ON SOME BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

1.1. **Statement of the Problem.** On the segment $I = [a, b]$ consider the system of linear functional differential equations

$$x_i'(t) = \sum_{k=1}^{n} \ell_{ik}(x_k)(t) + q_i(t) \qquad (i = 1, \dots, n) \tag{1}$$

and its particular case

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t) x_k(\tau_{ik}(t)) + q_i(t) \qquad (i = 1, \dots, n) \tag{1'}$$

with the boundary conditions

$$\int_a^b x_i(t) d\varphi_i(t) = c_i \qquad (i = 1, \dots, n). \tag{2}$$

Here $\ell_{ik} : C(I; \mathbb{R}) \to L(I; \mathbb{R})$ are linear bounded operators, p_{ik} and $q_i \in L(I; \mathbb{R})$, $c_i \in \mathbb{R}$ $(i, k = 1, \dots, n)$, $\varphi_i : I \to \mathbb{R}$ $(i = 1, \dots, n)$ are the functions with bounded variations, and $\tau_{ik} : I \to I$ $(i, k = 1, \dots, n)$ are measurable functions.

The simple but important particular case of the conditions (2) are the two–point boundary conditions

$$x_i(b) = \lambda_i x_i(a) + c_i \qquad (i = 1, \dots, n), \tag{3}$$

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the periodic boundary conditions

$$x_i(b) = x_i(a) + c_i \qquad (i = 1, \dots, n), \tag{4}$$

and the initial conditions

$$x_i(t_0) = c_i \qquad (i = 1, \dots, n), \tag{5}$$

where $t_0 \in I$ and $\lambda_i \in \mathbb{R}$ $(i = 1, \dots, n)$.

By a solution of the system (1) (of the system $(1')$) we understand an absolutely continuous vector function $(x_i)_{i=1}^n : I \to \mathbb{R}$ which satisfies the system (1) (the system $(1')$) almost everywhere on I. A solution of the system (1) (of the system $(1')$) which satisfies the condition (j), where $j \in \{2, 3, 4, 5\}$, is said to be a solution of the problem (1), (j).

As for the differential system

$$x_i'(t) = \sum_{k=1}^n p_{ik}(t)x_k(t) + q_i(t) \qquad (i = 1, \dots, n),$$

the boundary value problems have been studied in detail (see [4,5,8–10] and references therein). There are also a lot of interesting results concerning the problems (1), (k) and $(1')$, (k) $(k = 2, 3, 4, 5)$ (see [2,3,6,7,11–13]). In this paper, the optimal conditions for the unique solvability of the problems (1), (2) and $(1')$, (2) are established which are different from the previous results.

1.2. **Basic Notation.** Throughout this paper the following notation and terms are used:

$I = [a, b]$, $\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$;

δ_{ik} is the Kronecker's symbol, i.e.,

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k; \end{cases}$$

$\mathbb{R}^n$ is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with the components $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is the space of $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with the components $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$\mathbb{R}^n_+ = \{(x_i)_{i=1}^n \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}$;
$\mathbb{R}^{n\times n}_+ = \{(x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n} : x_{ik} \geq 0, i, k = 1, \ldots, n\}$;
the inequalities between vectors x and $y \in \mathbb{R}^n$, and between matrices X and $Y \in \mathbb{R}^{n\times n}$ are considered componentwise, i.e.,

$$x \leq y \Leftrightarrow (y - x) \in \mathbb{R}^n_+, \qquad X \leq Y \Leftrightarrow (Y - X) \in \mathbb{R}^{n\times n}_+;$$

$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n\times n}$;
X^{-1} is the inverse matrix to $X \in \mathbb{R}^{n\times n}$;
E is the unit matrix;
$C(I; \mathbb{R}^n)$ is the space of continuous[1] vector functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \sup\{\|x(t)\| : t \in I\};$$

$C(I; \mathbb{R}^n_+) = \{x \in C(I; \mathbb{R}^n) : x(t) \in \mathbb{R}^n_+ \text{ for } t \in I\}$;
$L(I; \mathbb{R}^n)$ is the space of summable vector functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\|_L = \int_I \|x(t)\| dt;$$

$L(I; \mathbb{R}^n_+) = \{x \in L(I; \mathbb{R}^n) : x(t) \in \mathbb{R}^n_+ \text{ for almost all } t \in I\}$;
$\widetilde{C}(I; \mathbb{R}^n)$ is the space of absolutely continuous vector functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\|_{\widetilde{C}} = \|x\|_C + \|x'\|_L;$$

$\mathcal{P}_\mathcal{I}$ is the set of linear operators $\ell : C(I; \mathbb{R}) \to L(I; \mathbb{R})$ mappings $C(I; \mathbb{R}_+)$ into $L(I; \mathbb{R}_+)$;
$\mathcal{L}_\mathcal{I}$ is the set of linear continuous operators $\ell : C(I; \mathbb{R}) \to L(I; \mathbb{R})$, for each of them there exists an operator $\overline{\ell} \in \mathcal{P}_\mathcal{I}$ such that for any $u \in C(I; \mathbb{R})$ the inequalities

$$|\ell(u)(t)| \leq \overline{\ell}(|u|)(t)$$

holds almost everywhere on I;
for any $u \in L(I; \mathbb{R})$

$$\eta(u)(t, s) = \int_t^s u(\xi) d\xi.$$

[1]The vector function $x = (x_i)_{i=1}^n : I \to \mathbb{R}^n$ is said to be continuous, bounded, summable, etc., if the components $x_i : I \to \mathbb{R}$ $(i = 1, \ldots, n)$ have such a property.

1.3. **Criterion on the Unique Solvability of the Problem (1), (2).** The results in general theory of boundary value problems (see [12], Theorems 1.1 and 1.4) yield the following

Theorem 1. *If $\ell_{ik} \in \mathcal{L}_I$ $(i,k = 1,\dots,n)$, then the boundary value problem* (1), (2) *with arbitrary $c_i \in \mathbb{R}$ and $q_i \in L(I;\mathbb{R})$ $(i = 1,\dots,n)$ is uniquely solvable if and only if the corresponding homogeneous problem*

$$x_i'(t) = \sum_{k=1}^{n} \ell_{ik}(x_k)(t) \qquad (i = 1,\dots,n), \tag{1_0}$$

$$\int_a^b x_i(s)d\varphi_i(s) = 0 \qquad (i = 1,\dots,n) \tag{2_0}$$

has only the trivial solution. If the latter condition is fulfilled, then the solution of the problem (1), (2) *admits the representation*

$$x_i(t) = \sum_{k=1}^{n} y_{ik}(t)c_k + g_i(q_1,\dots,q_n)(t) \qquad (i = 1,\dots,n), \tag{6}$$

where $y_{ik} \in \widetilde{C}(I;\mathbb{R})$ $(i,k = 1,\dots,n)$, and $g_i : L(I;\mathbb{R}^n) \to \widetilde{C}(I;\mathbb{R})$ $(i = 1,\dots,n)$ are linear continuous operators such that the vector function $(\sum\limits_{k=1}^{n} y_{ik}c_k)_{i=1}^n$ is the solution of the problem (1_0), (2), *and the vector function $(g_i(q_1,\dots,q_n))_{i=1}^n$ is the solution of the problem* (1), (2_0).

Remark 1. The operator $(g_i)_{i=1}^n : L(I;\mathbb{R}^n) \to \widetilde{C}(I;\mathbb{R}^n)$ is called *the Green's operator of the problem* (1_0), (2_0). According to Danford–Pettis Theorem (see [1], Ch. XI, §1, Theorem 6), there exists the unique matrix function $G = (g_{ik})_{i,k=1}^n : I \times I \to \mathbb{R}^{n\times n}$ with the essentially bounded components $g_{ik} : I \times I \to \mathbb{R}$ $(i,k = 1,\dots,n)$ such that

$$g_i(q_1,\dots,q_n)(t) \equiv \sum_{k=1}^{n} \int_a^b g_{ik}(t,s)q_k(s)ds \qquad (i = 1,\dots,n).$$

Consequently, the formula (6) can be rewritten as follows:

$$x_i(t) = \sum_{k=1}^{n} y_{ik}(t)c_k + \sum_{k=1}^{n} \int_a^b g_{ik}(t,s)q_k(s)ds \qquad (i = 1,\dots,n). \tag{$6'$}$$

This formula is called *the Green's formula* for the problem (1), (2), and the matrix G is called *the Green's matrix* of the problem (1_0), (2_0).

The aim of the following is to find effective criteria for the unique solvability of the above formulated problems. With a view to achieve this goal, we will need one lemma which is proved in Section 2.

2. Lemma on Boundary Value Problem for the System of Functional Differential Equations

Consider the system of differential inequalities

$$|y_i'(t) - \ell_i(y_i)(t)| \le \sum_{k=1}^{n} h_{ik}(t)\|y_k\|_C \qquad (i = 1, \dots, n) \tag{7}$$

with the boundary conditions

$$\int_a^b y_i(s)d\varphi_i(s) = 0 \qquad (i = 1, \dots, n), \tag{8}$$

where

$$\ell_i \in \mathcal{L}_I, \quad h_{ik} \in L(I; \mathbb{R}_+) \quad (i, k = 1, \dots, n),$$

$c_i \in \mathbb{R}$ $(i = 1, \dots, n)$, and $\varphi_i : I \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions with bounded variations.

Along with (7), (8) for every $i \in \{1, \dots, n\}$ consider the homogeneous problem

$$y'(t) = \ell_i(y)(t), \qquad \int_a^b y(s)d\varphi_i(s) = 0. \tag{9_i}$$

Lemma 1. *Let for every $i \in \{1, \dots, n\}$ the homogeneous problem (9_i) have only the trivial solution and there exist a matrix $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ such that*

$$r(A) < 1 \tag{10}$$

and

$$\int_a^b |g_i(t,s)|h_{ik}(s)ds \le a_{ik} \qquad \text{for } t \in I \quad (i, k = 1, \dots, n), \tag{11}$$

where g_i is the Green's function of the problem (9_i). Then the problem (7), (8) has only the trivial solution.

Proof. Let $(y_i)_{i=1}^n$ be a solution of (7), (8). Then for every $i \in \{1, \dots, n\}$, the function y_i is the solution of the problem

$$y'(t) - \ell_i(y)(t) = q_i(t), \qquad \int_a^b y(s)d\varphi_i(s) = 0, \tag{12}$$

where

$$q_i(t) \stackrel{def}{=} y_i'(t) - \ell_i(y_i)(t). \tag{13}$$

By the Green's formula we have

$$y_i(t) = \int_a^b g_i(t,s)q_i(s)ds \qquad \text{for } t \in I \quad (i = 1, \dots, n), \tag{14}$$

Due to (7) and (13),

$$\int_a^b |g_i(t,s)||q_i(s)|ds \le \sum_{k=1}^n \int_a^b |g_i(t,s)|h_{ik}(s)\|y_k\|_C ds \quad \text{for } t \in I \quad (i = 1, \dots, n).$$

In view of (11) and the last inequalities from (14) we obtain

$$|y_i(t)| \le \sum_{k=1}^n a_{ik}\|y_k\|_C \qquad \text{for } t \in I \quad (i = 1, \dots, n). \tag{15}$$

Consequently, (15) yields

$$(E - A)\big(\|y_i\|_C\big)_{i=1}^n \le 0. \tag{16}$$

Since A is a nonnegative matrix satisfying (10), there exists the nonnegative inverse matrix $(E - A)^{-1}$. Then by (16) we obtain $y_i(t) \equiv 0$ $(i = 1, \dots, n)$. $\square$

3. Existence and Uniqueness Theorems

Throughout the following we will assume that $\ell_{ik} \in \mathcal{L}_I$ $(i, k = 1, \dots, n)$ and for any $u \in C(I; \mathbb{R})$ the inequalities

$$|\ell_{ik}(u)(t)| \le \overline{\ell}_{ik}(|u|)(t) \qquad (i, k = 1, \dots, n)$$

hold almost everywhere on I, where $\overline{\ell}_{ik} \in \mathcal{P}_{\mathcal{I}}$ $(i, k = 1, \dots, n)$.

Theorem 2. *Let there exist operators $\ell_i, \widetilde{\ell}_{ik} \in \mathcal{L}_I$ $(i, k = 1, \dots, n)$, functions $h_{ik} \in L(I; \mathbb{R}_+)$ $(i, k = 1, \dots, n)$, and a matrix $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$ satisfying* (10) *such that:*

(i) any solution of the system (1_0) *is also a solution of the system*

$$x_i'(t) = \ell_i(x_i)(t) + \sum_{k=1}^{n} \widetilde{\ell}_{ik}(x_k)(t) \qquad (i = 1, \dots, n); \tag{17}$$

(ii) for any $y \in \widetilde{C}(I;\mathbb{R})$, *the inequalities*

$$|\widetilde{\ell}_{ik}(y)(t)| \le h_{ik}(t)\|y\|_C \qquad (i, k = 1, \dots, n) \tag{18}$$

holds almost everywhere on I*;*

(iii) for every $i \in \{1, \dots, n\}$ *the problem* (9_i) *has only the trivial solution and the inequalities* (11) *are fulfilled, where* g_i *is the Green's function of the problem* (9_i)*. Then the problem* (1), (2) *has a unique solution.*

Proof. Let $(y_i)_{i=1}^n$ be a solution of the problem (1_0), (2_0). Then by (17) and (18) it is also a solution of the problem (7), (8). Now it is obvious that all the assumptions of Lemma 1 are fulfilled. Therefore $y_i(t) \equiv 0$ $(i = 1, \dots, n)$. Thus the homogeneous problem (1_0), (2_0) has only the trivial solution and consequently, by Theorem 1, the problem (1), (2) has a unique solution. □

Corollary 1. *Let there exist operators* $\ell_i \in \mathcal{L}_I$ $(i = 1, \dots, n)$, *functions* $h_{ik} \in L(I;\mathbb{R}_+)$ $(i, k = 1, \dots, n)$, *and a matrix* $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ *satisfying* (10) *such that:*

(i) for any $y \in \widetilde{C}(I;\mathbb{R})$, *the inequalities*

$$|\ell_{ii}(y)(t) - \ell_i(y)(t)| \le h_{ii}(t)\|y\|_C \qquad (i = 1, \dots, n),$$
$$|\ell_{ik}(y)(t)| \le h_{ik}(t)\|y\|_C \qquad (i \ne k; i, k = 1, \dots, n)$$

holds almost everywhere on I*;*

(ii) for every $i \in \{1, \dots, n\}$ *the problem* (9_i) *has only the trivial solution and the inequalities* (11) *are fulfilled, where* g_i *is the Green's function of the problem* (9_i)*. Then the problem* (1), (2) *has a unique solution.*

Proof. Put

$$\widetilde{\ell}_{ii}(y)(t) \equiv \ell_{ii}(y)(t) - \ell_i(y)(t) \qquad (i = 1, \dots, n),$$
$$\widetilde{\ell}_{ik}(y)(t) \equiv \ell_{ik}(y)(t) \qquad (i \ne k; i, k = 1, \dots, n).$$

Then the assumptions of Theorem 2 are fulfilled. Consequently, the problem (1), (2) has a unique solution. □

Corollary 2. *Let*

$$\int_a^b \exp\left(\int_a^s \ell_{ii}(1)(\xi)d\xi\right) d\varphi_i(s) \ne 0 \qquad (i = 1, \dots, n) \tag{19}$$

and there exist a matrix $A = (a_{ik})_{i,k=1}^{n} \in \mathbb{R}_{+}^{n\times n}$ *satisfying* (10) *such that* (11) *is fulfilled, where* g_i *is the Green's function of the problem*

$$y'(t) = \ell_{ii}(1)(t)y(t), \qquad \int_a^b y(s)d\varphi_i(s) = 0 \tag{20_i}$$

and

$$h_{ik}(t) = \overline{\ell}_{ii}\big(|\eta(\overline{\ell}_{ik}(1))(t,\cdot)|\big)(t) + (1-\delta_{ik})\overline{\ell}_{ik}(1)(t) \quad (i,k = 1,\dots,n). \tag{21}$$

Then the problem (1), (2) *has a unique solution.*

Proof. The condition (19) is necessary and sufficient for the problem (20_i) to have only the trivial solution for every $i \in \{1,\dots,n\}$.

On the other hand, every solution $(x_i)_{i=1}^{n}$ of the system (1_0) satisfies

$$\begin{aligned} x_i'(t) &= \ell_{ii}(1)(t)x_i(t) + \ell_{ii}(x_i(\cdot) - x_i(t))(t) + \sum_{k=1}^{n}(1-\delta_{ik})\ell_{ik}(x_k)(t) = \\ &= \ell_{ii}(1)(t)x_i(t) + \sum_{k=1}^{n}\Big[\ell_{ii}\big(|\eta(\ell_{ik}(x_k))(t,\cdot)|\big)(t) + (1-\delta_{ik})\ell_{ik}(x_k)(t)\Big] \quad (i=1,\dots,n). \end{aligned} \tag{22}$$

Put

$$\ell_i(y)(t) \equiv \ell_{ii}(1)(t)y(t) \qquad (i=1,\dots,n),$$

$$\widetilde{\ell}_{ik}(y)(t) \equiv \ell_{ii}\big(|\eta(\ell_{ik}(y))(t,\cdot)|\big)(t) + (1-\delta_{ik})\ell_{ik}(y)(t) \qquad (i,k=1,\dots,n).$$

Then any solution of the system (1_0) is also a solution of the system (17). Now it is obvious that all the assumptions of Theorem 2 are fulfilled. Consequently, the problem (1), (2) has a unique solution. □

If

$$\ell_{ik}(y)(t) \equiv p_{ik}y(\tau_{ik}(t)) \qquad (i,k=1,\dots,n),$$

then the system (1) has the form (1′). In that case

$$\overline{\ell}_{ik}(y)(t) \equiv |p_{ik}|y(\tau_{ik}(t)) \qquad (i,k=1,\dots,n),$$

$$\overline{\ell}_{ii}\big(|\eta(\overline{\ell}_{ik}(1))(t,\cdot)|\big)(t) \equiv \left|p_{ii}(t)\int_t^{\tau_{ii}(t)}|p_{ik}(s)|ds\right|.$$

Therefore from Corollary 2 it follows

Corollary 2′. *Let*

$$\int_a^b \exp\left(\int_a^s p_{ii}(\xi)d\xi\right) d\varphi_i(s) \neq 0 \qquad (i = 1, \dots, n)$$

and there exist a matrix $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ *satisfying* (10) *such that* (11) *is fulfilled, where* g_i *is the Green's function of the problem*

$$y'(t) = p_{ii}(t)y(t), \qquad \int_a^b y(s)d\varphi_i(s) = 0$$

and

$$h_{ik}(t) = \left| p_{ii}(t) \int_t^{\tau_{ii}(t)} |p_{ik}(s)|ds \right| + (1 - \delta_{ik})|p_{ik}(t)| \tag{21′}$$

$$(i, k = 1, \dots, n).$$

Then the problem (1′), (2) *has a unique solution.*

Corollary 3. *Let* $\lambda_i \neq 1$, $\mu_i = \max\{1, |\lambda_i|\}$ $(i = 1, \dots, n)$ *and the matrix*

$$A = \left(\frac{\mu_i}{|1-\lambda_i|} \int_a^b \overline{\ell}_{ik}(1)(s)ds\right)_{i,k=1}^n$$

satisfies (10). *Then the problem* (1), (3) *has a unique solution.*

Proof. Since $\lambda_i \neq 1$ $(i = 1, \dots, n)$, for every $i \in \{1, \dots, n\}$ the problem

$$y'(t) = 0, \qquad y(b) = \lambda_i y(a) \tag{23_i}$$

has only the trivial solution. Moreover, the Green's function of (23_i) is of the form

$$g_i(t,s) = \begin{cases} \frac{\lambda_i}{\lambda_i - 1} & \text{for } a \le s \le t \le b, \\ \frac{1}{\lambda_i - 1} & \text{for } a \le t < s \le b. \end{cases}$$

Put

$$\ell_i(y)(t) \equiv 0, \qquad \widetilde{\ell}_{ik}(y)(t) \equiv \ell_{ik}(y)(t), \qquad h_{ik}(t) \equiv \overline{\ell}_{ik}(1)(t) \qquad (i, k = 1, \dots, n).$$

Then all the assumptions of Theorem 2 are fulfilled. Consequently, the problem (1), (3) has a unique solution □

Corollary 3′. *Let $\lambda_i \neq 1$, $\mu_i = \max\{1, |\lambda_i|\}$ $(i = 1, \dots, n)$ and the matrix*

$$A = \left(\frac{\mu_i}{|1-\lambda_i|} \int_a^b |p_{ik}(s)|ds \right)_{i,k=1}^n$$

satisfies (10). *Then the problem* (1′), (3) *has a unique solution.*

Corollary 4. *Let*

$$\int_a^b \ell_{ii}(1)(s)ds \neq 0 \qquad (i = 1, \dots, n) \tag{24}$$

and there exist a matrix $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$ satisfying (10) *such that* (11) *is fulfilled, where g_i is the Green's function of the problem*

$$y'(t) = \ell_{ii}(1)(t)y(t), \qquad y(b) = y(a), \tag{25_i}$$

and h_{ik} is defined by (21). *Then the problem* (1), (4) *has a unique solution.*

Proof. The condition (24) is necessary and sufficient for the problem (25_i) to have only the trivial solution for every $i \in \{1, \dots, n\}$ and its Green's function is of the form

$$g_i(t,s) = \begin{cases} \big(1 - \exp(\int_a^b \ell_{ii}(1)(\xi)d\xi)\big)^{-1} \exp(\int_s^t \ell_{ii}(1)(\xi)d\xi) \\ \qquad\qquad\qquad\qquad \text{for } a \leq s \leq t \leq b, \\ \big(\exp(-\int_a^b \ell_{ii}(1)(\xi)d\xi) - 1\big)^{-1} \exp(\int_s^t \ell_{ii}(1)(\xi)d\xi) \\ \qquad\qquad\qquad\qquad \text{for } a \leq t < s \leq b. \end{cases} \tag{26_i}$$

Now it is obvious that all the assumptions of Corollary 2 are fulfilled. Consequently, the problem (1), (4) has a unique solution. □

Corollary 4′. *Let*

$$\int_a^b p_{ii}(s)ds \neq 0 \qquad (i = 1, \dots, n) \tag{24′}$$

and there exist a matrix $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$ satisfying (10) *such that* (11) *is fulfilled, where g_i is the Green's function of the problem*

$$y'(t) = p_{ii}(t)y(t), \qquad y(b) = y(a),$$

and h_{ik} is defined by (21′). *Then the problem* (1′), (4) *has a unique solution.*

Corollary 5. *Let* (24) *be fulfilled and there exist* $\sigma_i \in \{-1, 1\}$, $\alpha_i \in]0, +\infty[$, $\alpha_{ik} \in [0, +\infty[$ $(i, k = 1, \dots, n)$ *such that the real part of every eigenvalue of the matrix* $A = (-\delta_{ik}\alpha_i + \alpha_{ik})_{i,k=1}^n$ *is negative and the inequalities*

$$\sigma_i \ell_{ii}(1)(t) \le -\alpha_i \qquad (i = 1, \dots, n), \tag{27}$$

$$\overline{\ell}_{ii}\big(|\eta(\overline{\ell}_{ik}(1))(t, \cdot)|\big)(t) + (1 - \delta_{ik})\overline{\ell}_{ik}(1)(t) \le \alpha_{ik} \tag{28}$$

$$(i, k = 1, \dots, n)$$

hold almost everywhere on I*. Then the problem* (1)*,* (4) *has a unique solution.*

Proof. At first note that according to Theorems 1.13 and 1.18 in [10] the negativeness of real parts of the eigenvalues of the matrix A yields the inequality

$$r(\overline{A}) < 1, \tag{29}$$

where

$$\overline{A} = \left(\frac{\alpha_{ik}}{\alpha_i}\right)_{i,k=1}^n .$$

On the other hand, from (27) it follows that for every $i \in \{1, \dots, n\}$ the problem (25_i) has only the trivial solution and its Green's function g_i is given by (26_i). Put

$$\Delta_i(s, t) = \exp(-\sigma_i\alpha_i(t - s)) \qquad (i = 1, \dots, n).$$

Then for every $i \in \{1, \dots, n\}$ from (26_i) and (27) we obtain

$$|g_i(t, s)| \le \begin{cases} \big[\sigma_i(1 - \Delta_i(a, b))\big]^{-1}\Delta_i(s, t) & \text{for } a \le s \le t \le b, \\ \big[\sigma_i(1 - \Delta_i(a, b))\big]^{-1}\Delta_i(a, b)\Delta_i(s, t) & \text{for } a \le t < s \le b. \end{cases} \tag{30}$$

Define the functions h_{ik} by (21). Then from (28) and (30) we get

$$\int_a^b |g_i(t, s)| h_{ik}(s) ds \le \tag{31}$$

$$\le \alpha_{ik}\big[\sigma_i(1 - \Delta_i(a, b))\big]^{-1}\left(\int_a^t \Delta_i(s, t)ds + \Delta_i(a, b)\int_t^b \Delta_i(s, t)ds\right) =$$

$$= \frac{\alpha_{ik}}{\alpha_i}\big[1 - \Delta_i(a, b)\big]^{-1}\big(1 - \Delta_i(a, t) + \Delta_i(a, b)\Delta_i(b, t) - \Delta_i(a, b)\big) =$$

$$= \frac{\alpha_{ik}}{\alpha_i} \qquad (i, k = 1, \dots, n).$$

Taking into account (29) we conclude that all the assumptions of Corollary 2 are fulfilled. Consequently, the problem (1), (4) has a unique solution. □

Corollary 5′. *Let* (24′) *be fulfilled and there exist* $\sigma_i \in \{-1,1\}$, $\alpha_i \in]0,+\infty[$, $\alpha_{ik} \in [0,+\infty[$ $(i,k=1,\dots,n)$ *such that the real part of every eigenvalue of the matrix* $A = (-\delta_{ik}\alpha_i + \alpha_{ik})_{i,k=1}^n$ *is negative and the inequalities*

$$\sigma_i p_{ii}(t) \le -\alpha_i \qquad (i=1,\dots,n),$$

$$\left| p_{ii}(t) \int_t^{\tau_{ii}(t)} |p_{ik}(s)|ds \right| + (1-\delta_{ik})|p_{ik}(t)| \le \alpha_{ik}$$

$$(i,k=1,\dots,n)$$

hold almost everywhere on I. *Then the problem* (1′), (4) *has a unique solution.*

The last two corollaries concern with the Cauchy problems (1), (5) and (1′), (5).

Corollary 6. *Let* $t_0 \in I$ *and there exist a nonnegative integer* m_0, *a natural number* $m > m_0$, *and* $\alpha \in]0,1[$ *such that*

$$\rho_{im}(t) \le \alpha \rho_{im_0}(t) \qquad \text{for } t \in I \qquad (i=1,\dots,n), \tag{32}$$

where

$$\rho_{i0}(t) \equiv 1 \qquad (i=1,\dots,n),$$

$$\rho_{ij}(t) = \sum_{k=1}^n \left| \int_{t_0}^t \overline{\ell}_{ik}(\rho_{kj-1})(s)ds \right| \qquad (i=1,\dots,n; j=1,2,\dots).$$

Then the problem (1), (5) *has a unique solution.*

Proof. For every $i \in \{1,\dots,n\}$ we define the following sequences of operators $\rho_{ij} : C(I;\mathbb{R}^n) \to C(I;\mathbb{R})$:

$$\rho_{i0}(u_1,\dots,u_n)(t) \overset{def}{=} u_i(t),$$

$$\rho_{ij}(u_1,\dots,u_n)(t) \overset{def}{=} \sum_{k=1}^n \left| \int_{t_0}^t \overline{\ell}_{ik}(\rho_{kj-1}(u_1,\dots,u_n))(s)ds \right| \qquad (j=1,2,\dots).$$

Then for any $(u_i)_{i=1}^n \in C(I;\mathbb{R}^n)$,

$$\rho_{ij}(u_1,\dots,u_n)(t) = \rho_{ij-j_0}(\rho_{1j_0}(u_1,\dots,u_n),\dots,\rho_{nj_0}(u_1,\dots,u_n))(t) \tag{33}$$

$$(i=1,\dots,n; j \ge j_0; j,j_0 = 0,1,2,\dots)$$

and

$$\rho_{ij}(1,\dots,1)(t)=\rho_{ij}(t) \qquad (i=1,\dots,n;\, j=0,1,2,\dots). \tag{34}$$

Now let $(y_i)_{i=1}^n$ be a solution of (1_0) satisfying the initial conditions

$$y_i(t_0)=0 \qquad (i=1,\dots,n).$$

Then for every nonnegative integer j,

$$|y_i(t)|\le \rho_{ij}(|y_1|,\dots,|y_n|)(t) \qquad (i=1,\dots,n). \tag{35_j}$$

By (34) from (35_{m_0}) we find

$$|y_i(t)|\le \rho_{im_0}(t)\sum_{k=1}^n \|y_k\|_C \qquad (i=1,\dots,n). \tag{36}$$

Let for every $i\in\{1,\dots,n\}$,

$$v_i(t)=\begin{cases}0 & \text{if } \rho_{im_0}(t)=0,\\ \frac{|y_i(t)|}{\rho_{im_0}(t)} & \text{if } \rho_{im_0}(t)\ne 0.\end{cases}$$

Then (36) yields

$$\gamma_i=\text{ess sup}\{v_i(t):t\in I\}<+\infty \qquad (i=1,\dots,n)$$

and

$$|y_i(t)|\le \gamma_i\rho_{im_0}(t)=\gamma_i\rho_{im_0}(1)(t) \qquad (i=1,\dots,n),$$

whence by (32), (33) and (35_{m-m_0}) for every $i\in\{1,\dots,n\}$ we get

$$\begin{aligned}|y_i(t)|&\le \rho_{im-m_0}(|y_1|,\dots,|y_n|)(t)\le\\ &\le \gamma\rho_{im-m_0}(\rho_{1m_0}(1),\dots,\rho_{nm_0}(1))(t)=\\ &=\gamma\rho_{im}(1)(t)=\gamma\rho_{im}(t)\le\gamma\alpha\rho_{im_0}(t),\end{aligned}$$

where $\gamma=\max\{\gamma_1,\dots,\gamma_n\}$. Hence we obtain

$$v_i(t)\le\alpha\gamma \qquad (i=1,\dots,n)$$

and, consequently,

$$\gamma\le\alpha\gamma.$$

Since $\alpha\in]0,1[$, we have $\gamma=0$, which implies $y_i(t)\equiv 0$ $(i=1,\dots,n)$. Consequently, the problem (1), (5) has a unique solution. □

If $m=2$, $m_0=1$, then Corollary 6 yields the following result for the problem $(1')$, (5):

Corollary 6′. *Let $t_0 \in I$ and $\alpha \in]0,1[$ be such that*

$$\sum_{k=1}^{n} \left| \int_{t_0}^{t} |p_{ik}(s)| \sum_{j=1}^{n} \left| \int_{t_0}^{\tau_{ik}(s)} |p_{kj}(\xi)| d\xi \right| ds \right| \leq \alpha \sum_{k=1}^{n} \left| \int_{t_0}^{t} |p_{ik}(s)| ds \right| \qquad \text{for } t \in I$$

$$(i = 1, \dots, n).$$

Then the problem (1′), (5) has a unique solution.

At the end of this subsection we give the examples verifying the optimality of the above formulated conditions in the existence and uniqueness theorems.

Example 1. Let $n = 2$, $\lambda_1 \in [-1, 1[$, $\lambda_2 \in]-\infty, -1[\cup]1, +\infty[$. On the segment $I = [0, 1]$ consider the system (1′) with the boundary conditions (3), where

$$p_{1k}(t) = \begin{cases} \delta_{1k}(\lambda_1 - 1) & \text{for } 0 \leq t \leq \frac{1}{2} \\ (1 - \delta_{1k})(\lambda_1 - 1) & \text{for } \frac{1}{2} < t \leq 1 \end{cases} \qquad (k = 1, 2),$$

$$p_{2k}(t) = \begin{cases} \delta_{2k}(\frac{\lambda_2 - 1}{\lambda_2}) & \text{for } 0 \leq t \leq \frac{1}{2} \\ (1 - \delta_{2k})(\frac{\lambda_2 - 1}{\lambda_2}) & \text{for } \frac{1}{2} < t \leq 1 \end{cases} \qquad (k = 1, 2),$$

$$\tau_{11}(t) \equiv \tau_{21}(t) \equiv 0, \qquad \tau_{12}(t) \equiv \tau_{22}(t) \equiv 1,$$

$q_i \in L(I; \mathbb{R})$, and $c_i \in \mathbb{R}$ $(i = 1, 2)$. Then all the assumptions of Corollary 3′ with $\mu_1 = 1$, $\mu_2 = |\lambda_2|$,

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

are fulfilled except the condition (10) instead of which we have

$$r(A) = 1. \tag{37}$$

On the other hand, the homogeneous problem

$$x_i'(t) = \sum_{k=1}^{2} p_{ik}(t) x_k(\tau_{ik}(t)) \qquad (i = 1, 2), \tag{38}$$

$$x_1(1) = \lambda_1 x_1(0), \qquad x_2(1) = \lambda_2 x_2(0) \tag{39}$$

has the nontrivial solution

$$x_1(t) = (\lambda_1 - 1)t + 1, \qquad x_2(t) = \frac{\lambda_2 - 1}{\lambda_2} t + \frac{1}{\lambda_2}.$$

This example shows that the strict inequality (10) in Corollaries 3 and 3′ cannot be replaced by the nonstrict one.

Example 2. On the segment $I = [a, b]$ consider the problem (1′), (4) with $n \geq 2$, constant coefficients $p_{ii}(t) \equiv p_{ii} = -1$, $p_{ik}(t) \equiv p_{ik} = \frac{1}{n-1}$ $(i \neq k; i, k = 1, \ldots, n)$, and $\tau_{ik} : I \to I$ $(i, k = 1, \ldots, n)$ arbitrary measurable functions. Then the vector $\gamma = (\gamma_i)_{i=1}^n \in \mathbb{R}^n$, where $\gamma_1 = \gamma_2 = \cdots = \gamma_n \neq 0$ satisfies the equality

$$P\gamma = 0,$$

where $P = (p_{ik})_{i,k=1}^n$, i.e., P has a zero eigenvalue. Thus all the assumptions of Corollary 5′ are fulfilled with $\sigma_i = 1$, $\alpha_i = |p_{ii}|$, $\alpha_{ik} = (1 - \delta_{ik})p_{ik}$, $(i, k = 1, \ldots, n)$, i.e., $A = P$, except the negativeness of real part of every eigenvalue of the matrix A.

On the other hand, the vector $(\gamma_i)_{i=1}^n$ is a nontrivial solution of the homogeneous problem

$$x_i'(t) = \sum_{k=1}^n p_{ik} x_k(\tau_{ik}(t)), \qquad x_i(b) = x_i(a).$$

This example shows that in Corollaries 5 and 5′ the requirement on the negativeness of the real part of every eigenvalue of the matrix A cannot be weakened.

Example 3. Let $I = [0, 1]$, $t_0 = 0$, $\tau_{ik}(t) \equiv 1$ $(i, k = 1, \ldots, n)$

$$p_{ik}(t) = \begin{cases} 1 & \text{for } t \in [\frac{k-1}{n}, \frac{k}{n}[\\ 0 & \text{for } t \in I \setminus [\frac{k-1}{n}, \frac{k}{n}[\end{cases} \qquad (i, k = 1, \ldots, n),$$

and consider the problem (1′), (5). Put

$$\rho_{i0}(t) \equiv 1, \qquad \rho_{ij}(t) = \sum_{k=1}^n \int_0^t p_{ik}(s)\rho_{ij-1}(\tau_{ik}(s))ds$$

$$(i = 1, \ldots, n; j = 1, 2, \ldots).$$

Then

$$\rho_{ij}(t) = t \qquad (i = 1, \ldots, n; j = 1, 2, \ldots)$$

and for every nonnegative integer m_0 and every natural number $m > m_0$ we have

$$\rho_{im}(t) \leq \rho_{im_0}(t) \qquad \text{for } t \in I \qquad (i = 1, \ldots, n).$$

On the other hand,

$$x_i(t) = t \qquad (i = 1, \ldots, n)$$

is a nontrivial solution of the homogeneous problem

$$x_i'(t) = \sum_{k=1}^n p_{ik}(t) x_k(\tau_{ik}(t)), \qquad x_i(t_0) = 0.$$

The last example shows that in Corollaries 6 and 6′ we cannot choose $\alpha = 1$.

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