0$$

and

$$\deg\{QN,\ \Omega \cap\ Ker L,\ 0\} \neq 0.$$

Then $Lx = Nx$ *has at least one solution in* $\overline{\Omega}$.

Recall that a linear mapping L: Dom $L \subset X \to Z$ with Ker $L = L^{-1}(0)$ and Im $L = L(\mathrm{Dom} L)$, will be called a Fredholm mapping if the following two conditions hold:

(i). Ker L has a finite dimension;

(ii). Im L is closed and has a finite codimension.

Recalled also that the codimension of Im L is the dimension of $Z/$Im L, i.e., the dimension of the cokernel coker L of L.

When L is a Fredholm mapping, its (Fredholm) index is the integer

$$\mathrm{I}nd\, L = \dim \mathrm{Ker}\ L - \mathrm{codim}\, \mathrm{Im}\, L.$$

We shall say that a mapping N is L-compact on Ω if the mapping $QN : \bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded, and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact, i.e., it is continuous and $K_P(I-Q)N(\bar{\Omega})$ is relatively compact, where K_P: Im $L \to$Dom $L\cap$Ker P is a inverse of the restriction L_P of L to Dom $L\cap$Ker P, so that $LK_P = I$ and $K_P L = I - P$.

THEOREM 2.1. *Assume that there exist a positive constant* M *and three non-negative constants* $\beta_1,\ \beta_2, \beta_3$ *such that*

$$|g(x_1, x_2, x_3)| \leq M + \beta_1|x_1| + \beta_2|x_2| + \beta_3|x_3| \ \ for\ \forall(x_1, x_2, x_3) \in R^3 \qquad (2.1)$$

and

$$|abc| - |bc|\beta_3 - |ac|\beta_2 - (2|ab| + 2\pi|ac|)\beta_1 > \beta_3\sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)}. \qquad (2.2)$$

Then Eq.(1.4) has at least one 2π-periodic solution.

Proof. In order to use Lemma 2.1 for Eq.(1.4), we take $X = \{x(t) \in C^2(R,R) : x(t+2\pi) = x(t)\}$ and $Z = \{z(t) \in C(R,R) : z(t+2\pi) = z(t)\}$, and denote $|x|_0 = \max_{t\in[0,2\pi]} |x(t)|$ and $|x|_2 = \max\{|x|_0, |x'|_0, |x''|_0\}$. Then X and Z are Banach spaces when they are endowed with norms $|\cdot|_2$ and $|\cdot|_0$, respectively.

Set

$$Lx = ax''(t), \quad Nx = -bx'(t) - cx(t) - g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) + p(t),$$

$$Px = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt, \quad x \in X, \ \ Qz = \frac{1}{2\pi}\int_0^{2\pi} z(t)dt, \quad z \in Z.$$

Since $\operatorname{Ker} L = R$ and $\operatorname{Im} L = \{x \in Z : \int_0^{2\pi} x(t)dt = 0\}$, $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \dim Z/\operatorname{Im} L = 1$. Therefore, L is a Fredholm mapping of index 0.

Corresponding to the operator equation

$$Lx = \lambda N x, \quad \lambda \in (0,1),$$

we have

$$ax''(t) + \lambda bx'(t) + \lambda cx(t) + \lambda g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) = \lambda p(t). \tag{2.3}$$

Let $x(t) \in X$ is a solution of Eq.(2.3) for a certain $\lambda \in (0,1)$. Integrating (2.3) from 0 to 2π, we have

$$\int_0^{2\pi} cx(t)dt = \int_0^{2\pi} [p(t) - g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3))]dt,$$

from which, it implies that there exists a $t^* \in (0, 2\pi)$ such that

$$2\pi c x(t^*) = \int_0^{2\pi} [p(t) - g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3))]dt.$$

Let $m = \max_{t\in[0,2\pi]} |p(t)|$. Then

$$\begin{aligned}
2\pi|cx(t^*)| &\le 2\pi(m+M) + \beta_1 \int_0^{2\pi} |x(t-\tau_1)|dt \\
&\quad + \beta_2 \int_0^{2\pi} |x'(t-\tau_2)|dt + \beta_3 \int_0^{2\pi} |x''(t-\tau_3)|dt \\
&= 2\pi(m+M) + \beta_1 \int_0^{2\pi} |x(t)|dt + \beta_2 \int_0^{2\pi} |x'(t)|dt + \beta_3 \int_0^{2\pi} |x''(t)|dt.
\end{aligned}$$

Since for $\forall t \in [0, 2\pi]$,

$$\begin{aligned}
x(t) &= x(t^*) + \int_{t^*}^{t} x'(s)ds, \\
|x(t)| &\le |x(t^*)| + \int_0^{2\pi} |x'(s)|ds \\
&\le \frac{1}{\sqrt{2\pi}|c|} \left[\sqrt{2\pi}(m+M) + \beta_1 \left(\int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} \right. \\
&\quad \left. + (2\pi|c| + \beta_2) \left(\int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} + \beta_3 \left(\int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \right].
\end{aligned}$$

Thus

$$\begin{aligned}
|c| \left(\int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} &\le \sqrt{2\pi}|c| \max_{t\in[0,2\pi]} |x(t)| \\
&\le \sqrt{2\pi}(m+M) + \beta_1 \left(\int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} \\
&\quad + \beta_3 \left(\int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \\
&\quad + (2\pi|c| + \beta_2) \left(\int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}}
\end{aligned}$$

from which, it follows that

$$\begin{aligned}
(|c| - \beta_1) \left(\int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} &\le \sqrt{2\pi}(m+M) + (2\pi|c| + \beta_2) \left(\int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \\
&\quad + \beta_3 \left(\int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}}. \qquad (2.4)
\end{aligned}$$

Multipling (2.3) by $x''(t)$ and integrating from 0 to 2π, we get

$$a\int_0^{2\pi}|x''(t)|^2dt-\lambda c\int_0^{2\pi}|x'(t)|^2dt$$
$$+\lambda\int_0^{2\pi}x''(t)[g(x(t-\tau_1),x'(t-\tau_2),x''(t-\tau_3))-p(t)]dt=0,$$

from which, it implies that

$$\begin{aligned}
&|a|\int_0^{2\pi}|x''(t)|^2dt\\
&\le|c|\int_0^{2\pi}|x'(t)|^2dt+\int_0^{2\pi}|x''(t)|\Bigg[m+M\\
&\quad+\beta_1|x(t-\tau_1)|+\beta_2|x'(t-\tau_2)|+\beta_3|x''(t-\tau_3)|\Bigg]dt\\
&\le|c|\int_0^{2\pi}|x'(t)|^2dt+\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac12}\Bigg[\sqrt{2\pi}(m+M)\\
&\quad+\beta_1\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac12}+\beta_2\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac12}+\beta_3\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac12}\Bigg].
\end{aligned}$$

Therefore,

$$\begin{aligned}
(|a|-\beta_3)\int_0^{2\pi}|x''(t)|^2dt\le|c|\int_0^{2\pi}|x'(t)|^2dt+\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac12}\Bigg[\sqrt{2\pi}(m+M)\\
+\beta_1\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac12}+\beta_2\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac12}\Bigg]. \qquad (2.5)
\end{aligned}$$

From (2.4) and (2.5), we have

$$\begin{aligned}
&(|c|-\beta_1)(|a|-\beta_3)\int_0^{2\pi}|x''(t)|^2dt\\
&\le|c|(|c|-\beta_1)\int_0^{2\pi}|x'(t)|^2dt\\
&\quad+\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac12}\Bigg[\sqrt{2\pi}|c|(m+M)\\
&\quad+\beta_1\beta_3\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac12}+(2\pi\beta_1+\beta_2)|c|\left(\int_0^{2\pi}|x'(t)|^2\right)\Bigg],
\end{aligned}$$

from which, it follows that

$$
\begin{aligned}
&(|ac| - \beta_3|c| - \beta_1|a|)\int_0^{2\pi}|x''(t)|^2dt \\
&\le |c|(|c|-\beta_1)\int_0^{2\pi}|x'(t)|^2dt \\
&\quad + \left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\left[\sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2(|ac| - \beta_3|c| - \beta_1|a|)\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}} \\
&\quad + \left\{\left[\sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\right]^2\right. \\
&\quad \left. + 4(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c|-\beta_1)\int_0^{2\pi}|x'(t)|^2dt\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{2.6}
$$

Using inequality $(a+b)^{\frac{1}{2}} \le a^{\frac{1}{2}} + b^{\frac{1}{2}}$, for $a \ge 0$ and $b \ge 0$,we have

$$
\begin{aligned}
&\left\{\left[\sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\right]^2\right. \\
&\quad \left. + 4(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c|-\beta_1)\int_0^{2\pi}|x'(t)|^2dt\right\}^{\frac{1}{2}} \\
&\le \sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}} \\
&\quad + 2\sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c|-\beta_1)}\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.7}
$$

By (2.6) and (2.7), we have

$$
\begin{aligned}
&(|ac| - \beta_3|c| - \beta_1|a|)\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi}|c|(m+M) + \Bigg[|c|(2\pi\beta_1 + \beta_2) \\
&\quad + \sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)}\Bigg]\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.8}
$$

Multipling (2.3) by $x'(t)$ and integrating from 0 to 2π, we obtain

$$
b\int_0^{2\pi} |x'(t)|^2 dt + \int_0^{2\pi} x'(t)[g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) - p(t)]dt = 0,
$$

from which, it implies that

$$
\begin{aligned}
|b|\int_0^{2\pi} |x'(t)|^2 dt &\le \left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} \Bigg[\sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi} |x(t-\tau_1)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \beta_2\left(\int_0^{2\pi} |x'(t-\tau_2)|^2 dt\right)^{\frac{1}{2}} + \beta_3\left(\int_0^{2\pi} |x''(t-\tau_3)|^2 dt\right)^{\frac{1}{2}}\Bigg] \\
&= \left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} \Bigg[\sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \beta_2\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} + \beta_3\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}}\Bigg].
\end{aligned}
$$

Thus

$$
\begin{aligned}
(|b| - \beta_2)\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} &\le \sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \beta_3\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}},
\end{aligned}
\tag{2.9}
$$

from which, together with (2.4), it implies that

$$
\begin{aligned}
&(|c| - \beta_1)(|b| - \beta_2)\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi}|c|(m+M) + \beta_3|c|\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + (2\pi\beta_1|c| + \beta_1\beta_2)\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.10}
$$

In view of (2.8) and (2.10), we can obtain

$$\begin{aligned}
&(|c|-\beta_1)(|b|-\beta_2)(|ac|-\beta_3|c|-\beta_1|a|)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\\
&\le \sqrt{2\pi}|c|(m+M)(|ac|-\beta_3|c|-\beta_1|a|)+\sqrt{2\pi}\beta_3c^2(m+M)\\
&\quad+\left\{(2\pi|c|\beta_1+\beta_1\beta_2)(|ac|-\beta_3|c|-\beta_1|a|)+\beta_3|c|\left[|c|(2\pi\beta_1+\beta_2)\right.\right.\\
&\quad\left.\left.+\sqrt{(|ac|-\beta_3|c|-\beta_1|a|)|c|(|c|-\beta_1)}\right]\right\}\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}},
\end{aligned}$$

from which, together with (2.2), it implies that there exists a positive constant R_1 such that

$$\int_0^{2\pi}|x'(t)|^2dt\le R_1. \tag{2.11}$$

By (2.6) and (2.11), there exists a positive constant R_2 such that

$$\int_0^{2\pi}|x''(t)|^2dt\le R_2. \tag{2.12}$$

From (2.4), (2.11) and (2.12), there exists a positive constant R_3 such that

$$\int_0^{2\pi}|x(t)|^2dt\le R_3. \tag{2.13}$$

Therefore, there exist three positive constants R_1^*, R_2^* and R_3^* such that $\forall t\in[0,2\pi]$,

$$|x(t)|\le R_1^* \quad |x'(t)|\le R_2^*, \quad |x''(t)|\le R_3^*.$$

Let $A=\max\{R_1^*,R_2^*,R_3^*,(m+M)/(|c|-\beta_1)\}$ and take $\Omega=\{x(t)\in X:|x|_2< A\}$. We now will show that N is L-compact on $\bar{\Omega}$. For any $x\in\bar{\Omega}$,

$$\begin{aligned}
|QNx|_0&\le\frac{1}{2\pi}\int_0^{2\pi}[|b|R_2^*+|c|R_1^*+m+M+\beta_1R_1^*+\beta_2R_2^*+\beta_3R_3^*]dt\\
&=M_1,
\end{aligned}$$

where $M_1 = |b|R_2^* + |c|R_1^* + m + M + \beta_1 R_1^* + \beta_2 R_2^* + \beta_3 R_3^*$. Hence, $QN(\bar{\Omega})$ is a bounded set in R. Obviously, $QNx : \bar{\Omega} \to Z$ is continuous. For $\forall z \in \text{Im } L \cap Z$,

$$(K_P z)(t) = \int_0^t ds \int_0^s z(u)du - \frac{1}{2\pi} \int_0^{2\pi} dt \int_0^t ds \int_0^s z(u)du$$

is continuous with respect to z, and

$$|K_P z|_0 \leq \frac{8}{3}\pi^2 \max_{t\in[0,2\pi]} |z(t)|,$$

$$\begin{aligned} |K_P(I-Q)Nx|_0 &\leq \frac{8}{3}\pi^2 |Nx|_0 + \frac{8}{3}\pi^2 |QNx|_0 \\ &\leq \frac{16}{3}\pi^2 |Nx|_0 \\ &\leq \frac{16}{3}\pi^2 M_1. \end{aligned}$$

For $\forall x \in \Omega$, we have

$$\begin{aligned} \left| \frac{d}{dt}(K_P(I-Q)Nx) \right|_0 &\leq \int_0^t |[(I-Q)Nx](t)|_0 dt \\ &\leq 2\pi |[(I-Q)Nx](t)|_0 \\ &\leq 4\pi\ |Nx|_0 \leq 4\pi M_1. \end{aligned}$$

Thus, the set $\{K_P(I-Q)Nx | x \in \bar{\Omega}\}$ is equicontinuous and uniformly bounded. Consequently, N is L-compact. This satisfies condition (a) in Lemma 2.1.

When $x \in \partial\Omega \cap KerL = \partial\Omega \cap R$, x is a constant with $|x| = A$. Then

$$\begin{aligned} QNx &= \frac{1}{2\pi} \int_0^{2\pi} [-bx'(t) - cx(t) - g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) + p(t)]dt \\ &= -cx - g(x, 0, 0) + \frac{1}{2\pi} \int_0^{2\pi} p(t)dt. \end{aligned}$$

Thus

$$\begin{aligned} |QNx|_0 &\geq |c| \left(|x| - \frac{|g(x,0,0)| + m}{|c|} \right) \\ &\geq |c| \left(A - \frac{m + M + \beta_1 A}{|c|} \right) > 0. \end{aligned}$$

Therefore, $QNx \neq 0, x \in \partial\Omega \cap R$.

Set for $0 \le \mu \le 1$

$$\begin{aligned}\phi(x,\mu) = \mu x(t) + (1-\mu)\Big[& x(t) + g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) \\ &- \frac{1}{2\pi}\int_0^{2\pi} p(t)dt\Big].\end{aligned}$$

When $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\mu \in [0,1]$, x is a constant with $|x| = A$. Without loss of generality, we suppose $x = A$. Now we consider two possible cases: (1) $x = A, c > 0$; (2) $x = A, c < 0$.

(1). When $x = A$ and $c > 0$,

$$\begin{aligned}\phi(x,\mu) &= cA + (1-\mu)\left[g(A,0,0) - \frac{1}{2\pi}\int_0^{2\pi} p(t)dt\right] \\ &\ge c\left[A - \frac{1-\mu}{c}\left(|g(A,0,0)| + \frac{1}{2\pi}\int_0^{2\pi}|p(t)|dt\right)\right] \\ &\ge c\left(A - \frac{m+M+\beta_1 A}{c}\right) > 0;\end{aligned}$$

(2). When $x = A$ and $c < 0$,

$$\phi(x,\mu) \le c\left(A - \frac{m+M+\beta_1 A}{|c|}\right) < 0.$$

Thus when $x = A, \phi(x,\mu) \neq 0$. Therefore,

$$\begin{aligned}\deg(QN, \Omega \cap KerL, 0) = \deg\Big\{& -cx(t) - g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) \\ &+ \frac{1}{2\pi}\int_0^{2\pi} p(t)dt, \Omega \cap \text{ Ker } L, 0\Big\} \\ = \deg(&-cx, \Omega \cap \text{ Ker } L, 0) \neq 0.\end{aligned}$$

By now we know that Ω verifies all the requirements in Lemma 2.1. This completes the proof of Theorem 2.1.

Example The second order neutral delay differential equation

$$10x''(t) + 100x'(t) + 5x(t) + \frac{1 + \frac{1}{2}x(t-1) + \frac{1}{2}x'(t-2) + \frac{1}{100}x''(t-3)}{1 + x^2(t-1)} = sint, \qquad (2.14)$$

satisfies all conditions in Theorem 2.1. Therefore, Eq.(2.14) has at least one 2π-periodic solution.

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