Oscillation Theorems for Nonlinear Differential Equations of Second Order

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Abstract

We establish new oscillation theorems for the nonlinear differential equation

$$[a(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)]' + q(t)f(x(t)) = 0, \quad \alpha > 0$$

where $a, q : [t_0, \infty) \to R$, $\psi, f : R \to R$ are continuous, $a(t) > 0$ and $\psi(x) > 0$, $xf(x) > 0$ for $x \neq 0$. These criteria involve the use of averaging functions.

1. Introduction

In this paper we are interested in obtaining results on the oscillatory behaviour of solutions of second order nonlinear differential equation

$$(E) \qquad [a(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)]' + q(t)f(x(t)) = 0$$

1991 *Mathematics Subject Classification.* 34C10, 34C15
Keywords. Oscillation, Nonlinear differential equations, Integral averages
Supported by Grant 04M03E of RFNS through Math.Inst. SANU

where $a, q : [t_0, \infty) \to R,\ \psi, f : R \to R$ are continuous, $\alpha > 0$ is a constant, $a(t) > 0$ and $\psi(x) > 0$, $xf(x) > 0$ for $x \neq 0$.

This nonlinear equation can be considered as a natural generalization of the half-linear equation

$$(HL) \qquad [a(t)|x'(t)|^{\alpha-1}x'(t)]' + q(t)|x(t)|^{\alpha-1}x(t) = 0,$$

which has been the object of intensive studies in recent years.

By a solution of (E) we mean a function $x \in C^1[T_x, \infty),\ T_x \geq t_0$, which has the property $|x'(t)|^{\alpha-1}x'(t) \in C^1[T_x, \infty)$ and satisfies (E). A solutions is said to be *global* if it exists on the whole interval $[t_0, \infty)$. The existence and uniqueness of solutions of (HL) subject to the initial condition $x(T) = x_0,\ x'(T) = x_1$ has been investigated by KUSANO and KITANO [12]. They have shown that the initial value problem has a unique global solution for any given values $x_0,\ x_1$ provided $q(t)$ is positive and locally of bounded variation on $[t_0, \infty)$.

The solution x of (E) which exists on some interval $(T_1, +\infty) \subset [t_0, \infty)$ is *singular solution of the first kind*, $x \in S_1$, if there exists $t^* \in (T_1, \infty)$ such that $\max\{(x(s)| : t \leq s \leq t^*\} > 0$ for $t_0 < t < t^*$ and $x(t) = 0$ for all $t \geq t^*$. The solution x of (E) which exists on some interval $(T_1, T_2) \subset [t_0, \infty)$ is *singular solution of the second kind*, $x \in S_2$, if $\limsup_{t \to T_2} x(t) = +\infty$. On the other hand, the solution x of (E) which exists on some interval $(T_x, +\infty),\ T_x \geq t_0$ is called *proper* if

$$\sup\{\,|x(t)| : t \geq T\} > 0 \quad \text{for all } T \geq T_x.$$

The existence of proper and singular solutions for the semilinear equations was investigated by MIRZOV [24] and for the nonlinear second order equation by KIGURADZE and CHANTURIA [11]. They established sufficient conditions that nonlinear and semilinear differential equation of the second order does not have singular solutions as well as that it has a proper solution and sufficient conditions for all global solutions to be proper. So, we shall suppose that the equation (E) has the proper solutions and our attention will be restricted only to those solutions.

A nontrivial solution of (E) is called *oscillatory* if it has arbitrarily large zeroes, otherwise it is said to be *nonoscillatory.* Equation (E) is called oscillatory if all its solution are oscillatory.

During the last two decades there has been a great deal of work on the oscillatory behavior of solutions of the equation (HL) (see Hsu, Yeh [10], Kusano, Naito [13], Kusano, Yoshida [14], Li, Yeh [16], [17], [18], [19], [20], Lian, Yeh, Li [22]). Wang in [27], [28] established oscillation criteria for the more general equation $[a(t)|x'(t)|^{\alpha-1}x'(t)]' + \Phi(t, x(t)) = 0$. Wong, Agarwal [30] considered a special case of this equation for $\Phi(t, x(t)) = q(t)\, f(x(t))$. We refer to that equation as to the equation (A). Afterward, in 1998. Hong in [9] generalized criteria of oscillation of half-linear differential equation due to Hsu, Yeh [10] to the nonlinear differential equation (E). Thereafter, our purpose here is to develop oscillation theory for a general case of the equation (E) in which $f(x)$ is not necessarily of the form $|\, x\, |^{\alpha-1}x,\ \alpha > 0$ and $\psi(x) \neq 1$, without any restriction on the sign of $q(t)$, which is of particular interest.

Some of the very important oscillation theorems for second order linear and nonlinear differential equations involve the use of averaging functions. As recent contribution to this study we refer to the papers of Grace, Lalli and Yeh [2], [3], Grace and Lalli [5], Grace [4], [6], [7], Li and Yeh [21], Philos [26], Wong and Yeh [29] and Yeh [31]. Using a general class of continuous functions

$$H : \mathcal{D} = \{\, (t, s) \,|\, t \geq s \geq t_0 \,\} \to R,$$

which is such that

$$H(t, t) = 0 \;\text{ for } t \geq t_0, \quad H(t, s) > 0 \;\text{ for all } (t, s) \in \mathcal{D}$$

and has a continuous and nonpositive partial derivative on $\mathcal{D}$ with respect to the second variable, Philos [26] presented oscillation theorems for linear differential equations of second order

$$x''(t) + q(t)x(t) = 0.$$

His results has been extended by Grace [7] and Li and Yeh [21] to the nonlinear differential equation

$$[a(t)\psi(x(t))x'(t)]' + q(t)f(x(t)) = 0.$$

In this paper, we are interested in extending the results of Grace to a broad class of second order nonlinear differential equations of type (E) by using a well-known inequality stated in Lemma 2.1.

2. Main results

Throughout this paper we assume that

$$(C_1) \qquad \frac{f'(x)}{(\psi(x)|f(x)|^{\alpha-1})^{\frac{1}{\alpha}}} \geq K > 0, \quad x \neq 0,$$

and in order to simplify notation we denote by

$$\beta = \frac{1}{\alpha K^{\alpha}} \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Notice that in the special case of the equation (HL), for $\psi(x) \equiv 1$ and $f(x) = |x|^{\alpha-1}x$, the condition (C_1) is satisfied.

We also need the following well-known inequality which is due to Hardy, Little and Polya [8].

Lemma 2.1 *If X and Y are nonnegative, then*

$$X^q + (q-1)Y^q - qXY^{q-1} \geq 0, \quad q > 1,$$

where equality holds if and only if $X = Y$.

Theorem 2.1 *Let condition (C_1) holds. Suppose that there exists a continuous function*

$$H : \mathcal{D} = \{\, (t,s) \,|\, t \geq s \geq t_0 \,\} \to R$$

such that

$$(H_1) \qquad H(t,t) = 0,\ t \geq t_0, \quad H(t,s) > 0,\ (t,s) \in \mathcal{D}$$

(H_2) $h(t,s) = -\dfrac{\partial H(t,s)}{\partial s}$ *is nonnegative continuous function on $\mathcal{D}$.*

If

$$(C_2) \quad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[q(s)H(t,s) - \beta\, a(s) \frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)} \right] ds = \infty,$$

then the equation (E) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of the equation (E). Without loss of generality, we assume that $x(t) \neq 0$ for $t \geq t_0$. We define

$$w(t) = \frac{a(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} \quad \text{for} \quad t \geq t_0.$$

Then, by taking into account (C_1), for every $s \geq t_0$, we obtain

$$\begin{aligned} w'(s) &= -q(s) - \frac{f'(x(s))|w(s)|^{\frac{\alpha+1}{\alpha}}}{(a(s)\psi(x(s))|f(x(s))|^{\alpha-1})^{\frac{1}{\alpha}}} \\ &\leq -q(s) - K\,\frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}. \end{aligned} \tag{1}$$

Multiplying (1) by $H(t,s)$ for $t \geq s \geq t_0$ and integrating from t_0 to t, we get

$$\int_{t_0}^t w'(s)H(t,s)\,ds \leq -\int_{t_0}^t q(s)H(t,s)\,ds - K\int_{t_0}^t H(t,s)\frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}\,ds.$$

Since,

$$\int_{t_0}^t w'(s)H(t,s)\,ds = -w(t_0)H(t,t_0) - \int_{t_0}^t w(s)\frac{\partial H(t,s)}{\partial s}\,ds, \tag{2}$$

we have

$$\begin{aligned} \int_{t_0}^t q(s)H(t,s)\,ds &\leq w(t_0)H(t,t_0) + \int_{t_0}^t |w(s)|h(t,s)\,ds \\ &\quad - K\int_{t_0}^t H(t,s)\frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}\,ds. \end{aligned} \tag{3}$$

Taking

$$\begin{aligned} X &= (K\,H(t,s))^{\frac{\alpha}{\alpha+1}}\,\frac{|w(s)|}{a^{\frac{1}{\alpha+1}}(s)}, \quad q = \frac{\alpha+1}{\alpha} \\ Y &= \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\frac{a^{\frac{\alpha}{\alpha+1}}(s)h^{\alpha}(t,s)}{[K\,H(t,s)]^{\frac{\alpha^2}{\alpha+1}}}, \end{aligned}$$

according to Lemma 2.1, we obtain for $t > s \geq t_0$

$$|w(s)|h(t,s) - K\,H(t,s)\frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)} \leq \beta\,a(s)\frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}.$$

Hence, (3) implies

$$\begin{aligned}(4)\qquad \frac{1}{H(t,t_0)}\int_{t_0}^{t} q(s)H(t,s)\,ds &\leq w(t_0)\\ &+\frac{\beta}{H(t,t_0)}\int_{t_0}^{t} a(s)\frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}\,ds,\end{aligned}$$

for all $t \geq t_0$. Consequently,

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[q(s)H(t,s) - \beta\,a(s)\frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}\right]ds \leq w(t_0), \quad t \geq t_0.$$

Taking the upper limit as $t \to \infty$, we obtain a contradiction, which completes the proof. $\square$

Corollary 2.1 *Let condition* (C_2) *in Theorem* 2.1 *be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} a(s)\frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}\,ds < \infty,$$
$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} q(s)H(t,s)\,ds = \infty$$

then the conclusion of Theorem 2.1 *holds.*

Remark 2.1 For $a(t) \equiv 1,\ \psi(x) \equiv 1,\ H(t,s) = t-s$ from Theorem 2.1. we derive Corollary 3.2. in [28]. Taking $H(t-s)^{\lambda}$ for some constant $\lambda > 1$, which obviously satisfies the conditions (H_1), (H_2), in the case of the equation (HL) as a special case of (E), Theorem 2.1. reduces to the oscillation criterion of Li and Yeh [16].

For illustration we consider the following example.

Example 2.1 Consider the differential equation

$$(E_1)\quad \left(\frac{|x(t)|^{3-\alpha}}{t^\nu}|x'(t)|^{\alpha-1}x'(t)\right)' + t^\lambda\left(\lambda\,\frac{2-\cos t}{t} + \sin t\right)x^3(t) = 0,$$

for $t \geq t_0$, where ν, λ, α are arbitrary positive constants and $\alpha \neq 2$. Then,

$$\frac{f'(x)}{(\psi(x)|f(x)|^{\alpha-1})^{\frac{1}{\alpha}}} = 3 \quad \text{for} \quad x \neq 0.$$

On the other hand, for any $t \geq t_0$, we have

$$\begin{aligned}\int_{t_0}^t q(s)\,ds &= \int_{t_0}^t d[s^\lambda(2-\cos s)] = t^\lambda(2-\cos t) - t_0^\lambda(2+\cos t_0)\\ &= t^\lambda(2-\cos t) - k_0 \geq t^\lambda - k_0.\end{aligned}$$

Taking $H(t,s) = (t-s)^2$, for $t \geq s \geq t_0$, we have

$$\begin{aligned}&\frac{1}{t^2}\int_{t_0}^t\left[(t-s)^2 q(s) - \beta\,2^{\alpha+1}\,\frac{(t-s)^{1-\alpha}}{s^\nu}\right]ds\\ &\quad= \frac{1}{t^2}\int_{t_0}^t\left[2\,(t-s)\left(\int_{t_0}^s q(u)\,du\right) - \beta\,2^{\alpha+1}\,\frac{(t-s)^{1-\alpha}}{s^\nu}\right]ds\\ &\quad\geq \frac{2}{t^2}\int_{t_0}^t (t-s)\left(s^\lambda - k_0\right)\,ds - \frac{\beta\,2^{\alpha+2}}{t_0^\nu\,t^2}\int_{t_0}^t (t-s)^{1-\alpha}\,ds\\ &\quad= \frac{2\,t^\lambda}{(\lambda+1)(\lambda+2)} + \frac{k_1}{t^2} + \frac{k_2}{t} - k_0 - \frac{k_3}{t^\alpha}\left(1-\frac{t_0}{t}\right)^{2-\alpha},\end{aligned}$$

where

$$k_1 = \frac{2\,t_0^{\lambda+2}}{\lambda+2} - k_0 t_0^2,\;\; k_2 = 2k_0t_0 - \frac{2\,t_0^{\lambda+1}}{\lambda+1},\;\; k_3 = \frac{\beta\,2^{\alpha+2}}{t_0^\nu(2-\alpha)}.$$

Consequently, condition (C_2) is satisfied. Hence, the equation (E_1) is oscillatory by Theorem 2.1.

Remark 2.2 We note that since $\int_0^\infty q(s)\,ds$ is not convergent the oscillation criteria in [9] fail to apply to the equation (E_1).

In the case of the half–linear differential equation we have the following corollary:

Corollary 2.2 *The equation (HL) is oscillatory if the condition (C_2) is satisfied for some continuous function $H(t,s)$ on $\mathcal{D}$ which satisfies (H_1) and (H_2).*

Remark 2.3 As in the previous example, we conclude that (HL) for $q(s) = t^\lambda \left(\lambda \frac{2-\cos t}{t} + \sin t\right)$, $a(s) = s^{-\nu}$ is oscillatory for λ and ν positive and $\alpha \neq 0$. On the other hand, criteria in [10], [13] and [18] (Section 2) can not be applied, since $q(t)$ is not positive function (assumed in [13]) and $\int_t^\infty q(s)\,ds < \infty$.

Theorem 2.2 *Let condition (C_1) holds and let the functions H and h be defined as in Theorem 2.1 such that conditions (H_1), (H_2),*

$$(H_3) \qquad 0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] \leq \infty,$$

and

$$(C_3) \qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t a(s)\frac{h^{\alpha+1}(t,s)}{H^\alpha(t,s)}\,ds < \infty$$

are satisfied. If there exists a continuous function φ on $[t_0,\infty)$ such that for every $T \geq t_0$

$$(C_4) \quad \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[q(s)H(t,s) - \beta\, a(s)\frac{h^{\alpha+1}(t,s)}{H^\alpha(t,s)}\right]ds \geq \varphi(T),$$

and

$$(C_5) \qquad \int_{t_0}^\infty \frac{\varphi_+^{\frac{\alpha}{\alpha+1}}(s)}{a^{\frac{1}{\alpha}}(s)}\,ds = \infty,$$

where $\varphi_+(s) = \max\{\varphi(s), 0\}$, then the equation (E) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of the equation (E), say $x(t) \neq 0$ for $t \geq t_0$. Next, we define the function w as in the proof of Theorem 2.1, so that we have (3) and (4). Then, for $t > T \geq t_0$ we have

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[q(s)H(t,s) - \beta\, a(s)\frac{h^{\alpha+1}(t,s)}{H^\alpha(t,s)}\right]ds \leq w(T).$$

Therefore, by conditions (C_4), we have

$$\varphi(T) \le w(T) \quad \text{for every} \quad T \ge t_0 \tag{5}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s)\,ds \ge \varphi(t_0). \tag{6}$$

We define functions

$$\begin{aligned} F(t) &= \frac{1}{H(t,t_0)} \int_{t_0}^{t} |w(s)|h(t,s)\,ds, \\ G(t) &= \frac{K}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \frac{|w(s)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(t)}\,ds. \end{aligned}$$

From (3), we get for $t \ge t_0$

$$G(t) - F(t) \le w(t_0) - \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s)\,ds, \tag{7}$$

so that (6) implies that

$$\begin{aligned} \liminf_{t\to\infty} [G(t) - F(t)] &\le w(t_0) - \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s)\,ds \\ &\le w(t_0) - \varphi(t_0) < \infty. \end{aligned} \tag{8}$$

Now, consider a sequence $\{T_n\}_{n=1}^{\infty}$ in (t_0, ∞) with $\lim_{n\to\infty} T_n = \infty$ and such that

$$\lim_{n\to\infty} [G(T_n) - F(T_n)] = \liminf_{t\to\infty} [G(t) - F(t)].$$

Because of (8), there exists a constant M such that

$$G(T_n) - F(T_n) \le M, \quad n = 1, 2, \ldots \tag{9}$$

We shall next prove that

$$\int_{t_0}^{\infty} \frac{|w(s)|^{\frac{\alpha}{\alpha+1}}(s)}{a^{\frac{1}{\alpha}}(s)}\,ds < \infty. \tag{10}$$

If we suppose that (10) fails, there exists a $t_1 > t_0$ such that

$$\int_{t_0}^{t} \frac{|w(s)|^{\frac{\alpha}{\alpha+1}}(s)}{a^{\frac{1}{\alpha}}(s)}\,ds \geq \frac{\mu}{K\,\xi}, \qquad \text{for} \qquad t \geq t_1,$$

where μ is an arbitrary positive number and ξ is a positive constant such that

$$\inf_{s \geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] > \xi > 0. \tag{11}$$

Therefore, for all $t \geq t_1$

$$\begin{aligned} G(t) &= \frac{K}{H(t,t_0)} \int_{t_0}^{t} H(t,s)d\left(\int_{t_0}^{s} \frac{|w(\tau)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(\tau)}\,d\tau\right) \\ &= -\frac{K}{H(t,t_0)} \int_{t_0}^{t} \frac{\partial H}{\partial s}(t,s) \left(\int_{t_0}^{s} \frac{|w(\tau)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(\tau)}\,d\tau\right) ds \\ &\geq -\frac{K}{H(t,t_0)} \int_{t_1}^{t} \frac{\partial H}{\partial s}(t,s) \left(\int_{t_0}^{s} \frac{|w(\tau)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(\tau)}\,d\tau\right) ds \\ &\geq -\frac{\mu}{\xi H(t,t_0)} \int_{t_1}^{t} \frac{\partial H}{\partial s}(t,s)\,ds = \frac{\mu H(t,t_1)}{\xi H(t,t_0)} \end{aligned}$$

By (11), there is a $t_2 \geq t_1$ such that $\frac{H(t,t_1)}{H(t,t_0)} \geq \xi$ for all $t \geq t_2$, and accordingly $G(t) \geq \mu$ for all $t \geq t_2$. Since μ is arbitrary,

$$\lim_{t\to\infty} G(t) = \infty,$$

which ensures that

$$\lim_{n\to\infty} G(T_n) = \infty. \tag{12}$$

Hence, (9) gives

$$\lim_{n\to\infty} F(T_n) = \infty. \tag{13}$$

From (9) we derive for n sufficiently large

$$\frac{F(T_n)}{G(T_n)} - 1 \geq -\frac{M}{G(T_n)} > -\frac{1}{2}.$$

Therefore,

$$\frac{F(T_n)}{G(T_n)} > \frac{1}{2} \quad \text{for all large} \ \ n,$$

which by (13) ensures that

$$\lim_{n\to\infty} \frac{F^{\alpha+1}(T_n)}{G^{\alpha}(T_n)} = \infty. \tag{14}$$

On the other hand, by Hölder's inequality, we have for all $n \in N$

$$\begin{aligned}
F(T_n) &= \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} |w(s)|h(T_n,s)\,ds, \\
&= \int_{t_0}^{T_n} \left(\frac{K^{\frac{\alpha}{\alpha+1}}}{H^{\frac{\alpha}{\alpha+1}}(T_n,t_0)} \frac{|w(s)|H^{\frac{\alpha}{\alpha+1}}(T_n,s)}{a^{\frac{1}{\alpha+1}}(s)} \right) \\
&\qquad \times \left(\frac{K^{-\frac{\alpha}{\alpha+1}}}{H^{\frac{1}{\alpha+1}}(T_n,t_0)} \frac{h(T_n,s)a^{\frac{1}{\alpha+1}}(s)}{H^{\frac{\alpha}{\alpha+1}}(T_n,s)} \right) ds \\
&\leq \left(\frac{K}{H(T_n,t_0)} \int_{t_0}^{T_n} \frac{|w(s)|^{\frac{\alpha+1}{\alpha}} H(T_n,s)}{a^{\frac{1}{\alpha}}(s)}\,ds \right)^{\frac{\alpha}{\alpha+1}} \\
&\qquad \times \left(\frac{K^{-\alpha}}{H(T_n,t_0)} \int_{t_0}^{T_n} a(s) \frac{h^{\alpha+1}(T_n,s)}{H^{\alpha}(T_n,s)}\,ds \right)^{\frac{1}{\alpha+1}}
\end{aligned}$$

and accordingly

$$\frac{F^{\alpha+1}(T_n)}{G^{\alpha}(T_n)} \leq \frac{K^{-\alpha}}{H(T_n,t_0)} \int_{t_0}^{T_n} a(s) \frac{h^{\alpha+1}(T_n,s)}{H^{\alpha}(T_n,s)}\,ds.$$

So, because of (14), we have

$$\lim_{n\to\infty} \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} a(s) \frac{h^{\alpha+1}(T_n,s)}{H^{\alpha}(T_n,s)}\,ds = \infty,$$

which gives

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s) \frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}\,ds = \infty,$$

contradicting the condition (C_3). So, (10) holds. Now, from (5), we obtain

$$\int_{t_0}^{\infty} \frac{\varphi_+^{\frac{\alpha}{\alpha+1}}(s)}{a^{\frac{1}{\alpha}}(s)}\, ds \le \int_{t_0}^{\infty} \frac{|w(s)|^{\frac{\alpha}{\alpha+1}}(s)}{a^{\frac{1}{\alpha}}(s)}\, ds < \infty,$$

which contradicts (C_5). This completes the proof. $\square$

Theorem 2.3 *Let condition (C_1) holds and let the functions H and h be defined as in Theorem* 2.1 *such that conditions (H_1), (H_2), (H_3) and*

$$(C_6) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} |q(s)| H(t,s)\, ds < \infty$$

are satisfied. If there exists a continuous function φ on $[t_0,\infty)$ such that for every $T \ge t_0$

$$(C_7) \quad \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[q(s)H(t,s) - \beta\, a(s) \frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)} \right] ds \ge \varphi(T),$$

and condition (C_5) holds, then the equation (E) is oscillatory.

Proof. For the nonoscillatory solution $x(t)$ of the equation (E), as in the proof of Theorem 2.1, (3) and (4) are fulfilled. Thus, for $t > T \ge t_0$, we have

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[q(s)H(t,s) - \beta\, a(s) \frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)} \right] ds \le w(T),$$

so that, according to condition (C_7), (5) is satisfied. By conditions (C_7) is

$$\begin{aligned}\varphi(t_0) \le \liminf_{t\to\infty} &\frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s)\, ds \\ &- \liminf_{t\to\infty} \frac{\beta}{H(t,t_0)} \int_{t_0}^{t} a(s) \frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}\, ds,\end{aligned}$$

so that (C_6) implies

$$\liminf_{t\to\infty} \frac{\beta}{H(t,t_0)} \int_{t_0}^{t} a(s) \frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}\, ds < \infty.$$

Condition (C_6) together with (7) implies

$$\limsup_{t\to\infty} [G(t) - F(t)] \le w(t_0) - \liminf_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} q(s)H(t, s)\, ds < \infty.$$

This shows that there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in (t_0, ∞) with $\lim_{n\to\infty} T_n = \infty$, such that

$$\lim_{n\to\infty} [G(T_n) - F(T_n)] = \limsup_{t\to\infty} [G(t) - F(t)].$$

Following the procedure of the proof of Theorem 2.2, we conclude that (10) is satisfied. Then, we come to the contradiction as in the proof of Theorem 2.2. $\square$

We observe that Theorem 2.2 can be applied in some cases in which Theorem 2.1 is not applicable. Such a case is described in the following example.

Example 2.2 Consider the differential equation

$$(E_2) \qquad \left(t^{\nu}|x(t)|^{3-\alpha}|x'(t)|^{\alpha-1}x'(t)\right)' + t^{\lambda}\cos t\, x^3(t) = 0,$$

for $t \ge t_0$, where ν, λ, α are constants such that $\lambda < 0$, $\alpha > 0$, $\alpha \ne 2$ and $\nu < \alpha$. Then, condition (C_1) is satisfied. Moreover, taking $H(t, s) = (t - s)^2$, for $t > s \ge t_0$, we have

$$\begin{aligned}\frac{1}{t^2}\int_{t_0}^{t} s^{\nu}(t-s)^{1-\alpha}\, ds &\le \begin{cases} \dfrac{t^{\nu}}{t^2}\,\dfrac{(t-t_0)^{2-\alpha}}{2-\alpha}, & \nu > 0 \\[2ex] \dfrac{t_0^{\nu}}{t^2}\,\dfrac{(t-t_0)^{2-\alpha}}{2-\alpha}, & \nu < 0 \end{cases} \\ &= \begin{cases} \dfrac{t^{\nu-\alpha}}{2-\alpha}\left(1 - \dfrac{t_0}{t}\right)^{2-\alpha}, & \nu > 0 \\[2ex] \dfrac{t_0^{\nu}}{2-\alpha}\,\dfrac{1}{t^{\alpha}}\left(1 - \dfrac{t_0}{t}\right)^{2-\alpha}, & \nu < 0 \end{cases}\end{aligned}$$

Therefore, condition (C_3) is satisfied and for arbitrary small constant $\varepsilon > 0$, there exists a $t_1 \geq t_0$ such that for $T \geq t_1$

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_T^t [(t-s)^2 s^\lambda \cos s - \beta\, s^\nu\, (t-s)^{1-\alpha}]\, ds \geq -T^\lambda \sin T - \varepsilon.$$

Now, set $\varphi(T) = -T^\lambda \sin T - \varepsilon$ and consider an integer N such that $2N\pi + 5\pi/4 \geq \max\{t_1, (1+\sqrt{2}\varepsilon)^{1/\lambda}\}$. Then, for all integers $n \geq N$, we have

$$\varphi(T) \geq \frac{1}{\sqrt{2}} \quad \text{for every} \quad T \in \left[2n\pi + \frac{5\pi}{4}, 2n\pi + \frac{7\pi}{4}\right].$$

Taking into account that $\nu < \alpha$, we obtain

$$\begin{aligned}
\int_{t_0}^\infty \frac{\varphi_+^{\frac{\alpha}{\alpha+1}}(s)}{a^{\frac{1}{\alpha}}(s)}\, ds &\geq \sum_{n=N}^\infty (\sqrt{2})^{-\frac{\alpha}{\alpha+1}} \int_{2n\pi+5\pi/4}^{2n\pi+7\pi/4} s^{\frac{\nu}{\alpha}}\, ds \\
&\geq (\sqrt{2})^{-\frac{\alpha}{\alpha+1}} \sum_{n=N}^\infty \int_{2n\pi+5\pi/4}^{2n\pi+7\pi/4} \frac{ds}{s} \\
&= (\sqrt{2})^{-\frac{\alpha}{\alpha+1}} \sum_{n=N}^\infty \ln\left(1 + \frac{\frac{\pi}{2}}{2n\pi + \frac{5\pi}{4}}\right) = \infty.
\end{aligned}$$

Accordingly, all conditions of Theorem 2.2 are satisfied and hence the equation (E_2) is oscillatory.

On the other hand, the condition (C_2) is not satisfied for $\lambda < -1$, so that by Theorem 2.1 we conclude that (E_2) is oscillatory only for $-1 \leq \lambda < 0$.

Remark 2.4 It is interesting to note that by Corollary 3.1. in [28] we have that (E_2), where $\psi(x) \equiv 1$, is oscillatory for $\lambda \geq 0$ and $\nu < \alpha$. Therefore, by the previous deduction, we have that such equation is oscillatory for $\nu < \alpha$ and all λ.

Theorem 2.4 *Suppose that condition (C_1) holds and let the functions H and h be defined as in Theorem 2.1, such that conditions (H_1) and (H_2) hold. If there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$*

such that $\rho'(t) \geq 0$ for all $t \geq t_0$ and
(C_8)
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s) \left[q(s)H(t,s) - \frac{\beta\, a(s)}{H^{\alpha}(t,s)} G^{\alpha+1}(t,s) \right] ds = \infty,$$
where $G(t,s) = h(t,s) + \dfrac{\rho'(s)}{\rho(s)} H(t,s)$, then the equation (E) is oscillatory.

Proof. Let x be a solution on $[t_0,\infty)$ of the differential equation (E) with $x(t) \neq 0$ for all $t \geq t_0$. Now, we define
$$W(t) = \rho(t) \frac{a(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} \quad \text{for} \quad t \geq t_0.$$
Then, for every $t \geq t_0$, we obtain
$$W'(t) = -q(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} W(t) - \frac{f'(x(t))|W(t)|^{\frac{\alpha+1}{\alpha}}}{(a(t)\rho(t)\psi(x(t))|f(x(t))|^{\alpha-1})^{\frac{1}{\alpha}}}.$$
Therefore,
$$\begin{aligned} \int_{t_0}^{t} W'(s)H(t,s)\,ds &\leq -\int_{t_0}^{t} q(s)\rho(s)H(t,s)\,ds \\ &+ \int_{t_0}^{t} \frac{\rho'(s)}{\rho(s)} W(s)H(t,s)\,ds - K\int_{t_0}^{t} H(t,s)\frac{|W(s)|^{\frac{\alpha+1}{\alpha}}}{(a(s)\rho(s))^{\frac{1}{\alpha}}}\,ds. \end{aligned}$$
Using (2), we have
$$\begin{aligned} \int_{t_0}^{t} q(s)\rho(s)H(t,s)\,ds &\leq W(t_0)H(t,t_0) \\ &+ \int_{t_0}^{t} G(t,s)|W(s)|\,ds - K\int_{t_0}^{t} H(t,s)\frac{|W(s)|^{\frac{\alpha+1}{\alpha}}}{(a(s)\rho(s))^{\frac{1}{\alpha}}}\,ds. \end{aligned} \tag{15}$$
If we take
$$\begin{aligned} X &= (K\,H(t,s))^{\frac{\alpha}{\alpha+1}} \frac{|W(s)|}{(a(s)\rho(s))^{\frac{1}{\alpha+1}}}, \quad q = \frac{\alpha+1}{\alpha} \\ Y &= \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \frac{[a(s)\rho(s)]^{\frac{\alpha}{\alpha+1}}}{[K\,H(t,s)]^{\frac{\alpha^2}{\alpha+1}}} \left(h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \right)^{\alpha}, \end{aligned}$$

according to Lemma 2.1, we get

$$
\begin{aligned}
&|W(s)|\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s)\Big)-K\,H(t,s)\frac{|W(s)|^{\frac{\alpha+1}{\alpha}}}{[a(s)\rho(s)]^{\frac{1}{\alpha}}} \\
&\qquad \le \beta\,\frac{a(s)\rho(s)}{H^\alpha(t,s)}\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s)\Big)^{\alpha+1}.
\end{aligned}
\tag{16}
$$

From (15) and (16) we obtain

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big[&q(s)\rho(s)H(t,s)-\beta\,\frac{a(s)\rho(s)}{H^\alpha(t,s)} \\
&\times\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s)\Big)^{\alpha+1}\Big]\,ds\le W(t_0),
\end{aligned}
$$

which contradicts (C_8). □

Remark 2.5 For $\alpha=1$ Theorem 2.4 reduces to Theorem 1 in Grace [7].

Remark 2.6 If $\alpha=1$ and $H(t,s)=(t-s)^\gamma$ for some constant $\gamma>1$, Theorem 2.4 include as a special case Theorem 2 in Grace [4].

Corollary 2.3 *Let condition* (C_8) *in Theorem* 2.4 *be replaced by*

$$
(C_9)\quad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{a(s)\rho(s)}{H^\alpha(t,s)}\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s)\Big)^{\alpha+1}ds<\infty,
$$

$$
(C_{10})\qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}q(s)\rho(s)H(t,s)\,ds=\infty,
$$

then the conclusion of Theorem 2.4 *holds.*

Example 2.3 Consider the differential equation

$$
(E_3)\quad \Big(t^\nu|x(t)|^{3-\alpha}|x'(t)|^{\alpha-1}x'(t)\Big)'+[\lambda\,t^{\lambda-3}(2-\cos t)+t^{\lambda-2}\sin t]x^3(t)=0,
$$

for $t\ge t_0>0$, where λ is arbitrary positive constant and ν,α are constants such that $\nu<\alpha-2$, $\alpha<1$. Here, we choose $\rho(t)=t^2$ and

$H(t,s)=(t-s)^2$ for $t\ge s\ge t_0$. Then, since $\rho(t)q(t)=\frac{d}{ds}[s^\lambda(2-\cos s)]$, as in Example 2.1, we get

$$\int_{t_0}^t \rho(s)q(s)\,ds \ge t^\lambda - k_0$$

and therefore,

$$\frac{1}{t^2}\int_{t_0}^t (t-s)^2\rho(s)q(s)\,ds \ge \frac{2t^\lambda}{(\lambda+1)(\lambda+2)} + \frac{k_1}{t^2} + \frac{k_2}{t} - k_0,$$

where

$$k_1 = \frac{2\,t_0^{\lambda+2}}{\lambda+2} - k_0 t_0^2,\;\; k_2 = 2k_0t_0 - \frac{2\,t_0^{\lambda+1}}{\lambda+1}.$$

Hence, condition (C_{10}) is satisfied. On the other hand,

$$\begin{aligned}
&\frac{1}{t^2}\int_{t_0}^t \frac{s^{\nu+2}}{(t-s)^{2\alpha}}\Big(2(t-s)+\frac{2}{s}(t-s)^2\Big)^{\alpha+1}ds\\
&\quad = t^{\alpha-1}2^{\alpha+1}\int_{t_0}^t s^{\nu-\alpha+1}(t-s)^{1-\alpha}\,ds\\
&\quad \le 2^{\alpha+1}\left(1-\frac{t_0}{t}\right)^{1-\alpha}\frac{t^{\nu-\alpha+2}-t_0^{\nu-\alpha+2}}{\nu-\alpha+2},
\end{aligned}$$

so that condition (C_9) is also satisfied. Consequently, by Corollary 2.3, the equation (E_3) is oscillatory.

Using Theorem 2.4 and the same technique as in the proof of Theorem 2.2 and 2.3, we have the following two theorems which extend two Grace's theorems [7, Theorem 3 and 4].

Theorem 2.5 *Let condition (C_1) holds and let the functions H and h be defined as in Theorem 2.1 such that conditions (H_1), (H_2), (H_3) are satisfied. If there exists a nonnegative, differentiable, increasing function $\rho(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \frac{a(s)\rho(s)}{H^\alpha(t,s)}\Big(h(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s)\Big)^{\alpha+1}ds < \infty,$$

and there exists a continuous function φ *on* $[t_0, \infty)$ *such that for every* $T \geq t_0$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[q(s)\rho(s)H(t,s) - \beta \frac{a(s)\rho(s)}{H^\alpha(t,s)} \times \Big(h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \Big)^{\alpha+1} \Big] \, ds \geq \varphi(T),$$

and condition (C_5) *is satisfied, then the equation* (E) *is oscillatory.*

Theorem 2.6 *Let condition* (C_5) *holds and let the functions* H *and* h *be defined as in Theorem* 2.1 *such that conditions* (H_1), (H_2), (H_3) *are satisfied. If there exists a nonnegative, differentiable, increasing function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t |q(s)|\rho(s)H(t,s) \, ds < \infty$$

and there exists a continuous function φ *on* $[t_0, \infty)$ *such that for every* $T \geq t_0$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[q(s)H(t,s) - \beta \frac{a(s)\rho(s)}{H^\alpha(t,s)} \times \Big(h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \Big)^{\alpha+1} \Big] \, ds \geq \varphi(T),$$

and condition (C_5) *holds, then the equation* (E) *is oscillatory.*

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