A necessary and sufficient condition for the oscillation in a class of even order neutral differential equations

Satoshi Tanaka

Department of Mathematical Sciences, Faculty of Science
Ehime University, Matsuyama 790–8577, Japan [1]

Abstract

The even order neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + f(t, x(g(t))) = 0 \tag{1.1}$$

is considered under the following conditions: $n \geq 2$ is even; $\lambda > 0$; $\tau > 0$; $g \in C[t_0, \infty)$, $\lim_{t\to\infty} g(t) = \infty$; $f \in C([t_0, \infty) \times \mathbf{R})$, $uf(t, u) \geq 0$ for $(t, u) \in [t_0, \infty) \times \mathbf{R}$, and $f(t, u)$ is nondecreasing in $u \in \mathbf{R}$ for each fixed $t \geq t_0$. It is shown that equation (1.1) is oscillatory if and only if the non-neutral differential equation

$$x^{(n)}(t) + \frac{1}{1+\lambda} f(t, x(g(t))) = 0$$

is oscillatory.

Keywords: Oscillation, Neutral differential equation.
AMS Subject Classification: 34K11, 34K40.

1. Introduction and main results

We shall be concerned with the oscillatory behavior of solutions of the even order neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + f(t, x(g(t))) = 0. \tag{1.1}$$

[1] Present address of the author : Department of Liberal Arts and Engineering Science, Hachinohe National College of Technology, Hachinohe 039–1192, Japan; e-mail address: tanaka-g@hachinohe-ct.ac.jp

Throughout this paper, the following conditions are assumed to hold: $n \geq 2$ is even; $\lambda > 0$; $\tau > 0$; $g \in C[t_0, \infty)$, $\lim_{t\to\infty} g(t) = \infty$; $f \in C([t_0, \infty) \times \mathbf{R})$, $uf(t, u) \geq 0$ for $(t, u) \in [t_0, \infty) \times \mathbf{R}$, and $f(t, u)$ is nondecreasing in $u \in \mathbf{R}$ for each fixed $t \geq t_0$.

By a solution of (1.1), we mean a function $x(t)$ that is continuous and satisfies (1.1) on $[t_x, \infty)$ for some $t_x \geq t_0$. Therefore, if $x(t)$ is a solution of (1.1), then $x(t) + \lambda x(t - \tau)$ is n-times continuously differentiable on $[t_x, \infty)$. Note that, in general, $x(t)$ itself is not continuously differentiable.

A solution is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. This means that a solution $x(t)$ is oscillatory if and only if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ as $i \to \infty$ and $x(t_i) = 0$ $(i = 1, 2, \ldots)$, and a solution $x(t)$ is nonoscillatory if and only if $x(t)$ is eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if every solution of (1.1) is oscillatory.

There has been considerable investigation of the oscillations of even order neutral differential equations. For typical results we refer to the papers [1, 2, 4–8, 11, 12, 16, 18, 19, 21–25] and the monographs [3] and [9]. Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. See Hale [10].

Now consider the linear equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t - \tau)] + p(t)x(t - \sigma) = 0 \tag{1.2}$$

and the nonlinear equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t - \tau)] + p(t)|x(t - \sigma)|^{\gamma} \operatorname{sgn} x(t - \sigma) = 0. \tag{1.3}$$

Here and hereafter we assume that $\sigma \in \mathbf{R}$, $\gamma > 0$, $\gamma \neq 1$, $p \in C[t_0, \infty)$, $p(t) > 0$ for $t \geq t_0$.

For the case $0 < \lambda < 1$, Jaroš and Kusano [11, Theorems 3.1 and 4.1] proved that equation (1.2) is oscillatory if

$$\int^{\infty} t^{n-1-\varepsilon} p(t) dt = \infty \quad \text{for some } \varepsilon > 0, \tag{1.4}$$

and that equation (1.2) has a nonoscillatory solution if

$$\int^{\infty} t^{n-1} p(t) dt < \infty. \tag{1.5}$$

There is a difference between (1.4) and (1.5). Indeed, the case $p(t) = c\,t^{-n}$ $(c > 0)$ is an example such that both conditions (1.4) and (1.5) fail.

In particular, Y. Naito [19, Theorems 5.3 and 5.4] characterized the oscillation of equation (1.3) with $0 < \lambda < 1$ as follows: equation (1.3) with $0 < \lambda < 1$ is oscillatory if and only if

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} p(t)dt = \infty. \tag{1.6}$$

For the case $\lambda \geq 1$, it is known that if (1.5) holds, then equation (1.2) has a nonoscillatory solution, and that if

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} p(t)dt < \infty, \tag{1.7}$$

then equation (1.3) has a nonoscillatory solution. See, M. Naito [18] $(\lambda = 1)$, Chen [1] $(\lambda > 1)$, and also [21]. In the case $\lambda \geq 1$, sufficient conditions for (1.2) or (1.3) to be oscillatory were established in [4], [5], [7] and [8], under the condition

$$\int^{\infty} p(t)dt = \infty.$$

Recently, it has been obtained in [22] that if

$$\int^{\infty} t^{n-1-\varepsilon} \min\{p(t), p(t-\tau)\}dt = \infty \quad \text{for some } \varepsilon > 0, \tag{1.8}$$

then equation (1.2) with $\lambda \geq 1$ is oscillatory, and that if

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} \min\{p(t), p(t-\tau)\}dt = \infty, \tag{1.9}$$

then equation (1.3) with $\lambda \geq 1$ is oscillatory. However, as compared with the case $0 < \lambda < 1$, there are gaps between conditions (1.5) and (1.8), and between conditions (1.7) and (1.9).

In this paper we have the following oscillation theorem, which is able to narrow the above difference and gaps.

Theorem 1.1. *Equation* (1.1) *is oscillatory if and only if*

$$x^{(n)}(t) + \frac{1}{1+\lambda} f(t, x(g(t))) = 0 \tag{1.10}$$

is oscillatory.

The oscillatory behavior of solutions of non-neutral differential equations of the form

$$x^{(n)}(t) + f(t, x(g(t))) = 0$$

has been intensively studied in the last three decades. We refer the reader to [9, 14, 15, 17, 20] and the references cited therein. In Section 2, using the known oscillation results for the equations

$$x^{(n)}(t) + p(t)x(t-\sigma) = 0 \tag{1.11}$$

and

$$x^{(n)}(t) + p(t)|x(t-\sigma)|^{\gamma} \operatorname{sgn} x(t-\sigma) = 0, \tag{1.12}$$

we prove the following corollaries of Theorem 1.1.

Corollary 1.1. (i) *Equation* (1.2) *is oscillatory if*

$$\int^{\infty} t^{n-2} p(t) dt = \infty. \tag{1.13}$$

(ii) *Suppose that* (1.13) *fails. Equation* (1.2) *is oscillatory if*

$$\limsup_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s) ds > (1+\lambda)(n-1)!, \tag{1.14}$$

or if

$$\liminf_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s) ds > (1+\lambda)(n-1)!/4. \tag{1.15}$$

Equation (1.2) *has a nonoscillatory solution if*

$$\limsup_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s) ds < (1+\lambda)(n-2)!/4. \tag{1.16}$$

Corollary 1.2. *Equation* (1.2) *is oscillatory if* (1.4) *holds. Equation* (1.2) *has a nonoscillatory solution if* (1.5) *holds.*

Corollary 1.3. *Equation* (1.3) *is oscillatory if and only if* (1.6) *holds.*

It is possible to obtain oscillation results for equations of the form (1.1). However, for simplicity, we have restricted our attention to equations (1.2) and (1.3).

We give an example illustrating Corollary 1.1.

Example 1.1. We consider the linear neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + c\,t^\alpha x(t-\sigma) = 0, \tag{1.17}$$

where $c > 0$, $\alpha \in \mathbf{R}$. Applying Corollary 1.1, we conclude that: equation (1.17) is oscillatory if either $\alpha = -n$ and $c > (1+\lambda)(n-1)!/4$ or $\alpha > -n$; equation (1.17) has a nonoscillatory solution if either $\alpha = -n$ and $c < (1+\lambda)(n-2)!/4$ or $\alpha < -n$.

Let us consider the equation

$$\frac{d^n}{dt^n}[x(t) + \overline{\lambda} x(t-\overline{\tau})] + \overline{f}(t, x(\overline{g}(t))) = 0, \tag{1.18}$$

where $\overline{\lambda} > 0$, $\overline{\tau} > 0$, $\overline{g} \in C[t_0, \infty)$, $\lim_{t\to\infty} \overline{g}(t) = \infty$, $\overline{f} \in C([t_0,\infty) \times \mathbf{R})$, $u\overline{f}(t,u) \geq 0$ for $(t,u) \in [t_0,\infty) \times \mathbf{R}$.

From Theorem 1.1, we obtain the following comparison result.

Corollary 1.4. *Suppose that* $\overline{\lambda} \leq \lambda$, $\overline{g}(t) \geq g(t)$ *for* $t \geq t_0$, *and* $|\overline{f}(t,u)| \geq |f(t,u)|$ *for* $(t,u) \in [t_0,\infty) \times \mathbf{R}$. *If* (1.1) *is oscillatory, then* (1.18) *is oscillatory.*

The proof of Corollary 1.4 is deferred to the next section.

In Section 3 we investigate the relation between functions $u(t)$ and $u(t) + \lambda u(t-\tau)$. We show the "if" part and the "only if" part of Theorem 1.1 in Sections 4 and 5, respectively.

Such an approach as Theorem 1.1 has been conducted by Tang and Shen [23], and Zhang and Yang [25] for odd order neutral differential equations.

2. Proofs of Corollaries 1.1–1.4

In this section we prove Corollaries 1.1–1.4. It is known that equation (1.12) is oscillatory if and only if (1.6) holds. See, for example, Kitamura [14, Corollary 3.1]. Hence, Corollary 1.3 follows from Theorem 1.1.

The following oscillation result for equation (1.11) have been established by M. Naito [17, Theorems 2 and 4] and Kusano [15, Theorems 3 and 4].

Lemma 2.1. (i) *Equation* (1.11) *is oscillatory if* (1.13) *holds.*

(ii) *Suppose that* (1.13) *fails. Equation* (1.11) *is oscillatory if*

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\,ds > (n-1)!,$$

or if

$$\liminf_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds > (n-1)!/4.$$

Equation (1.11) *has a nonoscillatory solution if*

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds < (n-2)!/4.$$

Combining Theorem 1.1 with Lemma 2.1, we obtain Corollary 1.1.

Now let us show that Corollary 1.1 implies Corollary 1.2.

Suppose that (1.4) holds. From the result of Kitamura [14, Corollary 5.1] it follows that the equation

$$x^{(n)}(t) + \mu p(t)x(t-\sigma) = 0 \tag{2.1}$$

is oscillatory for all constant $\mu > 0$. M. Naito [17, Theorem 5] and Kusano [15, Theorem 2] have shown that equation (2.1) is oscillatory for all $\mu > 0$ if and only if either (1.13) holds or

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds = \infty.$$

This means that if (1.4) holds, then either (1.13) or (1.14) is satisfied, and so equation (1.2) is oscillatory.

Suppose next that (1.5) holds. Then

$$0 \le \lim_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds \le \lim_{t\to\infty}\int_t^\infty s^{n-1}p(s)ds = 0.$$

Consequently, if (1.5) holds, then (1.16) is saisfied, and so (1.2) has a nonoscillatory solution.

To prove Corollary 1.4, we need the following result which has been obtained by Onose [20].

Lemma 2.2. *If the differential inequality*

$$x^{(n)}(t) + f(t, x(g(t))) \le 0$$

has an eventually positive solution, then the differential equation

$$x^{(n)}(t) + f(t, x(g(t))) = 0$$

has an eventually positive solution.

Proof of Corollary 1.4. Assume that (1.15) has a nonoscillatory solution. Then Theorem 1.1 implies that

$$x^{(n)}(t) + \frac{1}{1+\overline{\lambda}}\overline{f}(t, x(\overline{g}(t))) = 0$$

has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all large t. For the case where $x(t) < 0$ for all large t, $y(t) \equiv -x(t)$ is an eventually positive solution of

$$y^{(n)}(t) + \frac{1}{1+\overline{\lambda}}\widetilde{f}(t, y(\overline{g}(t))) = 0,$$

where $\widetilde{f}(t, u) = -\overline{f}(t, -u)$, and hence the case $x(t) < 0$ can be treated similarly. From Lemma 4.1 below it follows that $x(t)$ is eventually nondecreasing. In view of the hypothesis of Corollary 1.4, we see that $x(\overline{g}(t)) \geq x(g(t))$ for all large $t \geq t_0$, and

$$-x^{(n)}(t) \geq \frac{1}{1+\lambda} f(t, x(\overline{g}(t))) \geq \frac{1}{1+\lambda} f(t, x(g(t)))$$

for all large $t \geq t_0$. By Lemma 2.2 and Theorem 1.1, equation (1.1) has a nonoscillatory solution. This completes the proof.

3. Relation between $u(t)$ and $u(t) + \lambda u(t-\tau)$

In this section we study the relation between functions $u(t)$ and $u(t) + \lambda u(t-\tau)$.

We use the notation:

$$(\Delta u)(t) = u(t) + \lambda u(t-\tau).$$

Lemma 3.1. *Let $\lambda \neq 1$ and $l \in \mathbf{N} \cup \{0\}$. Suppose that $u \in C[T-\tau, \infty)$, $\Delta u \in C^1[T, \infty)$, $(\Delta u)(t) \geq 0$, $(\Delta u)'(t) \geq 0$ for $t \geq T$, and $\lim_{t\to\infty}(\Delta u)'(t)/t^l = 0$. For the case $\lambda > 1$, assume moreover that $\lim_{t\to\infty} \lambda^{-t/\tau} u(t) = 0$. Then*

$$u(t) = \frac{1}{1+\lambda}(\Delta u)(t) + o(t^l) \quad (t \to \infty).$$

We divide the proof of Lemma 3.1 into the two cases $0 < \lambda < 1$ and $\lambda > 1$.

Proof of Lemma 3.1 for the case $0 < \lambda < 1$. Let $\lambda \in (0,1)$. We see that

$$u(t) = (\Delta u)(t) - \lambda u(t-\tau), \quad t \geq T,$$

so that

$$\begin{aligned} u(t) &= (\Delta u)(t) - \lambda[(\Delta u)(t-\tau) - \lambda u(t-2\tau)] \\ &= (\Delta u)(t) - \lambda(\Delta u)(t-\tau) + \lambda^2 u(t-2\tau), \quad t \geq T+\tau, \end{aligned}$$

and

$$\begin{aligned} u(t) &= (\Delta u)(t) - \lambda(\Delta u)(t-\tau) + \lambda^2[(\Delta u)(t-2\tau) - \lambda u(t-3\tau)] \\ &= (\Delta u)(t) - \lambda(\Delta u)(t-\tau) + \lambda^2(\Delta u)(t-2\tau) - \lambda^3 u(t-3\tau) \end{aligned}$$

for $t \geq T + 2\tau$. We have

$$u(t) = \sum_{i=0}^{m}(-\lambda)^i(\Delta u)(t-i\tau) + (-\lambda)^{m+1}u(t-(m+1)\tau) \tag{3.1}$$

for $t \geq T + m\tau$, $m = 0, 1, 2, \ldots$. Observe that

$$\begin{aligned} \sum_{i=0}^{m}(-\lambda)^i(\Delta u)(t-i\tau) &= \sum_{i=0}^{m}(-\lambda)^i\left[(\Delta u)(t) - \int_{t-i\tau}^{t}(\Delta u)'(s)ds\right] \\ &= \frac{1-(-\lambda)^{m+1}}{1+\lambda}(\Delta u)(t) \\ &\quad - \sum_{i=1}^{m}(-\lambda)^i\int_{t-i\tau}^{t}(\Delta u)'(s)ds \end{aligned} \tag{3.2}$$

for $t \geq T + m\tau$, $m = 1, 2, \ldots$. If $t \in [T+m\tau, T+(m+1)\tau]$, $m = 1, 2, \ldots$, then $(t-T)\tau^{-1} - 1 \leq m \leq (t-T)\tau^{-1}$. Hence we have

$$\begin{aligned} |u(t-(m+1)\tau)| &\leq \max_{s\in[T+m\tau, T+(m+1)\tau]}|u(s-(m+1)\tau)| \\ &= \max_{s\in[T-\tau,T]}|u(s)| \end{aligned} \tag{3.3}$$

and

$$|(-\lambda)^{m+1}| \leq \lambda^{m+1} \leq \lambda^{(t-T)\tau^{-1}-1+1} = \lambda^{(t-T)/\tau} \tag{3.4}$$

for $t \in [T+m\tau, T+(m+1)\tau]$, $m = 1, 2, \ldots$. Combining (3.1) with (3.2)–(3.4), we obtain

$$\begin{aligned} \left|u(t) - \frac{(\Delta u)(t)}{1+\lambda}\right| &\leq \frac{\lambda^{(t-T)/\tau}}{1+\lambda}(\Delta u)(t) + \sum_{i=1}^{m}\lambda^i\int_{t-i\tau}^{t}(\Delta u)'(s)ds \\ &\quad + \lambda^{(t-T)/\tau}\max_{s\in[T-\tau,T]}|u(s)| \end{aligned} \tag{3.5}$$

for $t \in [T+m\tau, T+(m+1)\tau]$, $m = 1, 2, \ldots$. We find that

$$\begin{aligned}
(3.6) \qquad \sum_{i=1}^{m} \lambda^i \int_{t-i\tau}^{t} (\Delta u)'(s)ds &= \sum_{i=1}^{m} \lambda^i \sum_{j=1}^{i} \int_{t-j\tau}^{t-(j-1)\tau} (\Delta u)'(s)ds \\
&= \sum_{j=1}^{m} \sum_{i=j}^{m} \lambda^i \int_{t-j\tau}^{t-(j-1)\tau} (\Delta u)'(s)ds \\
&= \sum_{j=1}^{m} \frac{\lambda^j - \lambda^{m+1}}{1-\lambda} \int_{t-j\tau}^{t-(j-1)\tau} (\Delta u)'(s)ds \\
&\leq \frac{1}{1-\lambda} \sum_{j=1}^{m} \lambda^j \int_{t-j\tau}^{t-(j-1)\tau} (\Delta u)'(s)ds
\end{aligned}$$

for $t \in [T+m\tau, T+(m+1)\tau]$, $m = 1, 2, \ldots$. Since $(t-s)\tau^{-1} \leq j \leq (t-s)\tau^{-1}+1$ for $s \in [t-j\tau, t-(j-1)\tau]$, we have

$$\begin{aligned}
(3.7) \qquad \sum_{j=1}^{m} \lambda^j \int_{t-j\tau}^{t-(j-1)\tau} (\Delta u)'(s)ds &\leq \sum_{j=1}^{m} \int_{t-j\tau}^{t-(j-1)\tau} \lambda^{(t-s)/\tau} (\Delta u)'(s)ds \\
&= \lambda^{t/\tau} \int_{t-m\tau}^{t} \lambda^{-s/\tau} (\Delta u)'(s)ds \\
&\leq \lambda^{t/\tau} \int_{T}^{t} \lambda^{-s/\tau} (\Delta u)'(s)ds
\end{aligned}$$

for $t \in [T+m\tau, T+(m+1)\tau]$, $m = 1, 2, \ldots$. From (3.5)–(3.7) it follows that

$$\begin{aligned}
(3.8) \qquad \left| u(t) - \frac{(\Delta u)(t)}{1+\lambda} \right| &\leq \frac{\lambda^{(t-T)/\tau}}{1+\lambda} (\Delta u)(t) + \frac{\lambda^{t/\tau}}{1-\lambda} \int_{T}^{t} \lambda^{-s/\tau} (\Delta u)'(s)ds \\
&\quad + \lambda^{(t-T)/\tau} \max_{s \in [T-\tau, T]} |u(s)|
\end{aligned}$$

for $t \geq T + \tau$. It can be shown that

$$(3.9) \qquad \lambda^{t/\tau} \int_{T}^{t} \lambda^{-s/\tau} (\Delta u)'(s)ds = o(t^l) \quad (t \to \infty).$$

Indeed,

$$\begin{aligned}
\lim_{t\to\infty} \frac{\int_T^t \lambda^{-s/\tau} (\Delta u)'(s)ds}{\lambda^{-t/\tau} t^l} &= \lim_{t\to\infty} \frac{\frac{d}{dt} \int_T^t \lambda^{-s/\tau} (\Delta u)'(s)ds}{\frac{d}{dt}[\lambda^{-t/\tau} t^l]} \\
&= \lim_{t\to\infty} \frac{(\Delta u)'(t)}{[\log \lambda^{-1/\tau} + lt^{-1}]t^l} = 0.
\end{aligned}$$

By $\lim_{t\to\infty}(\Delta u)'(t)/t^l = 0$, we see that $\lim_{t\to\infty}\lambda^{(t-T)/\tau}(\Delta u)(t) = 0$. Consequently, the conclusion follows from (3.8) and (3.9).

Proof of Lemma 3.1 for the case $\boldsymbol{\lambda > 1}$. Assume that $\lambda > 1$, and that $\lim_{t\to\infty}\lambda^{-t/\tau}u(t) = 0$. Let $t \geq T$ be fixed. Since

$$u(t) = \lambda^{-1}(\Delta u)(t+\tau) - \lambda^{-1}u(t+\tau),$$

we find that

$$\begin{aligned} u(t) &= \lambda^{-1}(\Delta u)(t+\tau) - \lambda^{-1}[\lambda^{-1}(\Delta u)(t+2\tau) - \lambda^{-1}u(t+2\tau)] \\ &= \lambda^{-1}(\Delta u)(t+\tau) - \lambda^{-2}(\Delta u)(t+2\tau) + \lambda^{-2}u(t+2\tau) \\ &= \lambda^{-1}(\Delta u)(t+\tau) - \lambda^{-2}(\Delta u)(t+2\tau) \\ &\quad + \lambda^{-2}[\lambda^{-1}(\Delta u)(t+3\tau) - \lambda^{-1}u(t+3\tau)] \\ &= \lambda^{-1}(\Delta u)(t+\tau) - \lambda^{-2}(\Delta u)(t+2\tau) + \lambda^{-3}(\Delta u)(t+3\tau) \\ &\quad - \lambda^{-3}u(t+3\tau), \end{aligned}$$

so that

$$u(t) = -\sum_{i=1}^{m}(-\lambda)^{-i}(\Delta u)(t+i\tau) + (-\lambda)^{-m}u(t+m\tau), \quad m \in \mathbf{N}. \tag{3.10}$$

We have

$$\begin{aligned} \sum_{i=1}^{m}(-\lambda)^{-i}(\Delta u)(t+i\tau) &= \sum_{i=1}^{m}(-\lambda)^{-i}\left[(\Delta u)(t) + \int_t^{t+i\tau}(\Delta u)'(s)ds\right] \\ &= \frac{-1+(-\lambda)^{-m}}{1+\lambda}(\Delta u)(t) \\ &\quad + \sum_{i=1}^{m}(-\lambda)^{-i}\int_t^{t+i\tau}(\Delta u)'(s)ds, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \left|\sum_{i=1}^{m}(-\lambda)^{-i}\int_t^{t+i\tau}(\Delta u)'(s)ds\right| &\leq \sum_{i=1}^{m}\lambda^{-i}\int_t^{t+i\tau}(\Delta u)'(s)ds \\ &= \sum_{i=1}^{m}\lambda^{-i}\sum_{j=1}^{i}\int_{t+(j-1)\tau}^{t+j\tau}(\Delta u)'(s)ds \\ &= \sum_{j=1}^{m}\sum_{i=j}^{m}\lambda^{-i}\int_{t+(j-1)\tau}^{t+j\tau}(\Delta u)'(s)ds \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \frac{\lambda^{-j+1} - \lambda^{-m}}{\lambda - 1} \int_{t+(j-1)\tau}^{t+j\tau} (\Delta u)'(s)ds \\
&\le \frac{\lambda}{\lambda - 1} \sum_{j=1}^{m} \lambda^{-j} \int_{t+(j-1)\tau}^{t+j\tau} (\Delta u)'(s)ds
\end{aligned}
$$

for $m \in \mathbf{N}$. If $s \in [t+(j-1)\tau, t+j\tau]$, then $(s-t)\tau^{-1} \le j \le (s-t)\tau^{-1} + 1$. Thus we obtain

$$
\begin{aligned}
\sum_{j=1}^{m} \lambda^{-j} \int_{t+(j-1)\tau}^{t+j\tau} (\Delta u)'(s)ds &\le \sum_{j=1}^{m} \int_{t+(j-1)\tau}^{t+j\tau} \lambda^{-(s-t)/\tau} (\Delta u)'(s)ds \\
&= \lambda^{t/\tau} \int_{t}^{t+m\tau} \lambda^{-s/\tau} (\Delta u)'(s)ds \\
&\le \lambda^{t/\tau} \int_{t}^{\infty} \lambda^{-s/\tau} (\Delta u)'(s)ds
\end{aligned} \tag{3.13}
$$

for $m \in \mathbf{N}$. We note here that $\lambda^{-t/\tau}(\Delta u)'(t)$ is integrable on $[T, \infty)$, because of $\lim_{t\to\infty} (\Delta u)'(t)/t^l = 0$. Put

$$
A(t) = \frac{\lambda}{\lambda - 1} \lambda^{t/\tau} \int_{t}^{\infty} \lambda^{-s/\tau} (\Delta u)'(s)ds.
$$

From (3.10)–(3.13) it follows that

$$
\left| u(t) - \frac{(\Delta u)(t)}{1+\lambda} \right| \le \frac{\lambda^{-m}}{1+\lambda} (\Delta u)(t) + A(t) + \lambda^{-m} |u(t+m\tau)| \tag{3.14}
$$

for $m \in \mathbf{N}$. By $\lim_{t\to\infty} \lambda^{-t/\tau} u(t) = 0$, we find that

$$
\begin{aligned}
\lim_{m\to\infty} \lambda^{-m} |u(t+m\tau)| &= \lim_{m\to\infty} \lambda^{t/\tau} \lambda^{-(t+m\tau)/\tau} |u(t+m\tau)| \\
&= \lambda^{t/\tau} \lim_{s\to\infty} \lambda^{-s/\tau} |u(s)| = 0.
\end{aligned}
$$

Therefore, letting $m \to \infty$ in (3.14), we see that

$$
\left| u(t) - \frac{1}{1+\lambda} (\Delta u)(t) \right| \le A(t)
$$

for each fixed $t \ge T$. In a similar fashion as in the proof of Lemma 3.1 for the case $0 < \lambda < 1$, we conclude that $A(t) = o(t^l)$ $(t \to \infty)$. This completes the proof.

For the case $\lambda = 1$, we have the following results.

Lemma 3.2. *Let $\lambda = 1$. Suppose that $u \in C[T-\tau,\infty)$, $u(t) > 0$ for $t \geq T-\tau$. If $(\Delta u)(t)$ is nondecreasing and concave on $[T,\infty)$, then there exists a constant α such that*

$$0 < \frac{1}{2}(\Delta u)(t) - \alpha \leq u(t) \leq \frac{1}{2}(\Delta u)(t) + \frac{1}{2}(\Delta u)(T+2\tau), \quad t \geq T+2\tau.$$

Lemma 3.3. *Let $\lambda = 1$. Suppose that $u \in C[T-\tau,\infty)$, $u(t) > 0$ for $t \geq T-\tau$. If $(\Delta u)(t)$ is nondecreasing and convex on $[T,\infty)$, then there exists a constant α such that*

$$0 < \frac{1}{2}(\Delta u)(t) - \alpha \leq u(t) \leq \frac{1}{2}(\Delta u)(t+\tau) + \frac{1}{2}(\Delta u)(T+2\tau), \quad t \geq T+2\tau.$$

Proof of Lemma 3.2. Since $(\Delta u)(t)$ is concave, we find that

$$\frac{1}{2}(\Delta u)(t+\tau) + \frac{1}{2}(\Delta u)(t-\tau) \leq (\Delta u)\left(\frac{t+\tau}{2} + \frac{t-\tau}{2}\right) = (\Delta u)(t)$$

for $t \geq T+\tau$, so that

$$\begin{aligned}(\Delta u)(t) - (\Delta u)(t-\tau) &\geq \left[\frac{1}{2}(\Delta u)(t+\tau) + \frac{1}{2}(\Delta u)(t-\tau)\right] \\ &\quad -(\Delta u)(t-\tau) \\ &= \frac{1}{2}[(\Delta u)(t+\tau) - (\Delta u)(t-\tau)]\end{aligned} \tag{3.15}$$

for $t \geq T+\tau$, and

$$\begin{aligned}(\Delta u)(t) - (\Delta u)(t-\tau) &\leq (\Delta u)(t) \\ &\quad - \left[\frac{1}{2}(\Delta u)(t) + \frac{1}{2}(\Delta u)(t-2\tau)\right] \\ &= \frac{1}{2}[(\Delta u)(t) - (\Delta u)(t-2\tau)]\end{aligned} \tag{3.16}$$

for $t \geq T+2\tau$. Observe that

$$\begin{aligned}u(t) - u(t-2\tau) &= u(t) + u(t-\tau) - [u(t-\tau) + u(t-2\tau)] \\ &= (\Delta u)(t) - (\Delta u)(t-\tau), \quad t \geq T+\tau.\end{aligned} \tag{3.17}$$

Combining (3.15) and (3.16) with (3.17), we have

$$u(t) - u(t-2\tau) \geq \frac{1}{2}[(\Delta u)(t+\tau) - (\Delta u)(t-\tau)], \quad t \geq T+\tau,$$

and

$$u(t) - u(t-2\tau) \le \frac{1}{2}[(\Delta u)(t) - (\Delta u)(t-2\tau)], \quad t \ge T + 2\tau.$$

If $t \in [T + (2m-1)\tau, T + (2m+1)\tau]$, $m = 1, 2, \ldots$, then

$$\begin{aligned}
u(t) &\ge \frac{1}{2}[(\Delta u)(t+\tau) - (\Delta u)(t-\tau)] + u(t-2\tau) \\
&\ge \frac{1}{2}[(\Delta u)(t+\tau) - (\Delta u)(t-\tau)] + \frac{1}{2}[(\Delta u)(t-\tau) - (\Delta u)(t-3\tau)] \\
&\quad + u(t-4\tau) \\
&= \frac{1}{2}[(\Delta u)(t+\tau) - (\Delta u)(t-3\tau)] + u(t-4\tau) \\
&\ \vdots \\
&\ge \frac{1}{2}[(\Delta u)(t+\tau) - (\Delta u)(t-(2m-1)\tau)] + u(t-2m\tau),
\end{aligned}$$

and

$$u(t) \ge \frac{1}{2}[(\Delta u)(t) - (\Delta u)(T+2\tau)] + \min_{T-\tau \le s \le T+\tau} u(s),$$

since $(\Delta u)(t)$ is nondecreasing. In the same way, we see that

$$\begin{aligned}
u(t) &\le \frac{1}{2}[(\Delta u)(t) - (\Delta u)(t-2m\tau)] + u(t-2m\tau) \\
&\le \frac{1}{2}[(\Delta u)(t) - (\Delta u)(t-2m\tau)] + (\Delta u)(t-2m\tau) \\
&= \frac{1}{2}[(\Delta u)(t) + (\Delta u)(t-2m\tau)] \\
&\le \frac{1}{2}[(\Delta u)(t) + (\Delta u)(T+2\tau)]
\end{aligned}$$

for $t \in [T + 2m\tau, T + 2(m+1)\tau]$, $m = 1, 2, \ldots$. Put

$$\alpha = \frac{1}{2}(\Delta u)(T+2\tau) - \min_{T-\tau \le s \le T+\tau} u(s).$$

We have

$$u(t) \ge \frac{1}{2}(\Delta u)(t) - \alpha, \quad t \ge T + \tau.$$

If $t \ge T + 2\tau$, then

$$\frac{1}{2}(\Delta u)(t) - \alpha \ge \frac{1}{2}(\Delta u)(T+2\tau) - \alpha = \min_{T-\tau \le s \le T+\tau} u(s) > 0.$$

This completes the proof.

Proof of Lemma 3.3. We see that

$$\frac{1}{2}(\Delta u)(t+\tau)+\frac{1}{2}(\Delta u)(t-\tau)\geq(\Delta u)\left(\frac{t+\tau}{2}+\frac{t-\tau}{2}\right)=(\Delta u)(t)$$

for $t\geq T+\tau$. By using (3.17) and the same arguments as in the proof of Lemma 3.2, we conclude that

$$u(t)-u(t-2\tau)\geq\frac{1}{2}[(\Delta u)(t)-(\Delta u)(t-2\tau)],\quad t\geq T+2\tau$$

and

$$u(t)-u(t-2\tau)\leq\frac{1}{2}[(\Delta u)(t+\tau)-(\Delta u)(t-\tau)],\quad t\geq T+\tau,$$

and we have

$$u(t)\geq\frac{1}{2}[(\Delta u)(t)-(\Delta u)(t-2m\tau)]+u(t-2m\tau)$$

for $t\in[T+2m\tau,T+2(m+1)\tau]$, $m=1,2,\ldots$, and

$$\begin{aligned}u(t)&\leq\frac{1}{2}[(\Delta u)(t+\tau)-(\Delta u)(t-(2m-1)\tau)]+u(t-2m\tau)\\&\leq\frac{1}{2}[(\Delta u)(t+\tau)-(\Delta u)(t-(2m-1)\tau)]+(\Delta u)(t-(2m-1)\tau)\\&=\frac{1}{2}[(\Delta u)(t+\tau)+(\Delta u)(t-(2m-1)\tau)]\end{aligned}$$

for $t\in[T+(2m-1)\tau,T+(2m+1)\tau]$, $m=1,2,\ldots$. In view of the nondecreasing nature of $(\Delta u)(t)$, we obtain

$$\begin{aligned}u(t)&\geq\frac{1}{2}(\Delta u)(t)-\frac{1}{2}(\Delta u)(T+2\tau)+\min_{T\leq s\leq T+2\tau}u(s)\\&\equiv\frac{1}{2}(\Delta u)(t)-\alpha,\quad t\geq T+2\tau\end{aligned}$$

and

$$u(t)\leq\frac{1}{2}[(\Delta u)(t+\tau)+(\Delta u)(T+2\tau)],\quad t\geq T+\tau.$$

It is easy to see that

$$u(t)\geq\frac{1}{2}(\Delta u)(T+2\tau)-\alpha=\min_{T\leq s\leq T+2\tau}u(s)>0,\quad t\geq T+2\tau.$$

The proof is complete.

From Lemmas 3.2 and 3.3, we obtain the next result.

Lemma 3.4. *Let $\lambda = 1$ and $l \in \mathbf{N}$. Suppose that $u \in C[T-\tau,\infty)$, $u(t) > 0$ for $t \geq T-\tau$. Assume moreover that $\Delta u \in C^2[T,\infty)$, $(\Delta u)(t) \geq 0$, $(\Delta u)'(t) \geq 0$ and either $(\Delta u)''(t) \leq 0$ or $(\Delta u)''(t) \geq 0$ for $t \geq T$, and $\lim_{t\to\infty}(\Delta u)'(t)/t^l = 0$. Then*

$$u(t) = \frac{1}{2}(\Delta u)(t) + o(t^l) \quad (t \to \infty).$$

Proof. For the case $(\Delta u)''(t) \leq 0$, the conclusion follows immediately from Lemma 3.2. Assume that $(\Delta u)''(t) \geq 0$. From Lemma 3.3 it follows that

$$\frac{1}{2}(\Delta u)(t) - \alpha \leq u(t) \leq \frac{1}{2}(\Delta u)(t+\tau) + \frac{1}{2}(\Delta u)(T+2\tau), \quad t \geq T+2\tau$$

for some constant α. By the mean value theorem, for each large $t \geq T$, there is a number $\eta(t)$ such that $t < \eta(t) < t+\tau$ and

$$(\Delta u)(t+\tau) = (\Delta u)(t) + \tau(\Delta u)'(\eta(t)).$$

Since $\lim_{t\to\infty}(\Delta u)'(t)/t^l = 0$, we have

$$\lim_{t\to\infty} \frac{(\Delta u)'(\eta(t))}{t^l} = \lim_{t\to\infty} \frac{(\Delta u)'(\eta(t))}{[\eta(t)]^l} \left[\frac{\eta(t)}{t}\right]^l = 0.$$

This completes the proof.

4. Proof of the "if" part of Theorem 1.1

In this section we prove the "if" part of Theorem 1.1.

We make use of the following well-known lemma of Kiguradze [13].

Lemma 4.1. *Let $w \in C^n[T,\infty)$ satisfy $w(t) \neq 0$ and $w(t)w^{(n)}(t) \leq 0$ for $t \geq T$. Then there exists an integer $k \in \{1,3,\ldots,n-1\}$ such that*

$$\begin{cases} w(t)w^{(i)}(t) > 0, & 0 \leq i \leq k-1, \\ (-1)^{i-k}w(t)w^{(i)}(t) \geq 0, & k \leq i \leq n, \end{cases} \tag{4.1}$$

for all large $t \geq T$.

A function $w(t)$ satisfying (4.1) for all large t is called a function of Kiguradze degree k. It is known ([11], [12], [13], [19]) that if $w(t)$ is a function of Kiguradze degree $k \in \{1, 3, \ldots, n-1\}$ and $w(t) > 0$ for all large t, then

$$\lim_{t\to\infty} w^{(i)}(t) = 0, \quad i = k+1, k+2, \ldots, n-1, \tag{4.2}$$

and that one of the following three cases holds:

$$\lim_{t\to\infty} w^{(k)}(t) = \text{const} > 0 \quad \text{and} \quad \lim_{t\to\infty} w^{(k-1)}(t) = \infty; \tag{4.3}$$

$$\lim_{t\to\infty} w^{(k)}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} w^{(k-1)}(t) = \infty; \tag{4.4}$$

$$\lim_{t\to\infty} w^{(k)}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} w^{(k-1)}(t) = \text{const} > 0. \tag{4.5}$$

Lemma 4.2. *Let $\lambda \neq 1$. Suppose that $u \in C[T-\tau, \infty)$, $\Delta u \in C^n[T, \infty)$ and $(\Delta u)(t) > 0$ for $t \geq T$. For the case $\lambda > 1$, assume moreover that $\lim_{t\to\infty} \lambda^{-t/\tau} u(t) = 0$. If $(\Delta u)(t)$ is a function of Kiguradze degree k for some $k \in \{1, 3, \ldots, n-1\}$, then there exist a constant α and an integer $l \in \{0, 1, 2, \ldots, n-1\}$ such that*

$$u(t) \geq \frac{1}{1+\lambda}(\Delta u)(t) - \alpha t^l > 0$$

for all large $t \geq T$.

Proof. We see that one of the following three cases holds:

$$\lim_{t\to\infty} (\Delta u)^{(k)}(t) = \text{const} > 0 \quad \text{and} \quad \lim_{t\to\infty} (\Delta u)^{(k-1)}(t) = \infty; \tag{4.6}$$

$$\lim_{t\to\infty} (\Delta u)^{(k)}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} (\Delta u)^{(k-1)}(t) = \infty; \tag{4.7}$$

$$\lim_{t\to\infty} (\Delta u)^{(k)}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} (\Delta u)^{(k-1)}(t) = \text{const} > 0. \tag{4.8}$$

Put

$$\begin{array}{lll} l = k, & \alpha = \dfrac{1}{2(1+\lambda)} \lim\limits_{t\to\infty} \dfrac{(\Delta u)(t)}{t^l} & \text{if (4.6) holds,} \\ l = k-1, & \alpha = 1 & \text{if (4.7) holds,} \\ l = k-1, & \alpha = \dfrac{1}{2(1+\lambda)} \lim\limits_{t\to\infty} \dfrac{(\Delta u)(t)}{t^l} & \text{if (4.8) holds.} \end{array}$$

We note here that $l \in \{0, 1, 2, \dots, n-1\}$, and that if (4.6) or (4.8) holds, then $\lim_{t\to\infty}(\Delta u)(t)/t^l = \lim_{t\to\infty}(\Delta u)^{(l)}(t)/l! = \text{const} > 0$. It is easy to verify that $\lim_{t\to\infty}(\Delta u)'(t)/t^l = 0$. Lemma 3.1 implies that

$$u(t) \geq \frac{1}{1+\lambda}(\Delta u)(t) - \alpha t^l$$

for all large t. By the choice of α, we conclude that $(1+\lambda)^{-1}(\Delta u)(t) - \alpha t^l > 0$ for all large t. The proof is complete.

Now let us show the "if" part of Theorem 1.1.

Proof of the "if" part of Theorem 1.1. It is sufficient to prove that if equation (1.1) has a nonoscillatory solution, then equation (1.10) has a nonoscillatory solution. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for all large t. Then $(\Delta x)(t) > 0$ and $(\Delta x)^{(n)}(t) \leq 0$ for all large t. In view of Lemma 4.1, we find that $(\Delta x)(t)$ is a function of Kiguradze degree k for some $k \in \{1, 3, \dots, n-1\}$, and hence $\lim_{t\to\infty}(\Delta x)^{(k)}(t) = \text{const}$. Since $0 < x(t) \leq (\Delta x)(t)$ for all large t, we have $\lim_{t\to\infty} \lambda^{-t/\tau} x(t) = 0$ if $\lambda > 1$. By Lemmas 3.2, 3.3 and 4.2, there are a constant α and an integer $l \in \{0, 1, 2, \dots, n-1\}$ such that

$$x(t) \geq \frac{1}{1+\lambda}(\Delta x)(t) - \alpha t^l > 0 \quad \text{for all large } t.$$

Put $w(t) = (1+\lambda)^{-1}(\Delta x)(t) - \alpha t^l$. Then $x(t) \geq w(t) > 0$ for all large t. From the monotonicity of f it follows that

$$-w^{(n)}(t) = -\frac{1}{1+\lambda}(\Delta x)^{(n)}(t) = \frac{1}{1+\lambda} f(t, x(g(t))) \geq \frac{1}{1+\lambda} f(t, w(g(t)))$$

for all large t. Lemma 2.2 implies that (1.10) has a nonoscillatory solution. The proof is complete.

5. Proof of the "only if" part of Theorem 1.1

In this section we give the proof of the "only if" part of Theorem 1.1. To this end, we require the following result concerning an "inverse" of the operator Δ.

Lemma 5.1. *Let T_* and T be numbers such that* $\max\{t_0, 1\} \leq T_* \leq T - \tau$, *and let $k \in \mathbf{N}$ and $M > 0$. Define the set Y as follows*:

$$Y = \{y \in C[T_*, \infty) : y(t) = 0,\ t \in [T_*, T],\ \textit{and } |y(t)| \leq Mt^k,\ t \geq T\}.$$

Then there exists a mapping Φ *on* Y *which has the following properties* (i)–(v):

(i) Φ *maps* Y *into* $C[T_*, \infty)$;

(ii) Φ *is continuous on* Y *in the* $C[T_*, \infty)$*-topology;*

(iii) Φ *satisfies* $(\Phi y)(t) + \lambda(\Phi y)(t - \tau) = y(t)$ *for* $t \geq T$ *and* $y \in Y$;

(iv) *if* $\lambda = 1$ *and* $y \in Y$ *is nondecreasing on* $[T_*, \infty)$, *then* $(\Phi y)(t) \geq 0$ *for* $t \geq T_*$;

(v) *if* $\lambda > 1$, *then* $\lim_{t\to\infty} \lambda^{-t/\tau}(\Phi y)(t) = 0$ *for* $y \in Y$.

Here and hereafter, $C[T_*, \infty)$ is regarded as the Fréchet space of all continuous functions on $[T_*, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*, \infty)$.

We divide the proof of Lemma 5.1 into the two cases $0 < \lambda \leq 1$ and $\lambda > 1$.

Proof of Lemma 5.1 for the case $0 < \lambda \leq 1$. For each $y \in Y$, we define the function Φy on $[T_*, \infty)$ by

$$(\Phi y)(t) = \begin{cases} \displaystyle\sum_{i=0}^{m} (-\lambda)^i y(t - i\tau), & t \in [T + m\tau, T + (m+1)\tau), \\ & m = 0, 1, \ldots, \\ 0, & t \in [T_*, T). \end{cases}$$

(i) Let $y \in Y$. Note that $y(T) = 0$. It is obvious that $(\Phi y)(t)$ is continuous on $[T_*, \infty) - \{T + m\tau : m = 0, 1, 2, \ldots\}$. We observe that

$$\lim_{t\to T-0} (\Phi y)(t) = 0 = y(T) = \lim_{t\to T+0} (\Phi y)(t),$$

and that if $m \geq 1$, then

$$\begin{aligned} \lim_{t\to T+m\tau-0} (\Phi y)(t) &= \sum_{i=0}^{m-1} (-\lambda)^i y(T + m\tau - i\tau) \\ &= \sum_{i=0}^{m-1} (-\lambda)^i y(T + m\tau - i\tau) + (-\lambda)^m y(T) \\ &= \sum_{i=0}^{m} (-\lambda)^i y(T + m\tau - i\tau) \\ &= \lim_{t\to T+m\tau+0} (\Phi y)(t). \end{aligned}$$

Consequently, $(\Phi y)(t)$ is continuous on $[T_*, \infty)$.

(ii) It suffices to show that if $\{y_j\}_{j=1}^{\infty}$ is a sequence in $C[T_*, \infty)$ converging to $y \in C[T_*, \infty)$ uniformly on every compact subinterval of $[T_*, \infty)$, then $\{\Phi y_j\}$ converges to Φy uniformly on every compact subinterval of $[T_*, \infty)$. Clearly, $\{\Phi y_j\}$ converges to Φy uniformly on $[T_*, T]$. We claim that $\Phi y_j \to \Phi y$ uniformly on $I_m \equiv [T + m\tau, T + (m+1)\tau]$, $m = 0, 1, 2, \ldots$. Then we easily conclude that $\{\Phi y_j\}$ converges to Φy uniformly on every compact subinterval of $[T_*, \infty)$. Observe that

$$\begin{aligned}\sup_{t \in I_m} |(\Phi y_j)(t) - (\Phi y)(t)| &\leq \sum_{i=0}^{m} \lambda^i \sup_{t \in I_m} |y_j(t - i\tau) - y(t - i\tau)| \\ &\leq \sum_{i=0}^{m} \lambda^i \sup_{t \in I_{m-i}} |y_j(t) - y(t)|\end{aligned}$$

for $m = 0, 1, 2, \ldots$. Then we see that

$$\sup_{t \in I_m} |(\Phi y_j)(t) - (\Phi y)(t)| \to 0 \quad (j \to \infty), \quad m = 0, 1, 2, \ldots,$$

so that $\{\Phi y_j\}$ converges to Φy uniformly on I_m for $m = 0, 1, 2, \ldots$.

(iii) Let $y \in Y$. If $t \in [T, T + \tau)$, then $(\Phi y)(t - \tau) = 0$ and

$$(\Phi y)(t) = y(t) = y(t) - \lambda(\Phi y)(t - \tau).$$

It $t \in [T + m\tau, T + (m+1)\tau)$, $m = 1, 2, \ldots$, then

$$\begin{aligned}(\Phi y)(t) &= y(t) + \sum_{i=1}^{m} (-\lambda)^i y(t - i\tau) \\ &= y(t) - \lambda \sum_{i=1}^{m} (-\lambda)^{i-1} y(t - \tau - (i-1)\tau) \\ &= y(t) - \lambda \sum_{i=0}^{m-1} (-\lambda)^i y(t - \tau - i\tau) \\ &= y(t) - \lambda(\Phi y)(t - \tau),\end{aligned}$$

since $t - \tau \in [T + (m-1)\tau, T + m\tau)$.

(iv) Assume that $\lambda = 1$. Let $y \in Y$ be nondecreasing on $[T_*, \infty)$. Notice that $y(t) \geq y(T_*) = 0$ for $t \geq T_*$. It is easy to see that $(\Phi y)(t) = y(t) \geq 0$ for

$t \in [T, T+\tau)$ and $(\Phi y)(t) = 0$ for $t \in [T_*, T)$. Let $t \in [T+m\tau, T+(m+1)\tau)$, $m = 1, 2, \dots$. If $m \geq 1$ is odd, then

$$(\Phi y)(t) = \sum_{j=0}^{(m-1)/2} [y(t-2j\tau) - y(t-(2j+1)\tau)] \geq 0.$$

If $m \geq 2$ is even, then

$$(\Phi y)(t) = \sum_{j=0}^{(m/2)-1} [y(t-2j\tau) - y(t-(2j+1)\tau)] + y(t-m\tau) \geq 0.$$

Therefore we obtain $(\Phi y)(t) \geq 0$ for $t \geq T_*$. The proof for the case $0 < \lambda \leq 1$ is complete.

Proof of Lemma 5.1 for the case $\boldsymbol{\lambda > 1}$. For each $y \in Y$, we assign the function Φy on $[T_*, \infty)$ as follows:

$$(\Phi y)(t) = \begin{cases} -\sum_{i=1}^{\infty} (-\lambda)^{-i} y(t+i\tau), & t \in [T-\tau, \infty), \\ (\Phi y)(T-\tau), & t \in [T_*, T-\tau). \end{cases}$$

Let $y \in Y$. Then

$$|(-\lambda)^{-i} y(t+i\tau)| \leq \lambda^{-i} M (t+i\tau)^k \leq 2^{k-1} M \lambda^{-i} (t^k + i^k \tau^k) \tag{5.1}$$

for $t \geq T-\tau$, $i = 1, 2, \dots$. Thus we see that the series $\sum_{i=1}^{\infty} (-\lambda)^{-i} y(t+i\tau)$ converges uniformly on every compact subinterval of $[T-\tau, \infty)$, so that Φ is well-defined, and $(\Phi y)(t)$ is continuous on $[T_*, \infty)$ and satisfies

$$|(\Phi y)(t)| \leq \frac{2^{k-1} M}{\lambda - 1} t^k + L, \quad t \geq T-\tau$$

for each $y \in Y$, where $L = 2^{k-1} M \tau^k \sum_{i=1}^{\infty} \lambda^{-i} i^k$. This means that (i) and (v) follow. Now we show (ii) and (iii).

(ii) Take an arbitrary compact subinterval I of $[T-\tau, \infty)$. Let $\varepsilon > 0$. There is an integer $q \geq 1$ such that

$$\sum_{i=q+1}^{\infty} \lambda^{-i} M (t+i\tau)^k < \frac{\varepsilon}{3}, \quad t \in I. \tag{5.2}$$

Let $\{y_j\}_{j=1}^{\infty}$ be a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. There exists an integer $j_0 \geq 1$ such that

$$\sum_{i=1}^{q} \lambda^{-i} |y_j(t + i\tau) - y(t + i\tau)| < \frac{\varepsilon}{3}, \quad t \in I, \quad j \geq j_0.$$

It follows from (5.1) and (5.2) that

$$\begin{aligned}
|(\Phi y_j)(t) - (\Phi y)(t)| &\leq \sum_{i=1}^{q} \lambda^{-i} |y_j(t + i\tau) - y(t + i\tau)| \\
&\quad + \left| \sum_{i=q+1}^{\infty} (-\lambda)^{-i} y_j(t + i\tau) \right| \\
&\quad + \left| \sum_{i=q+1}^{\infty} (-\lambda)^{-i} y(t + i\tau) \right| \\
&\leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon, \quad t \in I, \quad j \geq j_0,
\end{aligned}$$

which implies that Φy_j converges Φy uniformly on I. We see that $\Phi y_j \to \Phi y$ uniformly on $[T_*, T - \tau]$, because of $(\Phi y)(t) = (\Phi y)(T - \tau)$ on $[T_*, T - \tau]$ for $y \in Y$. Consequently, we conclude that Φ is continuous on Y.

(iii) Let $y \in Y$. Observe that

$$\begin{aligned}
\lambda(\Phi y)(t - \tau) &= \sum_{i=1}^{\infty} (-\lambda)^{-(i-1)} y(t + (i-1)\tau) \\
&= y(t) + \sum_{i=1}^{\infty} (-\lambda)^{-i} y(t + i\tau) \\
&= y(t) - (\Phi y)(t), \quad t \geq T.
\end{aligned}$$

The proof for the case $\lambda > 1$ is complete.

Lemma 5.2. *Let $w \in C^n[T, \infty)$ be a function of Kiguradze degree k for some $k \in \{1, 3, \ldots, n-1\}$. Then $\lim_{t\to\infty} w(t + \rho)/w(t) = 1$ for each $\rho > 0$.*

Proof. We may assume that $w(t) > 0$ for all large t. Recall that $w(t)$ satisfies one of (4.3)–(4.5). If (4.3) holds, then

$$\lim_{t\to\infty} \frac{w(t + \rho)}{w(t)} = \frac{\lim_{t\to\infty} w^{(k)}(t + \rho)}{\lim_{t\to\infty} w^{(k)}(t)} = 1.$$

In exactly the same way, we have $\lim_{t\to\infty} w(t+\rho)/w(t) = 1$ for the case (4.5). Assume that (4.4) holds. By the mean value theorem, for each large fixed $t \geq T$, there is a number $\eta(t)$ such that

$$w(t+\rho) - w(t) = \rho w'(\eta(t)) \quad \text{and} \quad t < \eta(t) < t+\rho.$$

Thus we obtain

$$\frac{w(t+\rho)}{w(t)} - 1 = \rho \frac{w'(\eta(t))}{[\eta(t)]^{k-1}} \frac{t^{k-1}}{w(t)} \left[\frac{\eta(t)}{t}\right]^{k-1}.$$

By (4.4) we conclude that $\lim_{t\to\infty} w'(t)/t^{k-1} = 0$ and $\lim_{t\to\infty} w(t)/t^{k-1} = \infty$, so that $\lim_{t\to\infty} w(t+\rho)/w(t) = 1$.

Now we prove the "only if" part of Theorem 1.1.

Proof of the "only if" part of Theorem 1.1. We show that if equation (1.10) has a nonoscillatory solution, then equation (1.1) has a nonoscillatory solution. Let $z(t)$ be a nonoscillatory solution of (1.10). Without loss of generality, we may assume that $z(t)$ is eventually positive. Set $w(t) = (1+\lambda)z(t)$. Then $w(t)$ is an eventually positive solution of

$$w^{(n)}(t) + f(t, (1+\lambda)^{-1} w(g(t))) = 0. \tag{5.3}$$

Lemma 4.1 implies that $w(t)$ is a function of Kiguradze degree k for some $k \in \{1, 3, \ldots, n-1\}$, and one of the cases (4.3)–(4.5) holds. Hence, $\lim_{t\to\infty} w(t)/t^k = \text{const} \geq 0$. From Lemma 5.2 it follows that

$$w(t+2\tau) \leq \frac{3}{2} w(t), \quad t \geq T_1 \tag{5.4}$$

for some $T_1 \geq t_0$.

We can take a sufficiently large number $T \geq T_1$ such that $w^{(i)}(t) > 0$ $(i = 0, 1, 2, \ldots, k-1)$, $w(g(t)) > 0$ for $t \geq T$, and

$$T_* \equiv \min\{T-\tau, \inf\{g(t) : t \geq T\}\} \geq \max\{T_1,\, 1\}.$$

Recall (4.2). Integrating (5.3), we have

$$w(t) - P(t) = \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, \frac{w(g(r))}{1+\lambda}\right) dr ds$$

for $t \geq T$, where

$$P(t) = \frac{(t-T)^k}{k!} w^{(k)}(\infty) + \sum_{i=0}^{k-1} \frac{(t-T)^i}{i!} w^{(i)}(T), \quad t \geq T_*,$$

and $w^{(k)}(\infty) = \lim_{t\to\infty} w^{(k)}(t) \geq 0$.

Consider the set Y of functions $y \in C[T_*, \infty)$ which satisfies

$$y(t) = 0 \quad \text{for } t \in [T_*, T] \quad \text{and} \quad 0 \leq y(t) \leq w(t) - P(t) \quad \text{for } t \geq T.$$

Then Y is closed and convex. Note that there is a constant $M > 0$ such that $|y(t)| \leq Mt^k$ on $[T, \infty)$ for $y \in Y$, by $\lim_{t\to\infty} w(t)/t^k = \text{const} \geq 0$. Lemma 5.1 implies that there exists a mapping Φ on Y satisfying (i)–(v) of Lemma 5.1. Put

$$(\Psi y)(t) = (\Phi y)(t) + \frac{P(t)}{4(1+\lambda)}, \quad t \geq T_*, \quad y \in Y.$$

For each $y \in Y$, we define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \overline{f}(r, (\Psi y)(g(r)))drds, & t \geq T, \\ 0, & t \in [T_*, T], \end{cases}$$

where

$$\overline{f}(t,u) = \begin{cases} f(t, (1+\lambda)^{-1} w(g(t))), & u \geq (1+\lambda)^{-1} w(g(t)), \\ f(t,u), & 0 \leq u \leq (1+\lambda)^{-1} w(g(t)), \\ 0, & u \leq 0, \end{cases}$$

for $t \geq T$ and $u \in \mathbf{R}$. In view of the fact that

$$0 \leq \overline{f}(t,u) \leq f(t, (1+\lambda)^{-1} w(g(t))), \quad (t,u) \in [T, \infty) \times \mathbf{R},$$

we see that $\mathcal{F}$ is well defined on Y and maps Y into itself. Since Φ is continuous on Y, by the Lebesgue dominated convergence theorem, we can show that $\mathcal{F}$ is continuous on Y as a routine computation.

Now we claim that $\mathcal{F}(Y)$ is relatively compact. We note that $\mathcal{F}(Y)$ is uniformly bounded on every compact subinterval of $[T_*, \infty)$, because of $\mathcal{F}(Y) \subset Y$. By the Ascoli-Arzelà theorem, it suffices to verify that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*, \infty)$. Let I be an arbitrary compact subinterval of $[T, \infty)$. If $k = 1$, then

$$0 \le (\mathcal{F}y)'(t) \le \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!} f\left(s, \frac{w(g(s))}{1+\lambda}\right) ds, \quad t \ge T, \quad y \in Y.$$

If $k \ge 3$, then

$$0 \le (\mathcal{F}y)'(t) \le \int_T^t \frac{(t-s)^{k-2}}{(k-2)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, \frac{w(g(r))}{1+\lambda}\right) dr ds$$

for $t \ge T$ and $y \in Y$. Thus we see that $\{(\mathcal{F}y)'(t) : y \in Y\}$ is uniformly bounded on I. The mean value theorem implies that $\mathcal{F}(Y)$ is equicontinuous on I. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0$ for t_1, $t_2 \in [T_*, T]$, we conclude that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*, \infty)$.

By applying the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$, there exists a $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$.

Put $x(t) = (\Psi\widetilde{y})(t)$. Then we obtain

$$(\Delta x)(t) = \widetilde{y}(t) + \frac{P(t) + \lambda P(t-\tau)}{4(1+\lambda)}, \quad t \ge T, \tag{5.5}$$

by Lemma 5.1 (iii), and hence $(\Delta x)(t)$ is a function of Kiguradze degree k. Since $P(t)$ is nondecreasing in $t \in [T, \infty)$, we find that $P(t) \ge P(t-\tau) \ge P(T) = w(T) > 0$ for $t \ge T + \tau$, so that

$$0 < (\Delta x)(t) \le w(t) - P(t) + \frac{P(t) + \lambda P(t)}{4(1+\lambda)} = w(t) - \frac{3}{4}P(t) \tag{5.6}$$

for $t \ge T + \tau$. We will show that

$$0 < x(t) \le (1+\lambda)^{-1} w(t) \quad \text{for all large } t. \tag{5.7}$$

Then the proof of the "only if" part of Theorem 1.1 will be complete, since (5.5) and (5.7) imply that

$$\begin{aligned} \frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] = \widetilde{y}^{(n)}(t) = (\mathcal{F}\widetilde{y})^{(n)}(t) &= -\overline{f}(t, x(g(t))) \\ &= -f(t, x(g(t))) \end{aligned}$$

for all large t, which means $x(t)$ is a nonoscillatory solution of (1.1).

If $w^{(k)}(\infty) > 0$, then we put $l = k$, and if $w^{(k)}(\infty) = 0$, then we put $l = k - 1$. It can be shown that $\lim_{t\to\infty}(\Delta x)'(t)/t^l = 0$. Indeed, since

$$\begin{aligned}\lim_{t\to\infty}(\Delta x)^{(k)}(t) &= \lim_{t\to\infty}\widetilde{y}^{(k)}(t) + \lim_{t\to\infty}\frac{P^{(k)}(t) + \lambda P^{(k)}(t-\tau)}{4(1+\lambda)} \\ &= \lim_{t\to\infty}(\mathcal{F}\widetilde{y})^{(k)}(t) + \frac{w^{(k)}(\infty)}{4} = \frac{w^{(k)}(\infty)}{4},\end{aligned}$$

we see that if $l = k$, then $\lim_{t\to\infty}(\Delta x)'(t)/t^l = \lim_{t\to\infty}(\Delta x)^{(k)}(t)/(k!t) = 0$, and that if $l = k-1$, then $\lim_{t\to\infty}(\Delta x)'(t)/t^l = \lim_{t\to\infty}(\Delta x)^{(k)}(t)/(k-1)! = 0$.

First assume that $\lambda \neq 1$. From Lemma 4.2 it follows that $x(t) > 0$ for all large $t \geq T_*$. In view of Lemma 3.1 and the fact that $\lim_{t\to\infty} P(t)/t^l =$ const > 0, we have

$$x(t) \leq \frac{1}{1+\lambda}(\Delta x)(t) + \frac{3}{4(1+\lambda)}P(t)$$

for all large t. Hence, by (5.6), we obtain $x(t) \leq (1+\lambda)^{-1}w(t)$ for all large t.

Next we assume that $\lambda = 1$ and $l \neq 0$. Since $\widetilde{y}(t)(= (\mathcal{F}\widetilde{y})(t))$ is nondecreasing in $t \in [T_*, \infty)$, from Lemma 5.1 (iv), we see that $(\Phi\widetilde{y})(t) \geq 0$ for $t \geq T_*$, so that $x(t) \geq P(t)/[4(1+\lambda)]$ for $t \geq T_*$. Hence, $x(t) > 0$ for $t \geq T$. By using Lemma 3.4 and the same argument as in the case $\lambda \neq 1$, we can show that $x(t) \leq (1+\lambda)^{-1}w(t)$ for all large t.

Finally we suppose that $\lambda = 1$ and $l = 0$. Then $k = 1$ and $w^{(k)}(\infty) = 0$. Therefore, $P(t) = w(T)$ on $[T_*, \infty)$. As in the case $\lambda = 1$ and $l \neq 0$, we have $x(t) \geq P(t)/[4(1+\lambda)]$ for $t \geq T_*$, which implies that $x(t) > 0$ for $t \geq T_*$. Note that $(\Delta x)'(t) \geq 0$ and $(\Delta x)''(t) \leq 0$ for $t > T$, since $k = 1$. By Lemma 3.2, (5.6) and (5.4), we conclude that

$$\begin{aligned}x(t) &\leq \frac{1}{2}(\Delta x)(t) + \frac{1}{2}(\Delta x)(T+2\tau) \\ &\leq \frac{1}{2}\left[w(t) - \frac{3}{4}w(T) + w(T+2\tau) - \frac{3}{4}w(T)\right] \\ &\leq \frac{1}{2}w(t), \quad t \geq T + 2\tau.\end{aligned}$$

The proof is complete.

Acknowledgment

The author thanks the referee for many helpful suggestions.

References

[1] Y. Chen, Existence of nonoscillatory solutions of nth order neutral delay differential equations, *Funkcial. Ekvac.* **35** (1992), 557–570.

[2] Q. Chuanxi and G. Ladas, Oscillations of higher order neutral differential equations with variable coefficients, *Math. Nachr.* **150** (1991), 15–24.

[3] L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, Inc., New York, Basel and Hong Kong, 1995.

[4] J. R. Graef, M. K. Grammatikopoulos and P. W. Spikes, Asymptotic properties of solutions of nonlinear neutral delay differential equations of the second order, *Rad. Mat.* **4** (1988), 133–149.

[5] J. R. Graef and P. W. Spikes, On the oscillation of an nth-order nonlinear neutral delay differential equation, *J. Comput. Appl. Math.* **41** (1992), 35–40.

[6] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, Oscillations of second order neutral delay differential equations, *Rad. Mat.* **1** (1985), 267–274.

[7] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, Oscillation and asymptotic behavior of second order neutral differential equations, *Ann. Mat. Pura Appl.* **148** (1987), 29–40.

[8] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients, *Chinese Ann. Math. Ser. B* **9** (1988), 322–338.

[9] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Clarendon Press, Oxford, 1991.

[10] J. K. Hale, *Theory of Functional Differential Equations*, Springer Verlag, New York, 1977.

[11] J. Jaroš and T. Kusano, Oscillation theory of higher order linear functional differential equations of neutral type, *Hiroshima Math. J.* **18** (1988), 509–531.

[12] J. Jaroš and T. Kusano, Asymptotic behavior of nonoscillatory solutions of nonlinear functional differential equations of neutral type, *Funkcial. Ekvac.* **32** (1989), 251–263.

[13] I. T. Kiguradze, On the oscillatory character of solutions of the equation $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$, *Mat. Sb.* **65** (1964), 172–187. (Russian)

[14] Y. Kitamura, Oscillation of functional differential equations with general deviating arguments, *Hiroshima Math. J.* **15** (1985), 445–491.

[15] T. Kusano, On strong oscillation of even order differential equations with advanced arguments, *Hiroshima Math. J.* **11** (1981), 617–620.

[16] H. J. Li, Oscillatory theorems for second-order neutral delay differential equations, *Nonlinear Anal.* **26** (1996), 1397–1409.

[17] M. Naito, On strong oscillation of retarded differential equations, *Hiroshima Math. J.* **11** (1981), 553–560.

[18] M. Naito, An asymptotic theorem for a class of nonlinear neutral differential equations, *Czechoslovak Math. J.* **48** (1998), 419–432.

[19] Y. Naito, Nonoscillatory solutions of neutral differential equations, *Hiroshima Math. J.* **20** (1990), 231–258.

[20] H. Onose, A comparison theorem and forced oscillation, *Bull. Austral. Math. Soc.* **13** (1975), 13–19.

[21] S. Tanaka, Existence of positive solutions of higher order nonlinear neutral differential equations, *Rocky Mountain J. Math.* (to appear)

[22] S. Tanaka, Oscillation of solutions of even order neutral differential equations, *Dynam. Systems Appl.* (to appear)

[23] X. H. Tang and J. H. Shen, Oscillation and existence of positive solutions in a class of higher order neutral equations, *J. Math. Anal. Appl.* **213** (1997), 662–680.

[24] A. Zafer, Oscillation criteria for even order neutral differential equations, *Appl. Math. Lett.* **11** (1998), 21–25.

[25] B. G. Zhang and B. Yang, New approach of studying the oscillation of neutral differential equations, *Funkcial. Ekvac.* **41** (1998), 79–89.